Exact results on the dynamics of a multicomponent Bose-Einstein condensate

Pijush K. Ghosh
Department of Physics, Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan
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We study the time evolution of a two-dimensional multicomponent Bose-Einstein condensate in an external harmonic trap with arbitrary time-dependent frequency. We show analytically that the time evolution of the total mean-square radius of the wave packet is determined in terms of the same solvable equation as in the case of a single-component condensate. The dynamics of the total mean-square radius is also the same for a rotating as well as a nonrotating multicomponent condensate. We determine the criteria for the collapse of the condensate at a finite time. Generalizing our previous work on a single-component condensate, we show explosion-implosion duality in the multicomponent condensate.

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The successful creation and observation of Bose-Einstein condensation (BEC) in dilute alkali-metal atoms have opened up a plethora of possibilities to test otherwise intractable many-body quantum phenomena in the laboratory [1]. The Gross-Pitaevskii equation (GPE), the mean-field description of the BEC, is successful enough in explaining most of the observed results as well as in predicting additional phenomena. The methods involved in studying the GPE are mainly numerical and/or approximate: perturbative and variational. The exact and analytical results of a nonlinear equation, if known, not only act as a guide to determine the validity of different approximate and numerical methods, they also give rise to counterintuitive results in some cases. Unfortunately, no exact solution of the GPE is known except in one dimension.

The two-dimensional GPE, like its counterparts in higher dimensions, is not exactly solvable. However, due to an underlying dynamical O(2,1) symmetry [2], the time evolution of certain moments related to the two-dimensional GPE can be described exactly [3]. This result is valid even if the condensate is considered in a time-dependent harmonic trap. This leads to the prediction of explosion-implosion duality [4] and extended parametric resonance [4,5] in the two-dimensional BEC. Both of these phenomena are universal for any nonrelativistic theory having dynamical O(2,1) symmetry [3,4,6]. Interestingly enough, apart from the two-dimensional BEC, the same explosion-implosion duality can also be observed in supernova explosions and in laser induced implosions in plasma [7,8]. This shows the importance of exact methods, based on an underlying symmetry, in relating diverse areas of physics such as the BEC and the supernova explosion.

The results described above are for a single-component condensate, where the spin degrees of freedom have been frozen though the use of a magnetic trap. Recently, a spinor condensate with independent spin degrees of freedom has also been created and observed in the laboratory [9]. Similarly, the two-component condensate, where two different hyperfine states of the same atomic species are condensed simultaneously, has been experimentally realized [10]. The spinor condensate [11–14] and the two-component condensate [15] have a very rich structure compared to the single-component condensate. This is manifested in the existence of topological defects like skyrmions, domain walls, vortices, and Alice strings in these condensates [16].

The purpose of this note is to extend the studies of Refs. [3,4] on the two-dimensional single-component BEC to the two dimensional multicomponent BEC. The experimentally realizable two-component and spinor condensates can be obtained as special cases of this general multicomponent BEC. We study the exact time evolution of the second moment of the two-dimensional multicomponent condensate in an arbitrary time-dependent harmonic trap. This particular second moment can be identified as the *total mean-square radius* of the condensate. We show that the dynamics of the second moment is determined by the same solvable equation as in the case of a single-component condensate. No matter how many components there are, or how they interact among themselves, or even whether they are rotating or nonrotating, the dynamics of the total mean-square radius is universally determined by the same equation. The detailed information on the system is encoded, through the Hamiltonian, into a constant of motion appearing in this universal equation. Thus, the dynamics of the system can be studied in terms of the same set of initial conditions for any number of components. We determine the criteria for the collapse of the condensate of this system. We also show that the multicomponent BEC, in its full generality, exhibits an explosion-implosion duality and extended parametric resonance for special choices of the time dependence of the trap. All these results are exact and analytical.

