Entanglement purification in cavity QED using local operations

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We study a physical implementation of an entanglement purification protocol using a cavity quantum electrodynamics based proposal, in both the microwave and the optical domain. The protocol consists of local quantum operations of each particle of an entangled system with one auxiliary particle (ancilla). After the above interaction a measurement on ancillas is carried out. In the microwave region the quantum information is stored in field states inside two distant cavities. We also give a procedure for quantifying the degree of entanglement between quantum fields, which allows verifying the efficiency of the purification process. In the optical domain, we study a setup of cold trapped ions inside cavities, where quantum bits are defined by two electronic levels of ions. This latter proposal is extended to create multiparticle entangled states among distant quantum systems. Entanglement is achieved through a set of local measurements on pairs of entangled particles.

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I. INTRODUCTION

Quantum entanglement plays a fundamental role in the quantum information theory [1–3]. The entanglement concept has been central to the quantum theory since the famous work of Einstein, Podolsky, and Rosen [4]. Most of the quantum information protocols related to transmission, processing, and storing of information make use of the capability of creating and manipulating quantum entanglement among distant quantum bits. In recent years, there has been an extensive study of theoretical implications behind quantum entanglement and practical schemes to produce it in a variety of physical systems. The entangled particles provide a means for implementing quantum communication channels among nodes of a quantum network.

The degree of entanglement can be fully or partially destroyed, for instance, due to the presence of interactions with the environment, or imperfect quantum logic operations. Thus, the problem of improving the quality of quantum entanglement as a means for protecting or preserving quantum information has been studied recently. In quantum information theory, these processes are usually called quantum purification protocols [5,6]. These protocols consider the existence of a large number of entangled particles, and that each one of the components of the entangled system is located in one node of a quantum network. In a sequential process, many of these partially entangled systems are disregarded and the degree of entanglement of the remainder systems is higher than the initial one. If there exist many partially entangled systems at the same time, the purification can be implemented by performing bilateral controlled-NOT operations [5]. These can be generalized for many particle operations, which requires implementing collective measurement on the particles belonging to a given node of the network [21]. This will be addressed at the end of this paper. However, while establishing entanglement between two nodes of a quantum network, the most common scenario is to have access to only one entangled pair at a same time. For instance, this occurs in the experimental implementation of a quantum cryptography protocol [7] based on Ekert's proposal [8]. Recently, a proposal for a physical implementation of an entanglement purification protocol has been analyzed for Gaussian continuous variable entangled states. The proposal makes use of high finesse cavities and cavity enhanced Kerr nonlinearities [9]. In the case of entangled photons, an alternative method for entanglement purification operating on two pairs at the same state, without the use of (controlled-NOT) gates, has been given recently [10].

Alternatively, enhancement of entanglement in a single copy of a mixed state can be implemented by filtering operations, by making use of ancillary systems [11]. In this context, Horodecki *et al.* [12] gave an example of a mixed state that can be quasidistilled by using local quantum operations and classical communications (LQCC). In this case, the matrix was composed of a maximally entangled and an orthogonal product state. The above-mentioned matrices are of rank two, i.e., in which a single copy can be quasidistilled by LQCC as has been recently shown by Verstraete *et al.* [13]. Here we study the implementation of a purification protocol based on local operations on ancillary systems interacting with a partially entangled pair in the cavity QED context. This process corresponds to a purification protocol based on filtering operations that can be described by using positive operator valued measurement (POVM) [14], which consists of a unitary evolution between one particle of an entangled pair and an ancillary system, taking place at both nodes of the network. After the evolution, a measurement is applied on the ancilla. The initial state of the ancilla is chosen so as to optimize the purification protocol. A schematic diagram of the protocol is depicted in Fig. 1. In Sec. II we study a physical implementation of this purification protocol in the context of cavity quantum electrodynamics, at the microwave region [15–17]. In these cavities quantum bits are de-

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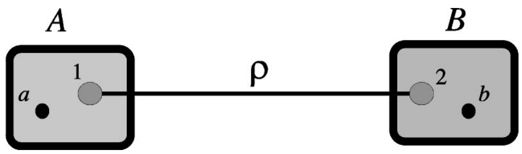

