

Robustness of multiparty entanglement

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(Received 20 September 2001; published 9 May 2002)

We study the robustness of multiparty entanglement under local decoherence, modeled by partially depolarizing channels acting independently on each subsystem. Surprisingly, we find that the robustness of (GHZ) entanglement increases with the number of qubits. An n -qubit Greenberger-Horne-Zeilinger (GHZ) state can stand more than 55% local depolarization in the limit $n \rightarrow \infty$. GHZ states are more robust than other generic states of three and four qubits. We also study spin-squeezed states in the limit $n \rightarrow \infty$ and find that they too can stand considerable local depolarization.

DOI: 10.1103/PhysRevA.65.052327

PACS number(s): 03.65.Yz, 03.65.Ud

Entanglement is certainly one of the most dramatically nonclassical features of quantum physics. There are states of composite quantum systems that cannot be decomposed into probabilistic combinations of product states: there is no way of writing the density matrix of such a state ρ in the form $\rho = \sum_i p_i \sigma_1^i \otimes \sigma_2^i \otimes \cdots \otimes \sigma_n^i$, where σ_k^i is a state of the k th subsystem and the $p_i > 0$ are probabilities. Such states ρ are called inseparable or entangled. One could say that for entangled systems the whole is indeed more than the sum of its parts. Only entangled states can exhibit quantum nonlocality [1]. In this case, there is no way of reproducing the predictions of quantum physics with classical systems, unless there is instantaneous communication over arbitrary distances.

Entanglement has recently been studied extensively in the context of quantum computation and quantum communication [2]. Quantum computation would require the creation and maintenance of highly complex entangled states of many subsystems. This is a difficult task because of decoherence [3]. A system interacts with its surroundings, which creates entanglement between system and environment, at the same time reducing the entanglement within the system itself.

Such considerations lead to a natural physical question: how common is complex entanglement—entanglement between many subsystems—in nature? Are there natural systems that contain substantial large-scale entanglement between their constituents? There seem to be two conflicting intuitions in response to this question. On the one hand, entanglement seems to be very fragile under decoherence, which is emphasized by the difficulty of building a quantum computer. On the other hand, measure-theoretic considerations suggest that the set of inseparable states is much larger than the set of separable ones [4], which seems to imply that entanglement should be relatively common.

One might hope to gain some insight into the above question by analyzing the behavior of complex entangled states under decoherence, in particular, by studying how fast the entanglement disappears. This is not an easy task, because it requires practical criteria for inseparability. However, for

some special sets of states such criteria have been found [5,6], which we will use in the following.

We will consider the evolution of certain multiqubit states under the action of partially depolarizing channels, to be defined below, acting independently on every qubit. The multiqubit states are our models for complex entanglement, and the depolarizing channel is our toy model for decoherence, corresponding to *local* and *homogeneous* noise. We ask how much local depolarization is possible such that the states are still entangled, in this way quantifying their robustness under decoherence.

We are particularly interested in how the robustness of the entanglement changes with the size of the system. One might expect that for n , the number of subsystems, going to infinity, the entanglement should become more and more fragile. We will show that for Greenberger-Horne-Zeilinger (GHZ)-type entanglement the exact opposite is the case. The states become *more* robust under local depolarization when the number of qubits is increased.

Our notion of robustness is different from the one of [7], who considered global mixing of entangled states with locally prepared noise. Admixing locally prepared noise is different from local decoherence; it corresponds to a scenario where the local “decohering agents” communicate when to add noise, and when to leave the system undisturbed.

It also differs from the notion of entanglement persistency proposed in [8], which is defined as the minimum number of local measurements needed in order to definitely disentangle a given state. This definition involves an optimization over all possible local measurements. It thus corresponds to intelligent and informed adversaries who optimize coordinated decohering measurements in order to most efficiently destroy the entanglement of a state. The depolarizing channels which we consider here, applied to each subsystem, correspond to independent measurements in random bases on each local component of the entangled state.