Consider the following Lagrangian in $2+1$ dimensions:

$$\mathcal{L}=\sum_a \left(i\psi_a^* \partial_\tau \psi_a - \frac{1}{2m}|\nabla \psi_a|^2 \right) - \frac{1}{2}\sum_{abcd} g_{abcd}\psi_a^*\psi_b^*\psi_c\psi_d, \quad a,b,c,d=1,2,\ldots,n, \tag{1}$$

where n is the total number of components. The coupling constants g_{abcd} are related to the s-wave scattering length matrix. The possible values of g_{abcd}, and hence of the scattering length matrix, may be constrained by symmetry requirements. For example, the special case of a two-component condensate can be obtained by choosing $n=2$ and $g_{abcd}=\frac{1}{2}(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\overline{g}_{ab}$ so that the system has a global $\mathrm{U}(1)^2$ symmetry. A phase separation occurs for such a system if all the scattering lengths are positive and satisfy the inequality $g_{12}^2=g_{21}^2>g_{11}g_{22}$ [15]. Similarly, the spin-1 spinor condensate can be obtained by choosing $n=3$ and $g_{abcd}=\frac{1}{2}[g_1\delta_{ac}\delta_{bd}+g_2\Sigma_\alpha(S_\alpha)_{ac}(S_\alpha)_{bd}+(a\leftrightarrow b)]$, where the S_α's are three spin matrices. A positive g_2 defines an antiferromagnetic regime, while the ferromagnetic regime is characterized by a negative g_2. It is known that the ferromagnetic or antiferromagnetic nature of the interaction plays an important role in characterizing different properties of the condensate [11,12]. Both the phenomenon of phase separation and the ferromagnetic or antiferromagnetic nature of the ground state are specific to multicomponent condensates for $n\geqslant 2$. Further, note that we have additional terms describing the interaction among different components as we go from the single-component to the two-component, to the spinor, and to the general multicomponent condensate described by Eq. (1). However, to our surprise, the dynamics of the total mean-square radius is independent of such variation in the intercomponent interaction and universally determined by the same solvable equation as in the case of a single-component ($n=1$) condensate. Consequently, the criterion for the collapse of the condensate at a finite time is also the same for any n-component condensate. For the very special case of an additional global $\mathrm{U}(1)^n$ symmetry in Eq. (1), such a result has been obtained previously in Ref. [17]. We remark that our results are much more general. Moreover, the known results are reproduced in a very elegant way. We will consider only the most general form of $\mathcal{L}$ from now on, since our result is independent of particular details of the interaction.

All the coupling constants g_{abcd} have inverse-mass dimension in the natural units with $c=\hbar=1$. This allows us to have a scale and conformally invariant theory. The action $\mathcal{A}=\int d\tau d^2\mathbf{r}\mathcal{L}$ is invariant under the following time-dependent transformations [18–24]:

$$\mathbf{r}\rightarrow\mathbf{r}_h=\dot{\tau}(t)^{-1/2}\mathbf{r},\quad \tau\rightarrow t=t(\tau),\quad \dot{\tau}(t)=\frac{d\tau(t)}{dt},$$

$$\psi_a(\tau,\mathbf{r})\rightarrow\psi_a^h(t,\mathbf{r}_h)=\dot{\tau}^{d/4}\exp\left(-im\frac{\ddot{\tau}}{4\dot{\tau}}r_h^2\right)\psi_a(\tau,\mathbf{r}),\quad (2)$$

with the scale factor τ given by

$$\tau(t)=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \alpha\delta-\beta\gamma=1.\quad (3)$$

Note that all the components of the order parameter are multiplied by the same time dependent scale-factor and phase in the symmetry transformation above. One might naively think that the requirement of identical phase factors for all the components of the order parameter is due to the interaction term. However, this is not the case. Even if we consider the free theory, i.e., $g_{abcd}=0$ for all values of the indices, the requirement of identical phases in Eq. (2) is essential in order for it to be a symmetry transformation. This is precisely because the transformation of the scalar fields is coupled with that of the space-time coordinates. If we choose some special values for the coupling constants g_{abcd} such that the Lagrangian has an internal global symmetry, say, for example, $\mathrm{SU}(n)$, we certainly have the freedom of varying the phase factors up to a global $\mathrm{SU}(n)$ rotation. However, such additional internal symmetries are completely decoupled from the symmetry transformations described in Eqs. (2) and (3), and do not have any effect on our results.

