Nonadditive information measure and quantum entanglement in a class of mixed states of an N^n system

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Generalizing Khinchin's classical axiomatic foundation, a basis is developed for nonadditive information theory with the Tsallis entropy indexed by q. The classical nonadditive conditional entropy is introduced and then translated into quantum theory. To examine if this theory has points superior to the ordinary additive information theory with the von Neumann entropy corresponding to the limit $q\to 1$, separability of a one-parameter family of the Werner-Popescu states of the N^n system (i.e., the n-partite N-level system) is discussed. The nonadditive information theory with $q>1$ is shown to yield a limitation on separability that is stronger than the one derived from the additive theory. How the strongest limitation can be obtained in the limit $q\to\infty$ is also shown.

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I. INTRODUCTION

There is growing interest in the roles of nonadditive measures in quantum information theory. In Ref. [1], the inadequacy of the additive von Neumann entropy as a measure of the information content in a quantum state has been pointed out, for example. Also, there is a theoretical observation [2] that the measure of quantum entanglement may not be additive.

Recently, the explicit use of nonadditive information measures has been made in quantum information theory [3–6]. In particular, the problems of separability of mixed states and quantum entanglement have been discussed in Refs. [5,6]. These attempts are primarily based on the Tsallis entropy, which is defined by

$$\mathcal{S}_q[\hat{\rho}]=\frac{1}{1-q}(\mathrm{Tr}\,\hat{\rho}^q-1), \tag{1}$$

where $\hat{\rho}$ is the system density matrix and q is the positive entropic index. This quantity is regarded as a one-parameter generalization of the von Neumann entropy, which is obtained in the limit $\mathcal{S}_q[\hat{\rho}]\to\mathcal{S}[\hat{\rho}]=-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$ $(q\to 1)$. $\mathcal{S}_q[\hat{\rho}]$ possesses some important properties as an entropy. It is non-negative, definitely concave for all values of $q>0$, and fulfills the H theorem. Additivity is, however, to be replaced by *pseudoadditivity*, which means that, for a product state of a bipartite system (A,B), the total amount satisfies

$$\mathcal{S}_q[\hat{\rho}(A)\otimes\hat{\rho}(B)]=\mathcal{S}_q[\hat{\rho}(A)]+\mathcal{S}_q[\hat{\rho}(B)]+(1-q)\mathcal{S}_q[\hat{\rho}(A)]\mathcal{S}_q[\hat{\rho}(B)]. \tag{2}$$

Clearly, additivity holds only in the limit $q\to 1$. In recent years, $\mathcal{S}_q[\hat{\rho}]$ and its classical counterpart have been widely discussed in the area of *nonextensive statistical mechanics* [7]. Also, classical nonadditive measures have been employed to generalize the ordinary additive information theory (see Ref. [8], for example).

In this paper, we develop a basis for nonadditive quantum information theory with the Tsallis entropy and the associated conditional entropy. To examine if this theory has points superior to the ordinary additive quantum information theory with the von Neumann entropy, we apply it to the separability problem of a specific one-parameter family of the mixed states of the N^n system (i.e., the n-partite N-level system), which are extreme generalizations of the Werner-Popescu-type states of the 2×2 system [i.e., the bipartite spin-($\frac{1}{2}$) system] [9,10]. The nonadditive conditional entropy is found to be non-negative for a separable (or, classically correlated) subfamily of these states but may take negative values for the states with quantum entanglement. Based on this fact, we show that the nonadditive quantum information theory with $q>1$ gives rise to a limitation on separability, which is stronger than that derived from the additive theory corresponding to the limit $q\to 1$. We also show how the strongest limitation can be obtained in the limit $q\to\infty$. This result should be compared with the discussion in Ref. [11], in which it is stated that local information can never be sufficient for establishing separability of the Werner-Popescu-type states in odd dimensions.

II. CLASSICAL NONADDITIVE MEASURE AND ITS AXIOMATIC FOUNDATION

Though our interest is in quantum theory, it seems appropriate to mention here that, at the classical level, the Tsallis entropy has its mathematical characterization, such as the Shannon entropy [12,13]. Therefore, we wish to devote this section to a brief summary of this point.