FIG. 1. Schematic diagram for the purification protocol. The particles 1 and 2 are the components of the partially entangled pair; *a* and *b* denote the ancillary systems; *A* and *B* are the nodes of the quantum network.

fined as superpositions of the field number states $|0\rangle$ and $|1\rangle$. We use a cavity leakage of photons as a source of partial loss of entanglement. We restrict this to the short-time region, where partial coherence still survives. Of course, in the long run, this mechanism will also be responsible for energy loss; however, we assume that the time elapsed until the purification process is small enough. We consider as ancillary systems two-level atoms, which interact in the large detuning limit with cavity fields. The atoms are also subject to resonant interactions with classical fields outside the cavities. The atom-field interaction makes a fully entangled system that, after a measurement on atoms, allows a modified state for cavity fields.

The above protocol can also be implemented in the optical domain, using ions in linear traps, Sec. III. Trapped ions in a linear trap was the first proposition to achieve a quantum computer [18]. In this system a set of ions is cooled to the ground state of the collective center of mass mode and permits performing quantum logic gate operations. However, in this case ions inside a cavity must also permit creating entangled pairs in distant nodes of a network, so we require additional interactions of ions with cavity modes. As a further application, this proposal is extended to create multiparticle entangled states among distant quantum systems in Sec. IV. Entanglement is achieved through a set of local measurements on pairs of entangled particles.

II. ENTANGLEMENT OF CAVITIES: MICROWAVE REGION

We consider a system of two microwave cavities interacting with two-level atoms. The atom-field interaction is described by the following Hamiltonian ($\hbar=1$):

$$H=\omega_f a^\dagger a+\omega_{\rm at}\frac{\sigma_z}{2}+g(a^\dagger\sigma_-+a\sigma_+), \tag{1}$$

where ω_f and $\omega_{\rm at}$ are, respectively, the cavity and atomic transition frequencies and g is the atom-field coupling constant. Using this interaction in the resonant case $\omega_f=\omega_{\rm at}$, the distant cavities can be entangled. This is performed by sending a resonant two-level atom crossing sequentially both cavities [15–17]. The atom is initially prepared in its upper state $|e\rangle_{\rm at}$, and the cavities A and B are in their vacuum state $|0_A,0_B\rangle$. We choose the atom velocity such that the interaction time satisfies $t=\pi/2g$. Thus, after crossing the first cavity, the atom will remain in an entangled state with the field. The state of the atom-field system evolves as

$$|e\rangle_{\rm at}\otimes|0_A,0_B\rangle\rightarrow\frac{1}{\sqrt{2}}(|e\rangle_{\rm at}|0_A\rangle-|g\rangle_{\rm at}|1_A\rangle)\otimes|0_B\rangle, \tag{2}$$

where $|g\rangle_{\rm at}$ is the atomic ground state.

By applying a Stark field, we can modify the atom-field coupling constant at the second cavity such that the new interaction time is $t=3\pi/2g'$ [15], so that we use one single atom to entangle the two cavities. After this process, the atom is in its lower level and the cavity fields reach a maximally entangled state

$$|\psi^-\rangle_{AB}=(|0_A,1_B\rangle-|1_A,0_B\rangle)/\sqrt{2}, \tag{3}$$

which is one of the Bell's states basis. The whole system state evolves as

$$|e\rangle_{\rm at}\otimes|0_A,0_B\rangle\rightarrow|g\rangle_{\rm at}\otimes|\psi^-\rangle_{AB}, \tag{4}$$

where the atom can be disregarded because it is factorized. This is the initial condition for the action of field dissipation. In this process we have neglected all other sources of decoherence, thus allowing only a partial entanglement between cavity fields. In this mechanism for establishing entanglement we have assumed that logic operations are perfect.