Let us now introduce two moments I_1 and I_2 in terms of the density ρ and the current $\mathbf{j}$, as

$$\rho(\tau,\mathbf{r})=\sum_a \psi_a^*\psi_a,$$

$$\mathbf{j}(\tau,\mathbf{r})=-\frac{i}{2m}\sum_a(\psi_a^*\nabla\psi_a-\psi_a\nabla\psi_a^*),$$

$$(4)$$

$$I_1(\tau)=\frac{m}{2}\int d^2\mathbf{r}r^2\rho,$$

$$I_2(\tau)=\frac{m}{2}\int d^2\mathbf{r}\mathbf{r}\cdot\mathbf{j}.$$

We are dealing with a conservative system and the total number of particles $N(\tau)=\int d^2\mathbf{r}\rho$ is a constant of motion. The global U(1) symmetry of $\mathcal{L}$ can be enlarged to $\mathrm{U}(1)^n$ for certain special choices of g_{abcd}. The total number of particles for each species is conserved separately for this case. However, as emphasized earlier, such an additional internal symmetry does not have any significant effect on our results. Thus, only the conservation of the total number of particles N is important for our study. The moment I_1 is the sum of the mean-square radii corresponding to each and every component. This moment can be interpreted as the square of the width of the wave packet for the single-component condensate, when confined in an external harmonic trap [5]. However, for the multicomponent case, the moment I_1 cannot be identified as the total width of the wave packet. As emphasized in our previous work [3], the moment I_1 has been used extensively in the analysis of the nonlinear Schrödinger equation (NLSE) [5,6,25–27] and the BEC [28], and in optics [29]. The dynamics of I_1, when the system (1) is immersed in an external time-dependent harmonic trap, is the central subject of the investigation of this paper. We show that the dynamics of I_1 is universally determined by the same solvable equation as in the case of a single-component BEC.

Particular choices of $\tau(t)=t+\beta,\alpha^2t$, and $t/(1+\gamma t)$, correspond to time translation, dilatation, and special conformal transformation. The corresponding generators of these transformations, the Hamiltonian H, the dilatation generator D, and the conformal generator K are

$$H=\int d^2\mathbf{r}\left[\frac{1}{2m}\sum_a |\nabla\psi_a|^2 + \frac{1}{2}\sum_{abcd} g_{abcd}\psi_a^*\psi_b^*\psi_c\psi_d\right],$$

$$D=\tau H - I_2, \tag{5}$$

$$K=-\tau^2 H + 2\tau D + I_1.$$

These generators close under the algebra

$$[H,D]=iH, \quad [H,K]=2iD, \quad [K,D]=-iK \tag{6}$$

if we promote the fields ψ_a to the operators $\hat{\psi}_a$ with the following bosonic commutation relations among themselves:

$$[\hat{\psi}_a(\mathbf{r}),\hat{\psi}_b^*(\mathbf{r}')]=\delta_{ab}\delta(\mathbf{r}-\mathbf{r}'),$$

$$[\hat{\psi}_a(\mathbf{r}),\hat{\psi}_b(\mathbf{r}')]=[\hat{\psi}_a^*(\mathbf{r}),\hat{\psi}_b^*(\mathbf{r}')]=0. \tag{7}$$

The algebra given by Eq. (6) defines a conformal group, which is isomorphic to the group O(2,1) [18]. Thus, the system (1) has a dynamical O(2,1) symmetry with the interpretation of the fields ψ_a as the operators $\hat{\psi}_a$ satisfying Eq. (7). In this article, we will be considering only the fields ψ_a, not the operators $\hat{\psi}_a$. We do not make use of the relations (7) or the algebra given by Eq. (6) in our subsequent discussion; what is required for our study is the conserved Noether charges H, D, and K. We just mention, in passing, that the results described in this article are valid for any nonrelativistic theory with a dynamical O(2,1) symmetry.

The generators H, D, and K are constant in time and lead to the following equations:

$$\frac{dH}{d\tau}=0, \quad \frac{dI_1}{d\tau}=2I_2, \quad \frac{dI_2}{d\tau}=H. \tag{8}$$

For time-independent solutions, neither I_1 nor I_2 depends on τ. As a consequence, the energy of the static solutions of H vanishes. This is also the case for the single-component BEC in $2+1$ dimensions. Even though there are extra terms due to intercomponent interaction in the case of a multicomponent BEC, the vanishing of the energy is a universal consequence of the underlying O(2,1) symmetry. The second equation of (8) shows that the moment I_2 is proportional to the time variation of the moment I_1. Recalling that the moment I_1 is identified as the total mean-square radius of the condensate, the moment I_2 can be related to the speed of growth of the condensate. This interpretation is also evident in the definition of I_2 in Eq. (4) after decomposing the current $\mathbf{j}$ as a product of the density ρ and the velocity.