The axioms and the uniqueness theorem have been presented for the Tsallis entropy in Ref. [14], in which the nonadditive conditional entropy has been introduced for the first time. Thereby, the Shannon-Khinchin axiomatic framework [12,13] was generalized to nonadditive information theory. The set of axioms presented in Ref. [14] is the following. (I) $S_q(p_1,p_2,\ldots,p_W)$ is continuous with respect to all its arguments and takes its maximum for the equiprobability distribution $p_i=1/W$ $(i=1,2,\ldots,W)$, (II) $S_q(A,B)$

$=S_q(A)+S_q(B|A)+(1-q)S_q(A)S_q(B|A)$ for a composite system, (A,B), and (III) $S_q(p_1,p_2,\ldots,p_W,p_{W+1}=0) = S_q(p_1,p_2,\ldots,p_W)$. It can be shown [14] that the quantity S_q satisfying axioms (I)–(III) is, up to a multiplicative constant, uniquely given by

$$S_q(p_1,p_2,\ldots,p_W)\equiv S_q[p]=\frac{1}{1-q}\left[\sum_{i=1}^{W}(p_i)^q-1\right], \quad (3)$$

which is the classical counterpart of $\mathcal{S}_q[\hat{\rho}]$ in Eq. (1). Comparing this set of axioms with that of Khinchin [13], we see that the one and only difference is in axiom (II) (that is, Khinchin's second axiom is recovered in the limit $q\to 1$). There, $S_q(B|A)$ is the nonadditive conditional entropy defined as follows:

$$S_q(B|A)=\langle S_q(B|A_i)\rangle_q^{(A)}, \quad (4)$$

provided that $S_q(B|A_i)$ is the Tsallis entropy of the conditional probability distribution of B with A found in its ith state, $p_{ij}(B|A)=p_{ij}(A,B)/p_i(A)$ with the marginal probability distribution $p_i(A)=\Sigma_j p_{ij}(A,B)$. The symbol $\langle Q\rangle_q^{(A)}$ stands for the *normalized q-expectation value* [15] defined by

$$\langle Q\rangle_q^{(A)}=\sum_i Q_i P_i(A), \quad (5)$$

where

$$P_i(A)=\frac{[p_i(A)]^q}{\Sigma_i[p_i(A)]^q} \quad (6)$$

is the *escort distribution* associated with $p_i(A)$, which has originally been introduced in the context of statistical mechanical description of chaotic systems [16]. Though it is not intuitively clear why such a distribution introduced in the different area appears also in information theory, one should recall that statistical mechanics can be thought of as a branch of information theory, and accordingly it is desirable to formulate the theories in these two areas in a unified manner.

Analogously to quantum theory, it is immediate to see that the classical additive Shannon entropy is recovered in the limiting case: $S_q[p]\to S[p]=-\Sigma_{i=1}^{W}p_i\ln p_i$ $(q\to 1)$, in conformity with Khinchin's set of axioms, that is, axioms (I)–(III) with $q\to 1$.

We wish to emphasize that the classical nonadditive conditional entropy $S_q(B|A)$ is non-negative, since $S_q(B|A_i)$ in Eq. (4) is the non-negative Tsallis entropy of the conditional probability distribution, as mentioned above.

The generalized composition law in axiom (II) can be recognized in connection with Eqs. (4)–(6). In fact, Eq. (4) admits the following expression:

$$S_q(B|A)=\frac{S_q(A,B)-S_q(A)}{1+(1-q)S_q(A)}, \quad (7)$$

where $S_q(A)$ is the Tsallis entropy of the marginal probability distribution $p_i(A)$. This is identical to the generalized composition law.

Note that there exists the following correspondence relation between the Bayes law and the generalized composition law in axiom (II):

$$\begin{aligned} p_{ij}(A,B)&=p_i(A)p_{ij}(B|A)=p_j(B)p_{ij}(A|B)\\ &\leftrightarrow\\ S_q(A,B)&=S_q(A)+S_q(B|A)+(1-q)S_q(A)S_q(B|A)\\ &=S_q(B)+S_q(A|B)+(1-q)S_q(B)S_q(A|B). \end{aligned} \quad (8)$$

In the ordinary additive information theory, the Shannon entropy establishes the following relation between the Bayes law and the composition law:

$$\begin{aligned} p_{ij}(A,B)&=p_i(A)p_{ij}(B|A)\\ &=p_j(B)p_{ij}(A|B)\leftrightarrow S(A,B)\\ &=S(A)+S(B|A)=S(B)+S(A|B). \end{aligned}$$

In this, one sees a clear structure in multiplication and addition. Correspondingly, Eq. (8) may also be seen to be natural in view of pseudoadditivity of the Tsallis entropy [cf. Eq. (2)]. The generalized composition law is actually more satisfactory than pseudoadditivity for characterizing the nonadditive feature of the Tsallis entropy since it holds even when A and B are correlated and the factorization ansatz in Eq. (2) is not fulfilled.