We will consider qubits with basis states $|0\rangle$ and $|1\rangle$. The *completely* depolarizing channel is defined by the transformation $|i\rangle\langle j| \rightarrow \delta_{ij} \frac{1}{2} \mathbb{1}$, where $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$. When applied to a subsystem A of a composite system AB this corresponds to the transformation $\rho_{AB} \rightarrow \frac{1}{2} \mathbb{1} \otimes \text{Tr}_A \rho_{AB}$. All correlations between the system and its environment are destroyed, and the system itself is put into the completely mixed state.

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The *partially* depolarizing channel C_d is defined by applying the completely depolarizing channel with a probability d , and applying the identity transformation with a probability $1-d$. This corresponds to the transformation

$$|i\rangle\langle j| \rightarrow (1-d)|i\rangle\langle j| + d\delta_{ij}\frac{1}{2}\mathbb{1}. \quad (1)$$

If the density matrix of the system is written in the basis of Pauli matrices, one can easily convince oneself that Eq. (1) corresponds to multiplying all σ_i by the *scaling factor* $s = 1-d$, while the unit matrix $\mathbb{1}$ remains unaffected.

An interesting physical realization of a partially depolarizing channel is by random measurements. One can show that if a qubit is subjected to a Von Neumann measurement in a basis chosen uniformly at random, the effect on the density matrix corresponds to a partially depolarizing channel with $d = 2/3$.

We will refer to d as the amount of local depolarization. Denote the product of partially depolarizing channels with depolarization d on each qubit by $C_d^{\otimes n}$. For a given n -party entangled state ρ , we will be interested in the critical amount of depolarization d_{crit} where $C_d^{\otimes n}(\rho)$ becomes separable, or—in the absence of a necessary and sufficient condition—ceases to fulfill certain sufficient conditions for entanglement [6,9]. In both cases, $d_{crit}(\rho)$ will be a quantitative signature of the robustness of the entanglement in that state.

Our main interest is the behavior of the critical depolarization for large systems. There are two families of states where we have been able to obtain explicit results in the limit of n , the number of subsystems, going to infinity, namely, GHZ states and spin-squeezed states [10] of n qubits. Our main result is that, surprisingly, in both cases a very substantial amount of local depolarization does not destroy the multiparty entanglement. For GHZ states, the critical depolarization even increases with n . We have also compared GHZ states to other generic entangled states for $n=3$ and $n=4$ qubits, with (to us) counter-intuitive results.

For the following discussion, we find it convenient to define $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$, $\sigma_+ = |0\rangle\langle 1|$, and $\sigma_- = |1\rangle\langle 0|$. In this basis the partially depolarizing channel (1) corresponds to the following transformations:

$$\begin{aligned} P_{0,1} &\rightarrow \frac{1+s}{2}P_{0,1} + \frac{1-s}{2}P_{1,0} \\ \sigma_{+,-} &\rightarrow s\sigma_{+,-}. \end{aligned} \quad (2)$$

Recall that $s = 1-d$.

Let us first discuss the case of GHZ states. An n -qubit GHZ state $1/\sqrt{2}(|00\dots 0\rangle + |11\dots 1\rangle)$ has the density matrix

$$\rho = \frac{1}{2}(P_0^{\otimes n} + P_1^{\otimes n} + \sigma_+^{\otimes n} + \sigma_-^{\otimes n}). \quad (3)$$

Application of the channel (2) to every qubit multiplies the off-diagonal terms $\sigma_+^{\otimes n}$ and $\sigma_-^{\otimes n}$ by s^n . The diagonal terms $P_0^{\otimes n}$ and $P_1^{\otimes n}$ give rise to new diagonal terms of the form

$P_0^{\otimes k} \otimes P_1^{\otimes (n-k)}$ and all permutations thereof, for k ranging from 0 to n . The coefficients λ_k of these terms are given by

$$\lambda_k = \frac{1}{2} \left[\left(\frac{1+s}{2} \right)^k \left(\frac{1-s}{2} \right)^{n-k} + \left(\frac{1+s}{2} \right)^{n-k} \left(\frac{1-s}{2} \right)^k \right]. \quad (4)$$

In order to analyze the entanglement of states of the above form it is sufficient to study the entanglement for bipartite cuts [5], where some qubits constitute one subsystem and all the other qubits the other subsystem. For each such cut one considers the partial transposition [9]. As long as the partial transpose of a state has negative eigenvalues, the state is definitely entangled.