Defining $X=\sqrt{I_1}$, it is easy to find a decoupled equation for X from Eq. (8):

$$\frac{d^2X}{d\tau^2}=\frac{Q}{X^3}, \quad Q=I_1 H - I_2^2, \quad \frac{dQ}{d\tau}=0. \tag{9}$$

The constant of motion Q is the Casimir operator of the O(2,1) symmetry. Note that the information on the Hamiltonian H is solely contained in Q. Thus, the effect of the interaction, say, for example, the strongly repulsive or attractive intercomponent and intracomponent interactions, will be manifested through initial conditions on H. Equation (9) can be interpreted as the equation of motion of a particle moving in an inverse-square potential. Interestingly enough, this system also has a dynamical O(2,1) symmetry. This reduced system of a particle in an inverse-square potential is a well-studied problem and the solution is given by [18]

$$X^2=(a+b\tau)^2+\frac{Q}{a^2}\tau^2, \tag{10}$$

where a and b are integration constants. Although any exact solution of the equation of motion of the action $\mathcal{A}$ is not known, it is surprising to note how the exact time dependence of the moment I_1 can be obtained easily using the underlying symmetry. We would like to stress that we are able to determine the dynamics of the total mean-square radius of the condensate only. The dynamics of the individual mean-square radii associated with each component cannot be obtained using our method even when there is an additional $U(1)^n$ symmetry in the system or there is no intercomponent interaction.

The criterion for the collapse of the condensate at a finite and real time τ^* is $Q\leqslant 0$. In particular, the moment X^2 vanishes at a finite time τ^*,

$$\tau^*=\frac{a^2}{(a^2b^2+Q)}[-ab\pm\sqrt{-Q}], \tag{11}$$

which is real if $Q\leqslant 0$. Note that we have the freedom of making τ^* either positive or negative by choosing appropriate values for the integration constants a and b. With the interpretation of Eq. (9) as a particle moving in an inverse-square potential, the collapse of the condensate can be understood as the fall of the particle to the center for attractive interaction. Recall that the moment I_1 is semipositive definite by definition. Thus, the exact expression for Q implies that the condensate collapses for any initial condition if $H\leqslant 0$. On the other hand, if $H>0$, the condition for the collapse is given by

$$\left.\frac{dI_1}{d\tau}\right|_{\tau=0}\leqslant -2\sqrt{I_1|_{\tau=0}|H|}. \tag{12}$$

We have used the second equation of (8) in the exact expression for Q in deriving the above equation. As far as we are aware, this is the first instance in the literature where the criterion for the collapse of the condensate of the most general two-dimensional multicomponent NLSE with cubic nonlinearity is given. The criterion is independent of the total number of components n and any additional global internal symmetry. Thus, the well-known results on the single-component [25] and the multicomponent [17] NLSE's in two dimensions are easily reproduced from our general result.

Consider the following time-dependent transformation:

$$\tau \rightarrow t = t(\tau), \quad \dot{\tau}(t) = \frac{d\tau(t)}{dt},$$

$$\mathbf{r} \rightarrow \mathbf{r_h} = \dot{\tau}(t)^{-1/2} \begin{pmatrix} \cos f(t) & \sin f(t) \\ -\sin f(t) & \cos f(t) \end{pmatrix} \mathbf{r}, \tag{13}$$

$$\psi_a(\tau,\mathbf{r}) \rightarrow \psi_a^h(t,\mathbf{r}_h) = \dot{\tau}^{d/4} \exp\left(-im\frac{\ddot{\tau}}{4\dot{\tau}} r_h^2 \right) \psi_a(\tau,\mathbf{r}),$$

with arbitrary $\tau(t)$ and $f(t)$. Note that this transformation can be obtained by first using the transformation (2) and then a time-dependent rotation around the z axis with a time-dependent angle $f(t)$. For arbitrary $\tau(t)$ and $f(t)$, the transformation (13) is not a symmetry transformation of the action $\mathcal{A}$; instead, it maps $\mathcal{A}$ to a new action $\mathcal{A}_h = \int dt d^2\mathbf{r} \mathcal{L}_h$. The new Lagrangian $\mathcal{L}_h$ reads as