Let us further discuss the generalized composition law for a multipartite system. To be specific, here we consider a tripartite system (A,B,C) as a simple example. The Bayes multiplication rule reads

$$\begin{aligned} p_{ijk}(A,B,C)&=p_{jk}(B,C)p_{ijk}(A|B,C)\\ &=p_k(C)p_{jk}(B|C)p_{ijk}(A|B,C), \end{aligned} \quad (9)$$

and so on. Accordingly, the generalized composition law becomes

$$\begin{aligned} S_q(A,B,C)&=S_q(B,C)+S_q(A|B,C)+(1-q)\\ &\quad\times S_q(B,C)S_q(A|B,C)\\ &=S_q(C)+S_q(B|C)+S_q(A|B,C)\\ &\quad+(1-q)[S_q(C)S_q(B|C)\\ &\quad+S_q(B|C)S_q(A|B,C)\\ &\quad+S_q(A|B,C)S_q(C)\\ &\quad+(1-q)S_q(C)S_q(B|C)S_q(A|B,C)]. \end{aligned} \quad (10)$$

Therefore, we have

$$S_q(B|C)=\frac{S_q(A,B,C)-[S_q(C)+S_q(A|B,C)+(1-q)S_q(C)S_q(A|B,C)]}{1+(1-q)[S_q(C)+S_q(A|B,C)+(1-q)S_q(C)S_q(A|B,C)]}, \tag{11}$$

for example. It is of interest to observe that the system A plays only an auxiliary role in this equation since $S_q(B|C)$ does not directly contain information on A. This discussion can be generalized to an arbitrary multipartite system in an obvious way.

III. QUANTUM NONADDITIVE CONDITIONAL ENTROPY

Equation (7) is assumed to remain form invariant under its quantum-mechanical generalization, that is,

$$\mathcal{S}_q(B|A)=\frac{\mathcal{S}_q(A,B)-\mathcal{S}_q(A)}{1+(1-q)\mathcal{S}_q(A)}, \tag{12}$$

where $\mathcal{S}_q(A,B)=\mathcal{S}_q[\hat{\rho}(A,B)]$ and $\mathcal{S}_q(A)=\mathcal{S}_q[\hat{\rho}(A)]$ with $\hat{\rho}(A)$ the marginal density matrix given by the partial trace, $\hat{\rho}(A)=\mathrm{Tr}_B\,\hat{\rho}(A,B)$.

In classical theory, the nonadditive conditional entropy is always non-negative as already mentioned, whereas it can be negative in quantum theory, in general. An important point arising here is that the occurrence of negative values may actually be a *signature* of quantum entanglement. Consider a classically correlated state, or a separable state, of (A,B)

$$\hat{\rho}(A,B)=\sum_\lambda w_\lambda \hat{\rho}_\lambda(A)\otimes\hat{\rho}_\lambda(B), \tag{13}$$

where $w_\lambda\in[0,1]$ with $\Sigma_\lambda w_\lambda=1$. This state is a nonproduct state but is known to admit locally realistic hidden-variable models [9]. Let us assume both $\{\rho_\lambda(A)\}$ and $\{\rho_\lambda(B)\}$ be simultaneously diagonalizable for all values of λ. Then, $\hat{\rho}_\lambda(A)$ and $\hat{\rho}_\lambda(B)$ can be expressed in the diagonalizing orthonormal bases, $|a\rangle$ and $|b\rangle$, as follows:

$$\hat{\rho}_\lambda(A)=\sum_a r_\lambda(a)|a\rangle\langle a|,\quad \hat{\rho}_\lambda(B)=\sum_a s_\lambda(b)|b\rangle\langle b|, \tag{14}$$

where $r_\lambda(a),\ s_\lambda(b)\in[0,1]$ with $\Sigma_a r_\lambda(a)=\Sigma_b s_\lambda(b)=1$. The above assumption defines a subclass of the density matrices of the classically correlated states, and is not valid, in general. However, since we are limiting our discussion here to the Werner-Popescu-type states [defined later in Eqs. (18) and (30)], it may actually be valid in such a limited class of the states. With this assumption, the nonadditive quantum conditional entropy in Eq. (12) is calculated to be

