Phase fluctuations of an electron cyclotron: Nondissipative decoherence in a quantum stochastic oscillator

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The phase damping induced in the cyclotron mode of a trapped electron by the coupling to the axial fluctuations is studied analytically. This system, described as a nonlinear oscillator stochastically driven in frequency, allows testing the generality of some elements present in the phenomenology of decoherence. In our approach, the reduced density matrix is obtained by performing a statistical average from the propagator for each noise realization. For short times, the decay of the coherences presents a nonexponential form, rooted in the non-Gaussian character of the stochastic driving. For large times, the decay becomes purely exponential, the rate showing a complex dependence on the difference between the Fock indices. As the populations do not change, the asymptotic state corresponds to a nonthermalized statistical mixture.

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Decoherence in open quantum systems, i.e., the evolution of a quantum superposition into a statistical mixture in a ''preferred'' basis, has been the subject of intense theoretical research [1–7]. In diverse contexts, different decohering mechanisms have been identified, and a standard methodology, based on a system-plus-reservoir approach [8], has been widely applied. The problem, relevant to fundamental issues such as quantum measurement [8], quantum-to-classical transition [1], or generation of superpositions of macroscopically distinguishable states (cat states) [2], is also crucial for applications, such as quantum-information processing [9], which require highly coherent operations. Recently, some predictions of the theory relative to decoherence rates of mesoscopic superpositions have been experimentally verified in cavity radiation fields [3] and in trapped ions [2]. In some of these experiments, different reservoirs have been ''engineered'' by incorporating classical fluctuations in parameters of the microscopic systems. In particular, a phase reservoir and the associated nondissipative decoherence process have been simulated by randomly driving the frequency of a trapped ion [2]. Here, we present analytical work on a related mechanism of phase damping, namely, the dephasing induced in the cyclotron mode of an electron in a Penning trap by the fluctuations of the axial coordinate [10]. This system, modeled as an anharmonic oscillator driven in frequency by non-Gaussian noise, provides a realizable scenario for a variety of fundamental effects [9–16]. Our study, which includes the harmonic and Gaussian regimes as particular cases, uncovers important elements in the phenomenology of decoherence and gives a framework for testing their generality. Specifically, the emergence of nonexponential decay and nontrivial dependence of the dephasing on the *separation* of states is discussed.

An electron in a Penning trap is described in terms of three modes (magnetron, axial, and cyclotron) with widely different time scales. Following standard treatments [10], we first eliminate adiabatically the magnetron motion. The axial coordinate, z, is coupled to a measuring external circuit that introduces resistive damping and noise in the mode, driving it to equilibrium at a temperature T_z. A classical description of this process is allowed [see Eq. (3) below]; moreover, assuming that its relaxation time is much smaller than any other relevant time scale [10,14,15], we can interpret the stationary axial coordinate as an effective dynamical parameter for the cyclotron evolution, which is then governed by the Hamiltonian [11,12]

$$H=\hbar[\omega_c+\Delta\omega_c(t)]a^\dagger a-\frac{1}{2}\hbar\,\delta(a^\dagger a)^2. \tag{1}$$

Relativistic corrections enter this description through the nonlinear term, characterized by $\delta=\hbar\omega_c^2/mc^2$ (m is the mass of the electron), and through the coupling to z, incorporated into the dynamical shift in frequency $\Delta\omega_c^r(t)=-\omega_c E_z/2mc^2$, where E_z is the axial energy. This coupling between axial and cyclotron modes can be enhanced by means of a ''magnetic bottle,'' i.e., by including magnetic field inhomogeneities in the trap. In particular, the arrangement implemented in Ref. [11], leads to the interaction term $V=2\mu_B B_2(a^\dagger a+1/2+S_z/\hbar)z^2$, where μ_B is the Bohr magneton, B_2 is a parameter of the additional magnetic field, and S_z is the spin operator (a constant of motion in this scheme). This potential displaces ω_c in $\Delta\omega_c^m(t)=2\mu_B B_2 E_z/\hbar m\omega_z^2$. Given that both, relativistic and magnetically induced, shifts are proportional to E_z, and, therefore, to the square of the axial amplitude $|\overline{z}|^2$ $[z=\overline{z}\exp(i\omega_z t)]$ [10], we will make a unified treatment of both effects by considering in Eq. (1) a displacement of the form

$$\Delta\omega_c(t)=\lambda|\overline{z}|^2,$$

λ referring to the total coupling constant. Note that this interaction shifts the axial frequency proportionally to the cyclotron occupation number; the measurement of this shift is the usual way of obtaining information on the cyclotron state.