$$\mathcal{L}_h = \sum_a \left(i\psi_a^{h*}\partial_t \psi_a^h - \frac{1}{2m} |\boldsymbol{\nabla}_h \psi_a^h|^2 \right) - \frac{1}{2} \sum_{abcd} g_{abcd} \psi_a^{h*} \psi_b^{h*} \psi_c^h \psi_d^h - \sum_a \left(\frac{1}{2} m\omega(t) r_h^2 |\psi_a^h|^2 + \dot{f} \psi_a^{h*} L_z \psi_a^h \right), \tag{14}$$

where the z component of the angular momentum $L_z = -i\mathbf{r}_h \times \boldsymbol{\nabla}_h$, and the time-dependent frequency $\omega(t)$ of the harmonic trap is determined as

$$\ddot{b} + \omega(t) b = 0, \quad b(t) = \dot{\tau}^{-1/2}. \tag{15}$$

The Lagrangian $\mathcal{L}_h$ is that of a rotating multicomponent BEC in an arbitrary time-dependent harmonic well. Note that the external harmonic potentials are identical for all the components of the condensate. This is not by choice. In fact, we do not have the freedom of generating different harmonic potentials for different components using the transformation in Eq. (13). This is even true for the free theory, i.e., $g_{abcd} = 0$. The reason is that the transformation of the scalar fields is coupled with that of the space-time coordinates. Consequently, unphysical and unwanted terms will be generated in the Lagrangian $\mathcal{L}_h$ unless all the components of the condensate transform identically.

The solutions of $\mathcal{A}$ and $\mathcal{A}_h$ are related to each other through the transformations in Eq. (13) with $\tau(t)$ determined for a specific trap frequency by Eq. (15). The scale factor $\tau(t)$ can obviously be exactly determined for a large class of $\omega(t)$. However, the exact solutions are not known for either $\mathcal{A}$ or $\mathcal{A}_h$. This is a major problem in making use of the mapping relating $\mathcal{A}$ to $\mathcal{A}_h$ and vice versa. However, note that the dynamics of the moment I_1 is uniquely determined by Eq. (10) independent of whether any exact solution of $\mathcal{A}$ is known or not. Thus, the transformation (13) can be used to find the dynamics of the moment $I_{1,h} = \Sigma_a \int d^2\mathbf{r}_h r_h^2 |\psi_a^h|^2$ from I_1. In particular, they are related to each other by

$$X_h = \sqrt{I_{1,h}} = b(t) X(\tau(t)), \tag{16}$$

where $b(t)$ and $\tau(t)$ are determined from Eq. (15). Thus, even though the exact solution of the equation of motion of $\mathcal{A}_h$ is not known, the dynamics of X_h can be described exactly.

An alternative but useful expression for X_h can be determined from the following equation [3]:

$$\frac{d^2 X_h}{dt^2} + \omega(t) X_h = \frac{Q_h}{X_h^3}, \quad Q_h = I_{1,h} H_h - I_{2,h}^2, \tag{17}$$

where Q_h is a constant of motion. Both H_h and $I_{2,h}$ have the same expressions as H and I_2, respectively, with $(\tau, \mathbf{r}, \psi_a)$ replaced by $(t, \mathbf{r}_h, \psi_a^h)$. Note that Eq. (17) can also be interpreted as describing the motion of a classical particle in a combined harmonic and inverse-square potential. The particle falls to the center for an attractive $(Q_h < 0)$ inverse-square potential, independent of the time dependence of the harmonic trap. This implies that the condensate collapses at a finite time for $Q_h < 0$. Analyzing the exact expression for Q_h further, we find that the condensate collapses for any initial condition if $H_h \leqslant 0$. For $H_h > 0$, the condition for the collapse is given by

$$\left.\frac{dI_{1,h}}{dt}\right|_{t=0} < -2\sqrt{(I_{1,h} H_h)|_{t=0}}, \tag{18}$$

where the relation [3] $\dot{I}_{1,h} = 2 I_{2,h}$, valid for the system described by the action $\mathcal{A}_h$, has been used. As far as we are aware, this is the first time in the literature that a criterion for the collapse of the condensate in the most general two-dimensional multicomponent GPE with cubic nonlinearity and an arbitrary time-dependent harmonic trap is given. Note that the criterion is independent of the total number of components n and any additional internal global symmetry. The known results for the single-component [26] and multicomponent [17] GPE's with time-independent harmonic trap in $2+1$ dimensions are easily reproduced from this very general result. Further, the criterion for collapse in a system without or with a harmonic trap is also identical, except that there is no equality sign in Eq. (18) for the latter case [compare with Eq. (12)]. This is precisely because, for $Q, Q_h = 0$, Eqs. (9) and (17) describe the dynamics of a free particle and that of a particle in a time-dependent harmonic trap, respectively. Thus, nothing can be said conclusively about the dynamical (in)stability for the latter case, unless the time dependence of the frequency of the trap is explicitly specified. For a time-independent trap, the equality sign is recovered in Eq. (18); and, of course, the known result [17,26] is identically reproduced.