$$\mathcal{S}_q(B|A)=\frac{\sum_a\left[\sum_\lambda w_\lambda r_\lambda(a)\right]^q S_q(B|a)}{\sum_a\left[\sum_\lambda w_\lambda r_\lambda(a)\right]^q}, \tag{15}$$

where

$$S_q(B|a)=\frac{1}{1-q}\left\{\sum_b [\pi(b|a)]^q-1\right\}, \tag{16}$$

$$\pi(b|a)=\frac{\sum_\lambda w_\lambda r_\lambda(a)s_\lambda(b)}{\sum_\lambda w_\lambda r_\lambda(a)}. \tag{17}$$

Equation (15) is to be compared with Eq. (4). $\pi(b|a)$ in Eq. (17) has the same properties as the classical conditional probability distribution does: $\pi(b|a)\in[0,1]$, $\Sigma_b\pi(b|a)=1$. This is the reason why Eq. (16) is written in the notation of classical theory. Consequently, $\mathcal{S}_q(B|A)$ in Eq. (15) is non-negative for any classically correlated states satisfying the above-mentioned simultaneous diagonalizability assumption. In other words, negative values of the nonadditive conditional entropy indicate the existence of nonclassical correlation, i.e., quantum entanglement. In the recent works [5,6], this point has been discussed in detail for a class of the density matrices of the 2×2 system. In particular, the strongest limitation [17] on separability of a one-parameter family of the Werner-Popescu state [9,10] has been obtained by using the nonadditive conditional entropy [5]. Also, it has been shown how the nonadditive conditional entropy is superior to the conditional von Neumann entropy, $\mathcal{S}(B|A)=\lim_{q\to1}\mathcal{S}_q(B|A)$, and to the Bell inequality for constraining validity of local realism.

IV. QUANTUM ENTANGLEMENT IN A CLASS OF STATES OF MULTIPARTITE SYSTEMS

As the simplest multipartite generalization of the previous discussion about the 2×2 system [5,6], first let us consider the tripartite spin-$(\frac{1}{2})$ system. A one-parameter family of the Werner-Popescu-type state of such a system is given by

$$\hat{\rho}(A,B,C)=\frac{1-x}{8}\hat{I}_2(A)\otimes\hat{I}_2(B)\otimes\hat{I}_2(C)+x|\Psi_2^{(3)}\rangle\langle\Psi_2^{(3)}| \quad (x\in[0,1]), \tag{18}$$

where $\hat{I}_2$ denotes the 2×2 unit matrix and

$$|\Psi_2^{(3)}\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C + |1\rangle_A \otimes |1\rangle_B \otimes |1\rangle_C). \tag{19}$$

Since the subsystems A, B, and C appear symmetrically, there are essentially two different kinds of the marginal density matrices,

$$\begin{aligned}\hat{\rho}(B,C) &= \mathrm{Tr}_A\, \hat{\rho}(A,B,C)\\ &= \frac{1-x}{4}\hat{I}_2(B)\otimes\hat{I}_2(C) + \frac{x}{2}(|0\rangle_{BB}\langle 0|\otimes|0\rangle_{CC}\langle 0|\\ &\quad + |1\rangle_{BB}\langle 1|\otimes|1\rangle_{CC}\langle 1|),\end{aligned} \tag{20}$$

$$\hat{\rho}(C) = \mathrm{Tr}_{A,B}\, \hat{\rho}(A,B,C) = \tfrac{1}{2}\hat{I}_2(C). \tag{21}$$

The eigenvalues of $\hat{\rho}(A,B,C)$, $\hat{\rho}(B,C)$, and $\hat{\rho}(C)$ are, respectively, given by

$$\frac{1-x}{8}(\text{sevenfold degenerate}), \quad \frac{1+7x}{8} \quad [\hat{\rho}(A,B,C)], \tag{22}$$

$$\frac{1-x}{4}(\text{doubly degenerate}),$$

$$\frac{1+x}{4}(\text{doubly degenerate}) \quad [\hat{\rho}(B,C)], \tag{23}$$

$$\tfrac{1}{2}(\text{doubly degenerate}) \quad [\hat{\rho}(C)]. \tag{24}$$

Therefore, the two independent nonadditive conditional entropies are calculated to be