We aim at describing the influence of the axial noise on the cyclotron mode. The importance of the role played by the fluctuations of ω_c in the heating and decoherence of the system has been pointed out in previous work [11,12]. Here, to isolate this aspect of the dynamics, we assume that spontaneous emission, caused by the interaction with the electromagnetic field of the cavity, is inhibited through a proper tuning of the cavity frequency [11]. For the model system of Eq. (1), it is known that, in the absence of the noisy driving, an initial coherent state $|\alpha\rangle$ evolves at $t=\pi/\delta$ into a superposition of coherent states with opposed phases, namely, $(e^{i\pi/4}|\alpha\rangle - e^{-i\pi/4}|-\alpha\rangle)/\sqrt{2}$, a complete revival taking place at $t=4\pi/\delta$ [17–19,14]. These findings have led to conjectures on generation of cat states in this scenario. The study of the robustness of these features against noise is one of our objectives. In our approach, the exact time propagator for each noise realization is obtained first. Then, a statistical average is performed to find the reduced density matrix of the cyclotron mode [20], the results being particularized to specific axial-noise characteristics. Finally, the *fidelity* [4,5] of different states is calculated.

In the rotating frame defined by the unitary transformation $U_1=\exp(-i\omega_c a^\dagger a t)$, the time evolution for each stochastic realization is given by

$$|\psi(t)\rangle = \exp\left[-i\Theta(t)a^\dagger a + i\frac{1}{2}\delta(a^\dagger a)^2 t\right]|\psi(0)\rangle,$$

where $\Theta(t)$ is a variable defined by

$$\Theta(t) \equiv \int_0^t \Delta\omega_c(t')dt',$$

which depends on the axial dynamics and corresponds to a nonstationary random process [21]. The evolution of the reduced density matrix is obtained by averaging over fluctuations ($\langle \cdots \rangle_f$); hence, in the Fock states, it reads

$$\rho_{m,n}(t) = \rho_{m,n}(0)e^{-i(n^2-m^2)\delta t/2}\langle e^{i(n-m)\Theta(t)}\rangle_f. \qquad (2)$$

The effect of the axial output is encapsulated in $\langle \exp[i(n-m)\Theta(t)]\rangle_f$. The nonzero mean value of E_z leads to a linear time increase in $\langle \Theta(t)\rangle_f$ and consequently, to an oscillation in the coherences, which modulates the effect of the quartic potential. Moreover, from the time increasing variance of Θ, the decay of the coherences can be predicted. Obviously, the populations do not change as the Fock states are eigenstates of the complete Hamiltonian.

Now let us explicitly consider the statistical properties of $\Theta(t)$. The axial motion is typically monitored by driving it with a nearly resonant field [10]. In this case, the stochastic evolution of z is described by

$$\ddot{z} + \gamma_z \dot{z} + \omega_z^2 z = \eta(t), \qquad (3)$$

where $\eta(t)$ is Gaussian white noise with $\langle \eta(t)\rangle_f = 0$ and $\langle \eta(t)\eta(t')\rangle_f = (4\gamma_z k_B T_z/m)\delta(t-t')$, γ_z being the resistive damping rate [21]. For small noise and friction, analytical solutions of Eq. (3) can be found as, in the rotating frame defined by $\overline{z} = \exp(-i\omega_z t)z$, the evolution corresponds to an Ornstein-Uhlenbeck process [22]. It is then shown that the non-Gaussian variable $|\overline{z}|^2$ has an exponential correlation function; its stationary probability density parallels the Boltzmann distribution for E_z. $\Theta(t)$ is also characterized: the mean value, $\langle \Theta(t)\rangle_f = \Omega t$, with $\Omega = 2\lambda k_B T_z/m\omega_z^2$, shifts ω_c; the fluctuations around this average lead to the decay of the coherences. Both effects are properly accounted for by the average present in Eq. (2), which is obtained as [23,10]