A. Cavity losses as a source of entanglement loss

After that we have created a maximally entangled state, the state can be destroyed by the action of cavity losses. Here, we determine the action of cavity losses, which are described by independent thermal baths (at zero temperature), on the entangled state. We assume that cavities have different decay constants γ_i, where $i=1,2$. Thus, the master equation can be written as ($\hbar=1$):

$$\dot{\rho}=i[\rho,H]+\sum_{i=1,2}\frac{\gamma_i}{2}(2a_i\rho a_i^\dagger-a_i^\dagger a_i\rho-\rho a_i^\dagger a_i), \tag{5}$$

where $H=\Sigma_{i=1,2}\omega_i a_i^\dagger a_i$ is the fields' free Hamiltonian. This can be readily solved in the case of an initial maximally entangled state, which we assume to be $\rho(0)=|\psi^-\rangle\langle\psi^-|$ as was previously derived. Thus, the mixed state of quantum fields can be written as

$$\rho(t)=F|\psi_1\rangle\langle\psi_1|+\left(1-\frac{e^{-\gamma_1 t}+e^{-\gamma_2 t}}{2}\right)|00\rangle\langle00|, \tag{6}$$

with

$$|\psi_1\rangle=|\psi^-\rangle+c_+|\psi^+\rangle, \tag{7}$$

where $|\psi_1\rangle$ is not normalized and $c_+=(e^{-\gamma_1 t/2}-e^{-\gamma_2 t/2})/2\sqrt{F}$. The quantum fidelity F, defined as the overlap between the ideal quantum state and the actual state of a given system (in this case $F=\langle\psi^-|\rho|\psi^-\rangle$), will decay due to the interaction between the cavities and their environments. In this case we see that quantum fidelity evolves according to

$$F=\frac{(e^{-\gamma_1 t/2}+e^{-\gamma_2 t/2})^2}{4}. \tag{8}$$

This quantity indicates that the entanglement of the fields is vanishing in the course of the evolution. The state in Eq. (6) is a mixture of a partially entangled state $|\psi_1\rangle$ and a product state. As the product state is lying in an orthogonal subspace to the $|\psi_1\rangle$ state, $\rho(t)$ is a rank two mixed state, i.e., with only two nonzero eigenvalues. Thus, this matrix belongs to a family of mixed states in which a single copy can be quasipurified by local quantum operations and classical communication as has been recently shown by Verstraete *et al.* [13]. If the cavity decay constants are equal the problem reduces to a maximally entangled and an orthogonal product state. The source of the entanglement losses comes from cavity losses, which appear to be the main contribution in an experimental setup in the microwave regime [17]. We think that the proposal in the microwave region can be experimentally implemented by using a setup similar to that used in the controlled quantum phase gate by Rauschenbeutel *et al.* [19]. In this regime there can also appear some small random contributions from a nonmaximally initially entangled state, which appear from imperfect interactions between atoms and cavity fields (for instance, nonexact velocity selection of flying atoms). These random contributions would be a relevant restriction on the entanglement purification process using local operations, as it has been pointed out in Refs. [20,21], which need collective manipulations of qubits. However, this still can be done in the context of a microwave cavity as in the recent experiments by Haroche's group, where an experiment for creating controlled entanglement between two modes in a cavity mode has been recently reported [22]. Thus, our proposal can be extended to collective manipulation of partially entangled pairs by using two cavites with two modes.

B. Entanglement measures

If the entangled particles are in a pure state $|\psi_{AB}\rangle$, the partial von Newmann entropy is a good entanglement measure, which can be written as

$$E(|\psi_{AB}\rangle)=S(\rho_A)=S(\rho_B), \tag{9}$$

where $\rho_{A(B)}=\mathrm{Tr}_{B(A)}(\rho)$ are the reduced density matrices. However, this definition is not suitable for an initially mixed state and it can be nonzero for separable states. It is well known that in the general case a good entanglement measure is the entanglement of formation [23,24] which is defined by