$$\begin{aligned}\mathcal{S}_q(A,B|C) &= \frac{\mathcal{S}_q(A,B,C)-\mathcal{S}_q(C)}{1+(1-q)\mathcal{S}_q(C)}\\ &= \frac{1}{1-q}\left[\frac{7\left(\frac{1-x}{8}\right)^q + \left(\frac{1+7x}{8}\right)^q}{2\left(\frac{1}{2}\right)^q} - 1\right],\end{aligned} \tag{25}$$

$$\begin{aligned}\mathcal{S}_q(A|B,C) &= \frac{\mathcal{S}_q(A,B,C)-\mathcal{S}_q(B,C)}{1+(1-q)\mathcal{S}_q(B,C)}\\ &= \frac{1}{1-q}\left[\frac{7\left(\frac{1-x}{8}\right)^q + \left(\frac{1+7x}{8}\right)^q}{2\left(\frac{1-x}{4}\right)^q + 2\left(\frac{1+x}{4}\right)^q} - 1\right].\end{aligned} \tag{26}$$

In Figs. 1 and 2, we present the implicit plots of the equations $\mathcal{S}_q(A,B|C)=0$ and $\mathcal{S}_q(A|B,C)=0$, respectively. Both show that the value of x for the border of separability of the density matrix in Eq. (18) monotonically decreases with respect to the index q. From them, it is seen how $\mathcal{S}_q(A|B,C)$ puts limitation on separability, which is clearly stronger than that obtained from $\mathcal{S}_q(A,B|C)$. It is also seen in Fig. 2 that the regime in which local realism holds is given by

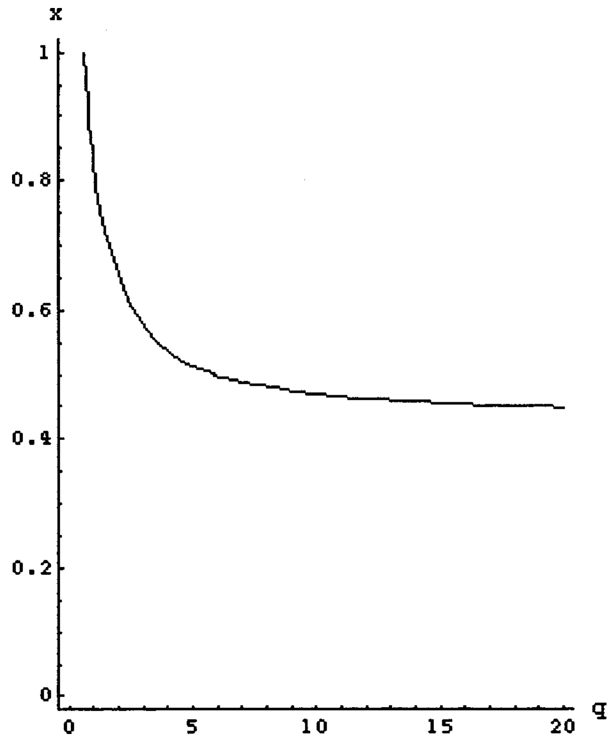

FIG. 1. Implicit plot of $S_q(A,B|C)=0$ with respect to $q \in [0,\infty)$ and $x\in[0,1]$. In the limit $q\to\infty$, x converges to $\frac{3}{7}$. All quantities are dimensionless.

$$0 \leqslant x < \tfrac{1}{5}. \tag{27}$$

Note that, in fact, this condition is obtained from Eq. (26) in the limit $q\to\infty$. [Actually, it is necessary and sufficient for separability of $\hat{\rho}(A,B,C)$ in Eq. (18). See the general discussion below.] Equation (27) should be compared with the following condition obtained from the conditional von Neumann entropy corresponding to $\mathcal{S}_q(A|B,C)=0$ in the limit $q\to 1$: $0\leqslant x < 0.682\,931\ldots$, which is clearly much more tolerant for separability than Eq. (27). Thus, one can appreciate superiority of the present nonadditive quantum information theory to the ordinary additive approach at least in the $2\times2\times2$ example (as well as the 2×2 example discussed in Ref. [5]).