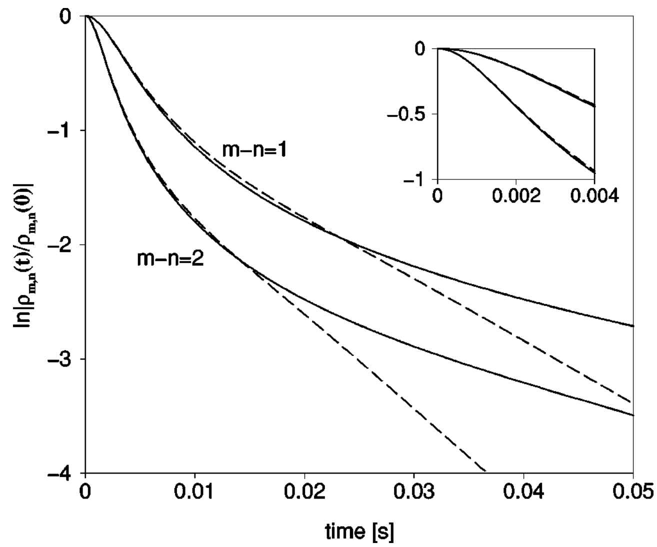

FIG. 1. Logarithmic plot of $|\rho_{m,n}(t)/\rho_{m,n}(0)|$ versus time for different values of $m-n$. $\Omega/2\pi = 47.75\ \mathrm{s}^{-1}$ and $\gamma_z = 10/3\ \mathrm{s}^{-1}$ (solid line) or $\gamma_z = 100/3\ \mathrm{s}^{-1}$ (dashed line). The inset magnifies the short-time region.

$$\langle e^{i(n-m)\Theta(t)}\rangle_f = \frac{4\gamma'_{m,n}\gamma_z}{(\gamma'_{m,n}+\gamma_z)^2 e^{(\gamma'_{m,n}-\gamma_z)t/2} - (\gamma'_{m,n}-\gamma_z)^2 e^{-(\gamma'_{m,n}+\gamma_z)t/2}}, \qquad (4)$$

with $\gamma'_{m,n} = [\gamma_z^2 - 4i\gamma_z(n-m)\Omega]^{1/2}$. Figure 1 illustrates the dependence of $\rho_{m,n}(t)$ on time. Remarkably, the decay, isolated from the deterministic and noise-induced oscillations, is nonexponential for short times. This feature, which is also present, with different origin, in other decoherence schemes [2,4], is rooted in the non-Gaussian character of the random driving, i.e., of $|\overline{z}|^2$. In effect, if we artificially assign a Gaussian statistic with an arbitrary correlation time to $|\overline{z}|^2$, a single exponential is found. Note that the applicability of nonexponential decay to implement *quantum Zeno effect*, suggested in other contexts [2,4], could also be feasible in this model. In Eq. (4), an exponential behavior for long times is also apparent, the decay rate, $\Gamma \equiv \mathrm{Re}(\gamma'_{m,n} - \gamma_z)/2$, depending in a complex way on T_z, λ, and $(n-m)$. These nontrivial characteristics of the dephasing are directly related to the non-Lorentzian line profiles calculated and measured in Ref. [10]. We emphasize that our results, with no restrictions, are applicable to the Penning trap if the experimental setup can be arranged to have z, during the whole process of decay, in the stationary state corresponding to the population of the initially prepared cyclotron state. If, on the contrary, z is out of equilibrium at some point in the process, the validity of our approach is restricted to the regime $\gamma_z \gg \Gamma$, where the assumed fast relaxation of z is justified.