$$E_f(\rho)\equiv\mathcal{E}(C(\rho)), \tag{10}$$

where $C(\rho)$ is the concurrence and ξ, which are defined by

$$\mathcal{E}(C(\rho))=h\left(\frac{1+\sqrt{1-C(\rho)^2}}{2}\right),$$

$$h(x)=-x\log_2 x-(1-x)\log_2(1-x), \tag{11}$$

$$C(\rho)=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}.$$

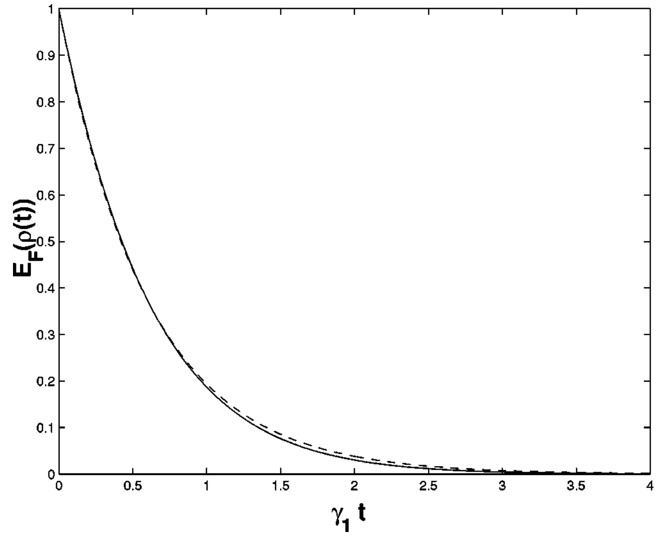

FIG. 2. Entanglement of formation, Eq. (10), as a function of an adimensional time $\gamma_1 t$ (solid line) and $P(t)^{3/2}$ (dashed line), with $\gamma_2/\gamma_1=1.2$.

Here, the λ_i's are the eigenvalues of the matrix $R=\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$, with $\tilde{\rho}$ being $\tilde{\rho}=(\sigma_{y_1}\otimes\sigma_{y_2})\rho^*(\sigma_{y_1}\otimes\sigma_{y_2})$. The quantity defined in Eq. (10) is the entanglement per copy needed to prepare a given mixed state. When we apply the above definitions to the density matrix of cavity fields given by Eq. (6), we get

$$R(t)=e^{-(\gamma_1+\gamma_2)t/2}|\psi_1'\rangle\langle\psi_1|. \tag{12}$$

Here the $|\psi_1'\rangle=|\psi_1\rangle/\sqrt{1+c_+^2}$ state is a normalized state lying in a subspace spanned by basis $\{|\psi^+\rangle,|\psi'\rangle\}$. Because the R matrix corresponds to a pure state, we readily get the value for the quantity called concurrence, which allows us to determine the entanglement evolution. This is,

$$C=\exp\left(-\frac{(\gamma_1+\gamma_2)t}{2}\right). \tag{13}$$

We can use the same scheme both for entangling cavities and for measuring the entanglement of the system. In order to achieve this goal, we inject one additional resonant two-level atom into the first cavity and another one into the second cavity. These atoms are initially prepared in their lower states and then go into the first and second cavity at a time t after preparing the state (4). The interaction time between atoms and field is $gt=\pi/2$. Here we assume that, during the entanglement transfer, cavity decay is negligible because the cavity decay time is much longer than the transfer time. Finally, the probability that the atom crossing the first or the atom crossing the second cavity reaches its upper state after the interaction is $P(t)=(e^{-\gamma_1 t}+e^{-\gamma_2 t})/2$. In Fig. 2 we see that this probability, even for short times, is closely related to the entanglement of the system. Indeed, we can write $E(\rho(t))\simeq P^{3/2}(t)$. Then, we can quantify the level of entanglement by a direct measurement of the probability of the

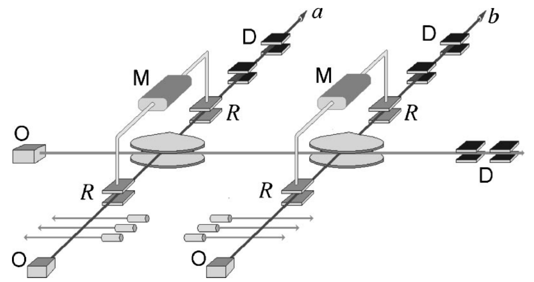

FIG. 3. Experimental setup in the microwave region. Atoms are produced in ovens (O). After that the atoms are excited to their lower level. The zones that are indicated by R are the Ramsey fields for preparing atomic coherent superpositions via a resonant interaction. These fields are connected to the same microwave generator (M) to avoid additional phase decoherence. Finally, the atoms go to the detectors (D), which permits measuring the atomic state after the conditional interaction with cavity fields.

upper state of the measurement atom. We shall make use of these features for measuring the quality of the purification process.