We have shown that the criterion for the collapse of the condensate in a $(2+1)$-dimensional system governed by the Lagrangian $\mathcal{L}_h$ is independent of the total number of components n. It is known [26] that the same criterion for the collapse of the condensate is also valid for the Lagrangian $\mathcal{L}_h$ in dimensions $d \geqslant 2+1$ with $n=1$, $\omega(t) = \omega_0 = \text{const}$, and $\dot{f} = 0$. So the criterion for collapse is independent of the underlying O(2,1) symmetry, which the cubic NLSE has only in $d = 2+1$. The dynamical O(2,1) symmetry only helps us in deriving the exact result in a much more simple and elegant way. Based on this observation, we conjecture that the criterion for the collapse of the condensate of $\mathcal{L}_h$ in dimensions $d \geqslant 2+1$, with $\omega(t)=\omega_0$ and $\dot{f}=0$, is independent of the total number of components n and the criterion is the same as stated in this article for $d=2+1$. Note that the physically interesting case of $d=3+1$ is also included in our conjecture.

The solution for X_h is given by

$$X_h^2 = u^2(t) + \frac{Q_h}{W^2} v^2(t), \quad W(t) = u\dot{v} - v\dot{u}, \tag{19}$$

where $u(t)$ and $v(t)$ are two independent solutions of Eq. (15) satisfying $u(t_0)=X_h(t_0)$, $\dot{u}(t_0)=\dot{X}_h(t_0)$, $\dot{v}(t_0)=0$, and $v(t_0)\neq 0$. The above solution is valid for arbitrary Q_h: positive, negative, or zero. We will be considering the case $Q_h \geqslant 0$ from now on, since we have already argued that the condensate collapses for $Q_h<0$. We obtained the same expressions (16) and (19) in Refs. [3,4] for the dynamics of the width of the wave packet of a single-component condensate in $2+1$ dimensions. So the results of the Refs. [3,4] are equally valid for the general multicomponent condensate in $2+1$ dimensions with the moment $I_1 = X_h^2$ identified as the total mean-square radius. In particular, (a) the system described by $\mathcal{L}_h$ has an explosion-implosion duality for $\dot{f}(t)=0$ and either $\omega(t)=0$ or $\omega(t)=t^{-2}$, (b) the condensate exhibits extended parametric resonance for a periodic $\omega(t)$ and arbitrary $f(t)$, and (c) the dynamic (in)stability of the system is independent of $f(t)$, i.e., the same for both rotating and nonrotating BEC's. We refer the readers to Refs. [3,4] for further details.

Finally, we conclude with the following comment. The results presented in this article for the multicomponent BEC are a generalization of what is already known for the single-component BEC in two dimensions. The results obtained in both these cases are also identical with the identification of the moment I_1 as the total mean square radius. In particular, the dynamics of the moment I_1 is determined from the same solvable equation as in the case of a single-component BEC with all the information about the Hamiltonian encoded into the constant of motion Q_h. Apart from its relevance to the ongoing experiments on BEC's, the importance of this result lies in its universality. No matter how many components there are, or how they interact among themselves, or even whether they are rotating or nonrotating, the dynamics of the total mean-square radius is universally determined by the same equation. This is indeed a counterintuitive result and may be realized in the laboratory in the near future.