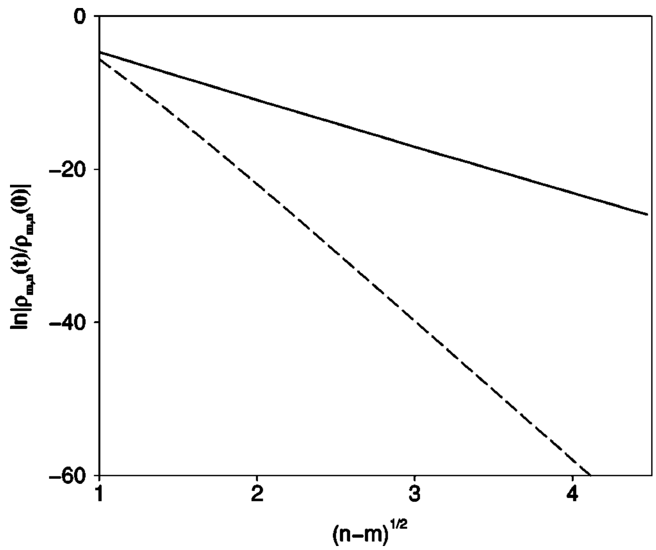

FIG. 2. Logarithmic plot of $|\rho_{m,n}(t)/\rho_{m,n}(0)|$ at $t=1$ s versus the square root of the separation between states, $(n-m)^{1/2}$. $\Omega/2\pi = 47.75\ \mathrm{s}^{-1}$ and $\gamma_z = 10/3\ \mathrm{s}^{-1}$ (solid line) or $\gamma_z = 100/3\ \mathrm{s}^{-1}$ (dashed line).

Let us now consider two limiting cases that correspond to qualitatively different behaviors.

(a) As the limit $\Omega \gg \gamma_z$ is reached, the non-Lorentzian shape of the lines becomes more apparent, and so does the nonexponential character of the decay for short times. Moreover, the rate for large times can be approximated as $\Gamma \sim (n-m)^{1/2}\gamma_z^{1/2}\Omega^{1/2}$. Figure 2 illustrates the linear dependence of Γ on $(n-m)^{1/2}$ for parameters of previous experimental realizations [10]; note that the effect of noise becomes stronger as γ_z increases.

(b) In the limit $\gamma_z \gg \Omega$, a Gaussian approximation for $|\overline{z}|^2$ is valid and, consistently with the previous discussion, the decay becomes exponential, the average being

$$\langle e^{i(n-m)\Theta(t)}\rangle_f = e^{-(n-m)^2\Omega^2 t/\gamma_z} e^{i(n-m)\Omega t}. \tag{5}$$

This behavior parallels that corresponding, in a Born-Markov approach [8], to an oscillator coupled to a phase reservoir through $H_{int} = a^\dagger a \Sigma_i (b_i + b_i^\dagger)$, where b_i denotes the annihilation operator of a bath mode. This interaction induces the decay, purely exponential, of the coherences, the rate scaling with $(n-m)^2$. As the populations do not change, the asymptotic state is a statistical mixture with nonthermalized weights. This model corresponds to a coupling $\Delta\omega_c(t) = \lambda z$ in our scheme; indeed, the found analogy is a consequence of the similar statistical properties of $|\overline{z}|^2$ and $\overline{z}$ in the limit considered. Note that a standard treatment of a system-plus-reservoir model, which implies a second-order approximation for the interaction, and, therefore, up to two-time correlations for the bath operators [8], cannot account for the effect of non-Gaussian noise.

A simple analysis of how the phase damping affects the stability of different initially prepared states can be made calculating the *fidelity*, defined as $F = \langle |\langle \psi(0)|\psi(t)\rangle|^2\rangle_f$.

(i) For a Fock state, i.e., for an eigenstate of the Hamiltonian, we trivially have $F=1$. The effect of noise is a random driving of the phase, the energy changing as ω_c is dis-