C. Purification protocol

As we described in Sec. I, we consider a purification protocol based on POVM [14]. Specifically, a given POVM is a set of non-negative Hermitian operators, acting on the Hilbert space of the system, which sum to the identity operator. This can be obtained by connecting the system to an auxiliary system, ancilla, followed by a measurement on the ancilla. The set of possible results of the measurement on the ancilla gives the POVM set. We consider two two-level atoms (ancillas) independently interacting with each cavity field. Here, the logical states of the ancillas a and b are $\{|+\rangle_{a(b)},|-\rangle_{a(b)}\}$, which are defined as

$$|\pm\rangle_{a(b)}=\frac{1}{\sqrt{2}}(|0\rangle_{a(b)}\pm|1\rangle_{a(b)}). \tag{14}$$

A diagram of the experimental setup is shown in Fig. 3. In the first step, an atom goes across the first Ramsey field $R(\pi/2)$, where the atom resonantly interacts with a classical field. This field prepares the atomic superposition $|+\rangle_{a(b)}$. This step corresponds to a physical implementation of a Hadamard transformation. Now, the atom crosses the cavity during a time τ. Here, the atom-field interaction is dispersive; therefore the evolution operator is given by

$$U_\varphi=e^{i\varphi a^\dagger a}|1\rangle_{cc}\langle 1|+|0\rangle_{cc}\langle 0|, \tag{15}$$

where $\varphi=g^2\tau/2\Delta$ is an adimensional interaction time, $\Delta=\omega_f-\omega_{\rm at}$ is the difference of frequency between field and atomic transitions. In order to get a conditional evolution of the ancilla depending on the state of the field, we choose $\varphi=\pi$. The result for each node is

$$\begin{aligned} U_\pi|0\rangle_C|\pm\rangle_c&=|0\rangle_C|\pm\rangle_c,\\ U_\pi|1\rangle_C|\pm\rangle_c&=|1\rangle_C|\mp\rangle_c. \end{aligned} \tag{16}$$

Here the index C denotes different cavities A or B, and c denotes the corresponding atomic ancillas a or b.

We have obtained a conditional evolution of the ancilla depending on the state of the field. This choice corresponds to a physical implementation of the controlled-NOT gate. After the atom leaves the cavity, we apply the second Ramsey field $R(\phi_C)$, which allows rotating the logical atomic state. Thus, we can write

$$\begin{aligned} U(\phi_C)|0\rangle_C|+\rangle_c&=\left(\sin\frac{\phi_C}{2}|1\rangle_c+\cos\frac{\phi_C}{2}|0\rangle_c\right)|0\rangle_C,\\ U(\phi_C)|1\rangle_C|-\rangle_c&=\left(\cos\frac{\phi_C}{2}|1\rangle_c-\sin\frac{\phi_C}{2}|0\rangle_c\right)|1\rangle_C, \end{aligned} \tag{17}$$

where $\phi_C=\Omega\tau_C+\pi/2$ is an adimensional interaction time, Ω is the Rabi frequency of the classical field, and τ_C is the interaction time between the ancilla c and the second Ramsey field. We have connected both Ramsey fields to the same microwave source to avoid the introduction of a random phase that can act as a phase decoherence source. The modified field density matrix can be obtained by applying the corresponding POVM field operators to the initial density matrix ρ. Thus, we get

$$\rho'_{(i,j)}=\frac{R_i(\phi_A)R_j(\phi_B)\rho R_j^\dagger(\phi_B)R_i^\dagger(\phi_A)}{\mathrm{Tr}(R_i(\phi_A)R_j(\phi_B)\rho R_j^\dagger(\phi_B)R_i^\dagger(\phi_A))}, \tag{18}$$