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[1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[2] We refer to the papers in Refs. [18] and [19] for the application of O(2,1) symmetry in physics. The models considered in these papers also have a dynamical O(2,1) symmetry.
[3] P. K. Ghosh, Phys. Rev. A **65**, 012103 (2002).
[4] P. K. Ghosh, e-print cond-mat/0109073 (unpublished).
[5] J. J. Garcia-Ripoll and V. M. Perez-Garcia, Phys. Rev. Lett. **83**, 1715 (1999); patt-sol/9904006.
[6] T. K. Ghosh, Phys. Lett. A **285**, 222 (2001).
[7] I. O'C Drury and J. T. Mendonca, Phys. Plasmas **7**, 5148 (2000).
[8] K. Walter, www.llnl.gov/str/BRemington.html; I. Hachisu *et al.*, Astrophys. J. Lett. **368**, L27 (1991); H. Sakagami and K. Nishihara, Phys. Fluids B **2**, 2715 (1990).
[9] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, and W. Ketterle, Nature (London) **369**, 345 (1998); H.-J. Miesner, D. M. Stamper-Kurn, J. Stenger, S. Inouye, A. P. Chikkatur, and W. Ketterle, Phys. Rev. Lett. **82**, 2228 (1999); D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, S. Inouye, J. Stenger, and W. Ketterle, *ibid.* **83**, 661 (1999).
[10] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett. **78**, 586 (1997); M. R. Matthews, D. S. Hall, D. S. Jin, J. R. Ensher, C. E. Wieman, E. A. Cornell, F. Dalfovo, C. Minniti, and S. Stringari, *ibid.* **81**, 243 (1998); D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, *ibid.* **81**, 1539 (1998).
[11] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. **67**, 1822 (1998).
[12] T.-L. Ho, Phys. Rev. Lett. **81**, 742 (1998).
[13] D. M. Stamper-Kurn and W. Ketterle, e-print cond-mat/0005001 (unpublished).
[14] A. J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).
[15] T.-L. Ho and V. B. Shenoy, Phys. Rev. Lett. **77**, 3276 (1996); B. D. Esry, C. H. Greene, J. P. Burke, Jr., and J. L. Bohn, *ibid.* **78**, 3594 (1997).
[16] S.-P. Yip, Phys. Rev. Lett. **83**, 4677 (1999); U. Leonhardt and G. E. Volovik, JETP Lett. **72**, 46 (2000); J. Ruostelkoski and J. R. Anglin, Phys. Rev. Lett. **86**, 3934 (2001); H. T. C. Stoof, E. Vliegen, and U. Al Khawaja, *ibid.* **87**, 120407 (2001); U. Al Khawaja and H. Stoof, Nature (London) **411**, 918 (2001); K.-P. Marzlin, W. Zhang, and B. C. Sanders, Phys. Rev. A **62**, 013602 (2000).
[17] T. Tsurumi and M. Wadati, J. Phys. Soc. Jpn. **67**, 93 (1998).
[18] V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cimento Soc. Ital. Fis., A **34**, 569 (1976).
[19] R. Jackiw, Ann. Phys. (N.Y.) **129**, 183 (1980); **201**, 83 (1990).
[20] C. R. Hagen, Phys. Rev. D **5**, 377 (1972).
[21] U. Niederer, Helv. Phys. Acta **45**, 802 (1972); **46**, 191 (1973); **47**, 119,167 (1974); **51**, 220 (1978).
[22] E. A. Kuznetsov and S. K. Turitsyn, Phys. Lett. **112A**, 273 (1985).
[23] S. Takagi, Prog. Theor. Phys. **84**, 1019 (1990); **85**, 463 (1991); **85**, 723 (1991).
[24] Pijush K. Ghosh, Phys. Rev. D **53**, 2248 (1996).

[25] V. E. Zakharov, Sov. Phys. JETP **35**, 908 (1972); V. E. Zakharov and V. S. Synakh, *ibid.* **41**, 465 (1975); M. I. Weinstein, Commun. Math. Phys. **87**, 567 (1983).
[26] T. Tsurumi and M. Wadati, J. Phys. Soc. Jpn. **66**, 3031 (1997); **66**, 3035 (1997); M. Wadati and T. Tsurumi, Phys. Lett. A **247**, 287 (1998); T. Tsurumi, H. Morise, and M. Wadati, Int. J. Mod. Phys. B **14**, 655 (2000).
[27] A. V. Rybin, G. G. Varzugin, M. Lindberg, J. Timonen, and R. K. Bullough, Phys. Rev. E **62**, 6224 (2000).
[28] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996); V. M. Perez-Garcia *et al.*, *ibid.* **77**, 5320 (1996).
[29] S. N. Vlasov, V. A. Petrischev, and V. I. Talanov, Radiophys. Quantum Electron. **14**, 1062 (1971); V. M. Perez-Garcia *et al.*, J. Opt. B: Quantum Semiclassical Opt. **2**, 353358 (2000).