The above result suggests that the nonadditive conditional entropy of the form

$$\begin{aligned}&\mathcal{S}_q(A_1|A_2,A_3,\ldots,A_n)\\ &\quad= \frac{\mathcal{S}_q(A_1,A_2,\ldots,A_n)-\mathcal{S}_q(A_2,A_3,\ldots,A_n)}{1+(1-q)\mathcal{S}_q(A_2,A_3,\ldots,A_n)}\end{aligned} \tag{28}$$

is to be examined in general. It should be noted that, *in the asymptotic evaluation of this quantity in the limit* $q\to\infty$, *it is sufficient to consider the largest eigenvalues of* $\hat{\rho}(A_1,A_2,\ldots,A_n)$ *and* $\hat{\rho}(A_2,A_3,\ldots,A_n)$, since $\mathcal{S}_q(A_1|A_2,A_3,\ldots,A_n)$ can also be expressed as follows:

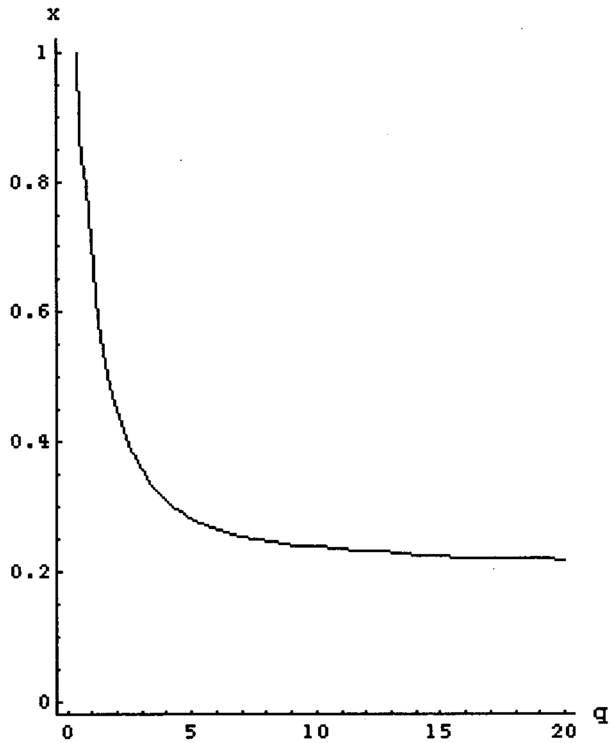

FIG. 2. Implicit plot of $S_q(A|B,C)=0$ with respect to $q \in [0,\infty)$ and $x\in[0,1]$. In the limit $q\to\infty$, x converges to $\frac{1}{5}$. All quantities are dimensionless.

$$\mathcal{S}_q(A_1|A_2,A_3,\dots,A_n)=\frac{1}{1-q}\left[\frac{\mathrm{Tr}\,\hat{\rho}^q(A_1,A_2,\dots,A_n)}{\mathrm{Tr}\,\hat{\rho}^q(A_2,A_3,\dots,A_n)}-1\right]. \tag{29}$$

Now, we consider the problem of separability of a one-parameter family of the Werner-Popescu-type state of the N^n system. The density matrix of this state is written as follows:

$$\hat{\rho}(A_1,A_2,\dots,A_n)=\frac{1-x}{N^n}\hat{I}_N(A_1)\otimes\hat{I}_N(A_2)\otimes\cdots\otimes\hat{I}_N(A_n) + x|\Psi_N^{(n)}\rangle\langle\Psi_N^{(n)}| \quad (x\in[0,1]), \tag{30}$$

where $\hat{I}_N$ is the $N\times N$ unit matrix and $|\Psi_N^{(n)}\rangle$ is given by

$$|\Psi_N^{(n)}\rangle=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle_{A_1}\otimes|k\rangle_{A_2}\otimes\cdots\otimes|k\rangle_{A_n}. \tag{31}$$

The marginal density matrix of interest is

$$\begin{aligned}\hat{\rho}(A_2,A_3,\dots,A_n)&=\mathrm{Tr}_{A_1}\hat{\rho}(A_1,A_2,A_3,\dots,A_n)\\ &=\frac{1-x}{N^{n-1}}\hat{I}_N(A_2)\otimes\hat{I}_N(A_3)\otimes\cdots\otimes\hat{I}_N(A_n)\\ &\quad+\frac{x}{N}\sum_{k=0}^{N-1}|k\rangle_{A_2A_2}\langle k|\\ &\quad\otimes|k\rangle_{A_3A_3}\langle k|\otimes\cdots\otimes|k\rangle_{A_nA_n}\langle k|.\end{aligned} \tag{32}$$