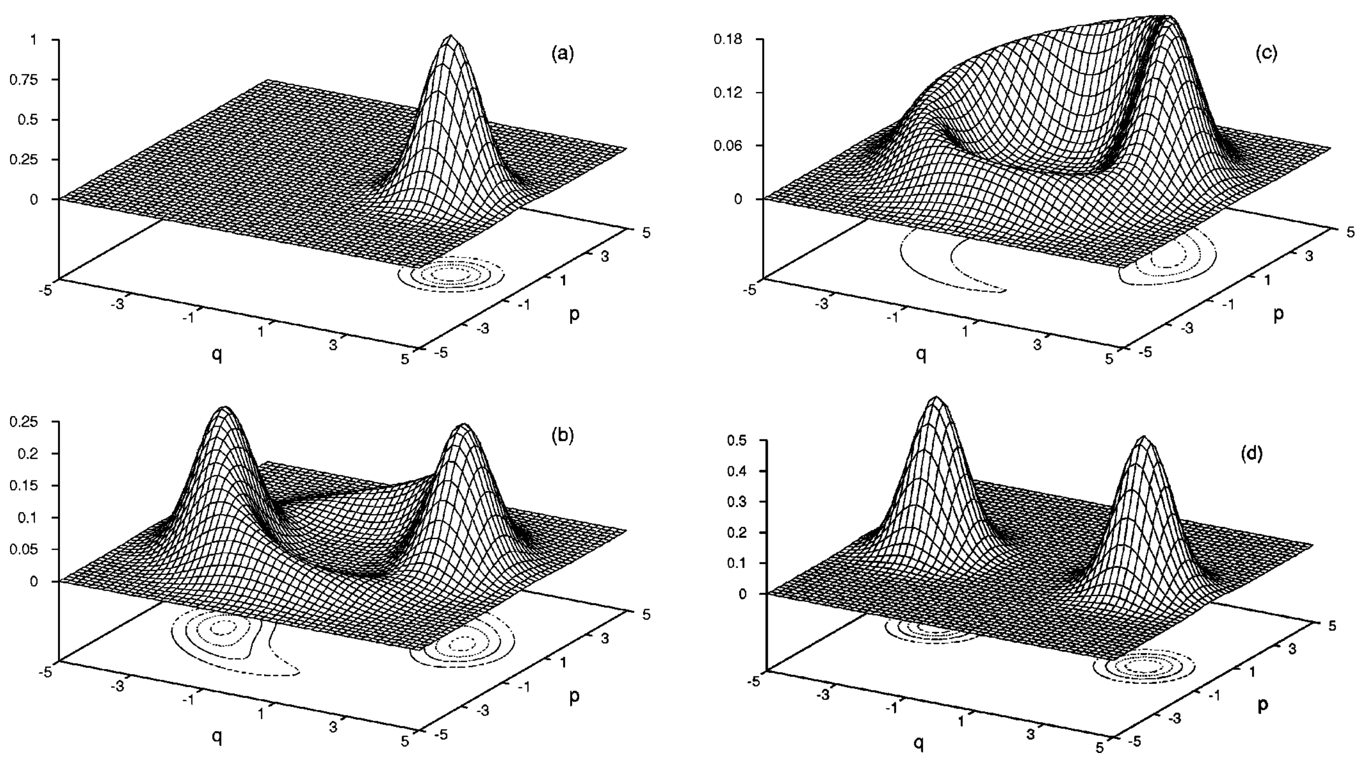

FIG. 3. Q function for a coherent state $|\alpha\rangle$ ($\alpha=3$) at $t=0$ (a), $t=\pi/\delta$ (b), $t=4\pi/\delta$ (c); and, in the deterministic case at $t=\pi/\delta$ (d). $\delta = 1100\ \mathrm{s}^{-1}$, $\Omega/2\pi = 47.75\ \mathrm{s}^{-1}$, and $\gamma_z = 10/3\ \mathrm{s}^{-1}$.

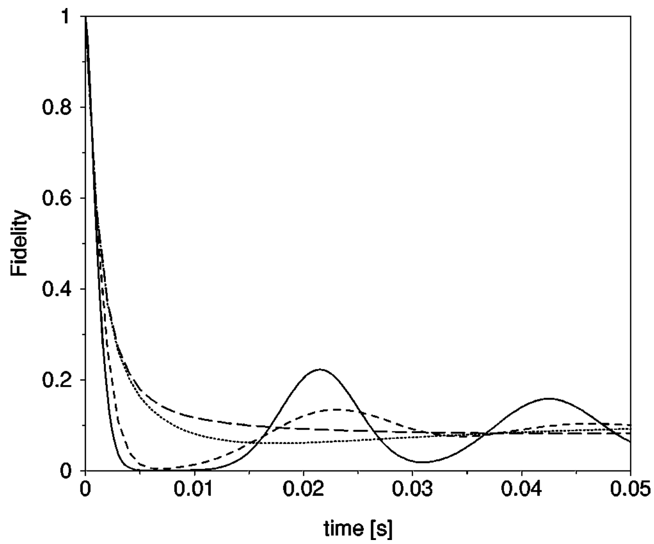

FIG. 4. Fidelity F of a coherent state $|\alpha\rangle$ ($\alpha=3$) as a function of time for $\gamma_z=10/3\ \mathrm{s}^{-1}$ (long-dashed line), $\gamma_z=100/3\ \mathrm{s}^{-1}$ (dotted line), $\gamma_z=10^3\ \mathrm{s}^{-1}$ (short-dashed line), and $\gamma_z=10^4/3\ \mathrm{s}^{-1}$ (solid line). $\Omega/2\pi=47.75\ \mathrm{s}^{-1}$.

placed by Ω. It is worth recalling that quantum nondemolition measurement on $a^\dagger a$ has been implemented in this system [11].