Take partial transposition of the first k qubits as an example. The diagonal terms do not change, while $\sigma_+^{\otimes n} = (|0\rangle\langle 1|)^{\otimes n}$ goes into an off-diagonal term between the states $|1\rangle^{\otimes k}|0\rangle^{\otimes (n-k)}$ and $|0\rangle^{\otimes k}|1\rangle^{\otimes (n-k)}$. It is then easy to see that the state will have nonpositive partial transpose for some bipartite cut, and thus will definitely be entangled, as long as

$$\frac{s^n}{2} > \lambda_k \quad (5)$$

for some k . In [5] it was moreover shown that if the state has positive partial transpose with respect to all bipartite cuts, then it is separable, so that the above condition for inseparability is both sufficient and necessary.

The smallest eigenvalue λ_k is the one for $k=n/2$, if n is even, or $k=m$, for $n=2m+1$. Therefore, we can immediately make a statement about entanglement after local depolarization for general n (consider even n for simplicity). The state is definitely entangled as long as

$$\frac{s^n}{2} > \left(\frac{1+s}{2} \right)^{n/2} \left(\frac{1-s}{2} \right)^{n/2}. \quad (6)$$

The critical value of s where the entanglement disappears is given by

$$s_{crit}(n) = \frac{1}{\sqrt{2^{2-2n} + 1}} \quad (7)$$

for even n . It is very remarkable that s_{crit} decreases with n . The entanglement in the GHZ state thus becomes more robust under local depolarization when the number of subsystems is increased. In the limit $n \rightarrow \infty$, $s_{crit}^{\infty} = 1/\sqrt{5} = 0.447$. This means that a very large GHZ state can stand more than 55% depolarization ($d_{crit}^{\infty} = 1 - s_{crit}^{\infty} = 0.553$) of every qubit before it becomes separable.

A natural question to ask is how the robustness of the GHZ state compares to other multiparty entangled states. One might expect GHZ states to be particularly fragile, since for example the loss of a single qubit from an n -qubit GHZ state destroys the entanglement completely. It turns out that under local depolarization GHZ entanglement is comparatively robust.

We do not know of any other family of multiqubit states for which there is a necessary and sufficient condition for entanglement. As a first step, we have therefore chosen to apply the partial transposition criterion [9] to some characteristic states of three and four qubits. In the four-qubit case

TABLE I. Critical values of local depolarization d for which several generic entangled states of three and four qubits become PPT (with positive partial transpose). For the four-qubit states the values for both the 1-3 and the 2-2 bipartite cuts are given. One sees that G_4 , the four-qubit GHZ state, stays NPT (with negative partial transpose, and thus entangled) up to the largest value of depolarization, $d_{crit}=0.489$. For the GHZ case PPT-ness and separability are equivalent. The states considered were the following: $|G_3\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$, $|G_4\rangle = 1/\sqrt{2}(|0000\rangle + |1111\rangle)$, $|W_3\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$, $|W_4\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$, $|X_4\rangle = 1/\sqrt{6}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle)$, and $|B_4\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$.

$n=3$	1-2 cut	
G_3	0.443	
W_3	0.425	
$n=4$	1-3 cut	2-2 cut
G_4	0.423	0.489
W_4	0.423	0.423
X_4	0.416	0.453
B_4	0.468	0.450

we have studied both the 1-3 and the 2-2 bipartite cuts. These results can be directly compared to the results in the GHZ case, where the partial transposition criterion happens to be necessary and sufficient. Our results are collected in Table I. The main—again surprising—conclusion is that GHZ states are comparatively robust.