It can be found that the eigenvalues of $\hat{\rho}(A_1,A_2,\dots,A_n)$ and $\hat{\rho}(A_2,A_3,\dots,A_n)$ are, respectively, given by

$$\frac{1-x}{N^n}\quad[(N^n-1)\text{-fold degenerate}],\quad \frac{1+(N^n-1)x}{N^n}\quad[\hat{\rho}(A_1,A_2,\dots,A_n)], \tag{33}$$

$$\frac{1-x}{N^{n-1}}\quad[(N^{n-1}-N)\text{-fold degenerate}],$$

$$\frac{1+(N^{n-2}-1)x}{N^{n-1}}(N\text{-fold degenerate})\quad[\hat{\rho}(A_2,A_3,\dots,A_n)]. \tag{34}$$

Using these eigenvalues, the nonadditive conditional entropy in Eq. (28) [or Eq. (29)] is calculated to be

$$\mathcal{S}_q(A_1|A_2,A_3,\dots,A_n)=\frac{1}{1-q}\,\frac{(N^n-1)\left(\frac{1-x}{N^n}\right)^q+\left[\frac{1+(N^n-1)x}{N^n}\right]^q}{(N^{n-1}-N)\left(\frac{1-x}{N^{n-1}}\right)^q+N\left[\frac{1+(N^{n-2}-1)x}{N^{n-1}}\right]^q}-1. \tag{35}$$

For fixed N and n, the value of x satisfying $\mathcal{S}_q(A_1|A_2,A_3,\dots,A_n)=0$ monotonically decreases with respect to q, as in the tripartite spin-$(\frac{1}{2})$ system discussed previously. Therefore, the nonadditive theory with $q>1$ yields a limitation on separability of the density matrix in Eq. (30) that is stronger than the one derived from the additive von Neumann theory corresponding to the limit $q\to1$. In the limit $q\to\infty$, evaluating the eigenvalues of $\hat{\rho}(A_1,A_2,\dots,A_n)$ and $\hat{\rho}(A_2,A_3,\dots,A_n)$, we find that the density matrix is separable if

$$0\leqslant x<\frac{1}{1+N^{n-1}}. \tag{36}$$

It has recently been shown using algebraic methods [18,19] that Eq. (36) is actually the necessary and sufficient condition. This is the main result of the present work. It indicates how the generalized nonadditive information theoretic approach sheds more light on characterizing quantum entanglement in a one-parameter family of the Werner-Popescu-type states of multipartite systems.

V. CONCLUSION

We have developed a basis for nonadditive generalization of the ordinary framework of quantum information. To examine if this theory has points superior to the ordinary additive theory with the von Neumann entropy, we have applied it to the problem of separability of a one-parameter family of the Werner-Popescu-type states of the N^n system. We have found that the present theory with the Tsallis entropy and the associated nonadditive conditional entropy with the entropic index q greater than unity leads to a limitation on separability of the state that is stronger than the one derived from the additive theory corresponding to the limit $q \rightarrow 1$. In particular, the necessary and sufficient condition for separability has been obtained in the limit $q \rightarrow \infty$.

It is logically clear that the present nonadditive quantum information theoretic approach does not lead to the necessary and sufficient condition for separability, in general. For example, it can be shown that this approach fails to give a criterion for separability of the marginal density matrix of the reduced 2×2 system derived from the W state of the $2 \times 2 \times 2$ system: there, the conditional entropy vanishes for all values of q but the Peres criterion tells us that the state is entangled [20].

Finally, we make a comment on the quantum-mechanical version of the Rényi entropy defined by $\mathcal{S}_q^R[\hat{\rho}] = (1 - q)^{-1} \ln \mathrm{Tr}\, \hat{\rho}^q$. This quantity is related to the Tsallis entropy as $\mathcal{S}_q^R = (1-q)^{-1} \ln[1 + (1-q)\mathcal{S}_q]$, and satisfies strict additivity. In Ref. [21], this quantity with $q=2$ is examined for characterizing quantum entanglement. However, we wish to point out that, unlike the Tsallis entropy, the Rényi entropy does not possess the information content and, in addition, it is not concave if $q > 1$. These may be seen as serious drawbacks from the unified viewpoint of information theory and statistical mechanics.

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