(ii) For a superposition of two Fock states, i.e., for $|\psi(0)\rangle=c_n|n\rangle+c_m|m\rangle$, we find

$$F=1-2|c_n|^2|c_m|^2\left[1-\frac{1}{2}e^{-i(n^2-m^2)\delta/2}\langle e^{i(n-m)\Theta(t)}\rangle_f+\text{c.c.}\right].$$

In the limit $\gamma_z\gg\Omega$ [see Eq. (5)], the nonoscillating part of F has a form similar to that detected for a trapped ion driven in frequency by zero-mean Gaussian noise [2]. An exponential dependence on $(n-m)^2$ and on the variance is present in both systems; in our case, it is the variance of $\overline{|z|}^2$ that is relevant. As opposed to the model of Ref. [2], our treatment requires no adiabaticity condition for the noise correlation time.

(iii) For an initial coherent state $|\alpha\rangle$, the evolution of the Q function reads

$$Q(q,p,t)=e^{-|\alpha|^2-|\beta|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha\beta^*)^n(\alpha^*\beta)^m}{n!m!}\times e^{-i(n^2-m^2)\delta t/2}\langle e^{i(n-m)\Theta(t)}\rangle_f,$$

with $\beta=q+ip$. In Fig. 3, this Q function is compared with that corresponding to the noiseless case. The generation of catlike states and the occurrence of revivals, present in the deterministic system and rooted in the nonlinear term, can still be identified for large δ/Γ; the shifts from the deterministic locations are due to the nonzero $\langle\Theta(t)\rangle_f$. To isolate the effect of noise from the influence of the nonlinearity, we perform the additional transformation $U_2=\exp[i\delta(a^\dagger a)^2t/2]$. In this new frame, the fidelity of $|\alpha\rangle$ is

$$F=\langle e^{-2|\alpha|^2[1-\cos\Theta(t)]}\rangle_f$$
$$=e^{-2|\alpha|^2}\left[I_0(2|\alpha|^2)+2\sum_{k=1}^{\infty}I_k(2|\alpha|^2)\times\langle\cos k\Theta(t)\rangle_f\right],$$

where $I_k(x)$ are the modified Bessel functions [24]. F is plotted as a function of time in Fig. 4. The partial revivals, which become less pronounced as Γ/Ω increases, are the combined effect of the frequency shift and the damping in phase. As the averages $\langle\cos k\Theta(t)\rangle_f$, which incorporate the decay of the different coherences present in $|\alpha\rangle$ ($k\equiv n-m$), go to zero, the asymptotic fidelity, $e^{-2|\alpha|^2}I_0(2|\alpha|^2)$, is reached.

In conclusion, the coupling of an electron cyclotron to the axial fluctuations, leads to a nondissipative decoherence process with nonstandard characteristics. Our study generalizes previous theoretical work on phase damping: nonlinearity and unrestricted random properties are simultaneously considered. As the nonlinear potential commutes with the number operator, its effect merely consists in adding an oscillation to the Fock state coherences. In contrast, nontrivial effects derive from the stochastic driving. Especially interesting is the nonexponential decay of the coherences, found for short times, and rooted in the non-Gaussian properties of the noise. For large times, a single exponential emerges, the rate presenting a complex dependence on T_z, λ, and $(n-m)$. In the limit $\gamma_z\gg\Omega$, where the noisy driving has approximate Gaussian character, the decay becomes purely exponential, the rate depending quadratically on both T_z and λ, and showing the standard linear dependence on $(n-m)^2$. The applicability of these results is not restricted to the Penning trap: as the study traces back some elements of the decoherence phenomenology to characteristics of the fluctuations, it can open the way to experimental tests in related systems.

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