The states considered were the following. For $n=3$, the state $|W_3\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$. For a symmetric state of three qubits, there is just one possible bipartite cut (1-2). The state $|W_3\rangle$ becomes PPT (with positive partial transpose) across this cut for a lower value of depolarization than the three-qubit GHZ state, i.e., at least according to the partial transposition criterion, it is less robust under local depolarization.

For $n=4$, we have studied three different states, namely, $|W_4\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$, $|X_4\rangle = 1/\sqrt{6}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle)$, and $|B_4\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$ [8], representing different kinds of four-qubit entanglement. There is only a single instance where one of them is more robust than the four-qubit GHZ state: the 1-3 cut for $|B_4\rangle$ remains NPT (with negative partial transpose) for higher values of depolarization. However, if the 2-2 cut is also taken into account, again the GHZ state remains NPT for higher d values.

It is worth noting that the state $|X_4\rangle$, which can also be written as $|X_4\rangle = 1/\sqrt{6}(|00\rangle|11\rangle + 2|\psi_+\rangle|\psi_+\rangle + |11\rangle|00\rangle)$ originally has more entanglement across the 2-2 cut than the GHZ state: the Von Neumann entropy of the reduced state is 1.252 compared to 1 for the GHZ state (for pure states, the Von Neumann entropy is a good measure of entanglement, cf. [2]). Nevertheless, it reaches the boundary of PPT states before the GHZ state. A more trivial example for the fact that the amount of entanglement and its robustness do not have a simple relation is provided by any number of independent shared two-qubit singlet states. Then the collective state is

exactly as robust (or fragile) as an individual singlet, which has $d_{crit} = 1 - s_{crit}(2) = 0.423$.

Remarkably, there is also no positive correlation between the Schmidt rank of the states with respect to the 2-2 cut (i.e. the number of terms in the corresponding Schmidt decomposition) and their robustness under local depolarization, cf. [11]. The state $|X_4\rangle$ has Schmidt rank 3, nevertheless it is less robust than the GHZ state, which has Schmidt rank 2. Again, an even simpler example is provided by a state of two independent shared singlets, $|S_4\rangle = \frac{1}{2}(|00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|11\rangle)$, which has Schmidt rank 4, but is less robust than the GHZ state of four qubits.

Let us note that the robustness of a state under local depolarization *can* be increased by local operations. To see this, let us consider the states $|G_4\rangle$ and $|B_4\rangle$, where we will focus on the bipartite entanglement between qubits one and two on the one hand and qubits three and four on the other hand. The state $|B_4\rangle$ can be written as $\frac{1}{2}[|00\rangle(|00\rangle + |11\rangle) + |11\rangle(|00\rangle - |11\rangle)]$. It can thus clearly be transformed into a GHZ state by acting only on qubits three and four. Table I shows that the critical depolarization with respect to the 2-2 bipartite cut is larger for the GHZ state than for the state $|B_4\rangle$.

There is one more class of states for which we were able to study the entanglement robustness for a general number of qubits n , in particular n tending towards infinity, namely, states exhibiting spin squeezing [10]. An n -qubit state $|\psi\rangle$ is called spin squeezed, if it satisfies

$$\xi^2 = \frac{n\langle J_x^2 \rangle}{\langle J_y \rangle^2 + \langle J_z \rangle^2} < 1 \quad (8)$$

for three appropriately chosen orthogonal directions x , y , and z . Here, $J_{x,y,z} = \frac{1}{2} \sum_{i=1}^n \sigma_{x,y,z}^{(i)}$ are the total angular momentum operators, and $\langle J_x^2 \rangle = \langle \psi | J_x^2 | \psi \rangle$, etc. It was shown in [6] that spin squeezing is a sufficient condition for entanglement. Such states can, e.g., be generated from an initial product state of the n qubits through time evolution with a Hamiltonian of the general form $H = \chi J_x^2$.

For sufficiently large n , there are states for which ξ^2 is very small, $\langle J_z \rangle = \zeta n/2$ with ζ close to 1, and $\langle J_x \rangle = \langle J_y \rangle = 0$ [10]. The local depolarizing channels (1) can be quite easily applied in the present situation. If the state is written in the basis of Pauli matrices, then simply all σ_i are multiplied by a factor $s = 1 - d$. One finds

$$\begin{aligned} \langle J_z \rangle &\rightarrow s \langle J_z \rangle \\ \langle J_x^2 \rangle &= \frac{n}{4} + \frac{1}{4} \sum_{i \neq j} \langle \sigma_x^{(i)} \sigma_x^{(j)} \rangle \rightarrow \frac{n}{4} + \frac{s^2}{4} \sum_{i \neq j} \langle \sigma_x^{(i)} \sigma_x^{(j)} \rangle \\ &= (1 - s^2) \frac{n}{4} + s^2 \langle J_x^2 \rangle, \end{aligned} \quad (9)$$

where again $s = 1 - d$, and thus for the squeezing parameter ξ_s^2 after depolarization

$$\xi_s^2 = \frac{1-s^2}{\zeta^2 s^2} + \xi_0^2, \quad (10)$$

where ξ_0^2 is the original squeezing parameter.

This means that under local depolarization the states definitely remain entangled for all scaling factors larger than

$$s_{crit} = \frac{1}{\sqrt{1 + \zeta^2(1 - \xi_0^2)}}. \quad (11)$$

This is equal to $1/\sqrt{2} = 0.707$ in the limit of $\zeta = 1$ and $\xi_0^2 = 0$, which is approached for $n \rightarrow \infty$. This means that in this limit the spin-squeezing entanglement can stand more than 29% of local depolarization. Note that Eq. (8) is a sufficient, but not a necessary, condition for entanglement, so the states may well be entangled for even larger values of depolarization.

In the present paper, we have used local partially depolarizing channels as our model for decoherence, corresponding to local and homogeneous noise. Our results emphasize that the robustness of multiparty entangled states will in general depend on the nature of the decoherence process. In particular, GHZ states have turned out to be surprisingly robust under local depolarization. On the other hand, GHZ entanglement is of course fragile under some other operations, such as the loss of a single qubit, or the measurement of a single qubit in the $\{|0\rangle, |1\rangle\}$ basis. For the latter scenario it is however worth noting that the measurement has to be performed *with certainty* in order to destroy the entanglement. To show this consider the state generated from the n -qubit GHZ state $|G_n\rangle$ by performing local measurements in the $\{|0\rangle, |1\rangle\}$ basis on every qubit with a probability p . *Certain* measurement corresponds to $p = 1$. This state is given by

$$\frac{1 - (1-p)^n}{2} (P_0^{\otimes n} + P_1^{\otimes n}) + (1-p)^n |G_n\rangle\langle G_n|. \quad (12)$$

It is entangled for all $p < 1$, as can be seen from its partial transpose. Of course, the coefficient of the inseparable term decreases exponentially.

One should certainly be careful in drawing general conclusions from a single simple model. Nevertheless, our results show that multiparty entanglement can be surprisingly robust under decoherence, and that the robustness can even increase with the number of parties. This suggests that large-scale entanglement could be more frequent in natural systems than one might have expected. Persistent entanglement would definitely require the presence of an entanglement-generating process—interactions between the subsystems, possibly in combination with external excitation—that competes with the decoherence. The competition between entanglement generation and decoherence in multiparty systems is a promising and fascinating topic for future research.

C.S. would like to thank H. Briegel, L. Hardy, and L. Henderson for a very useful discussion. C.S. has been supported by the QIPC program of the European Union (Project QuComm) and the Austrian Science Foundation (FWF, Project No. S6503). J.K.'s effort is sponsored by the Defense Advanced Research Projects Agency (DARPA) and Air Force Laboratory, Air Force Material Command, USAF, under Agreement No. F30602-01-2-0524. J.K. also thanks the Erwin-Schrödinger Institute, Vienna, where part of this research was conducted, for support and hospitality. Both authors are grateful for the hospitality of the Benasque Center for Science.

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