where the POVM operators are defined as the diagonal matrix elements of the evolution operator in the atomic space, namely, $R_j(\phi_C)\equiv {}_c\langle j|U(\phi_C)U_\pi|j\rangle_c$. We can explicitly write these operators as

$$R_0(\phi_C)=\cos\frac{\phi_C}{2}|0\rangle_{CC}\langle 0|+\sin\frac{\phi_C}{2}|1\rangle_{CC}\langle 1|, \tag{19}$$

$$R_1(\phi_C)=-\sin\frac{\phi_C}{2}|0\rangle_{CC}\langle 0|+\cos\frac{\phi_C}{2}|1\rangle_{CC}\langle 1|. \tag{20}$$

In this case the resulting density matrix, Eq. (18), corresponds to measuring the atoms a and b in state $|i\rangle$ and $|j\rangle$, respectively. The normalization constant $\mathrm{Tr}(R_i(\phi_A)R_j(\phi_B)\rho R_j^\dagger(\phi_B)R_i^\dagger(\phi_A))$ is the probability of the above measurement. The conditional evolution of ancillas needed for purification is obtained by a sequence of one and two qubit gates, as is shown in Fig. 4. In this case the initial condition is given as a direct sum of two orthogonal density matrices ($|\psi^-\rangle\langle\psi^-|$ and $|00\rangle\langle 00|$) under both the inner product $A\cdot B=\mathrm{Tr}(AB)$ and the evolution operator, so the system evolution will be split up into two independent matrices. This allows getting the evolution of matrix elements. In order to get a purified density matrix, the new fidelity must be greater than the initial one, i.e., $F'>F$. For instance, we consider the case when both atoms are measured in their lower state. This is,

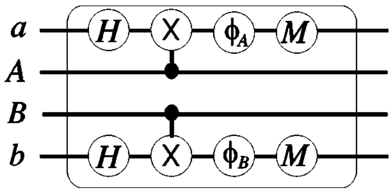

FIG. 4. Quantum circuit of the purification protocol. This circuit consists of local operations at different nodes of the quantum network, where H denotes a Hadamard transformation, which is followed by a controlled-NOT gate, with the cavity field and atom as source and target, respectively. There is a phase gate, which allows rotating the atomic state in a given angle ϕ. Finally, we have included the measurement on the ancillary system.

$$F'(\phi_A,\phi_B)=\frac{\left[\sqrt{F}\sin\left(\frac{\phi_A+\phi_B}{2}\right)+c_+\sin\left(\frac{\phi_A-\phi_B}{2}\right)\right]^2}{P_{00}}, \tag{21}$$

where

$$P_{00}=\left[\sqrt{F}\sin\left(\frac{\phi_A+\phi_B}{2}\right)+c_+\sin\left(\frac{\phi_A-\phi_B}{2}\right)\right]^2 +\left[c_+\sin\left(\frac{\phi_A+\phi_B}{2}\right)+\sqrt{F}\sin\left(\frac{\phi_A-\phi_B}{2}\right)\right]^2 +4\left(1-\frac{e^{-\gamma_1 t}+e^{-\gamma_2 t}}{2}\right)\cos^2\left(\frac{\phi_A}{2}\right)\cos^2\left(\frac{\phi_B}{2}\right) \tag{22}$$

is the probability of measuring both atoms in their lower level. If we assume that both atoms have the same interaction time, $\phi_A=\phi_B=\phi$, we get

$$F'=\frac{F(1-\cos\phi)}{1+\cos\phi(1-e^{-\gamma_1 t}-e^{-\gamma_2 t})}. \tag{23}$$

In this case, we get the same kind of state as in the initial condition, with a modified fidelity. In Fig. 5 we show that $F'>F$ for interaction times $\phi\in(\pi/2,3\pi/2)$, which corresponds to an improvement of the entanglement between cavity fields. The most interesting feature of this protocol is that, in principle, there is no limit on the value of F for getting purification. The maximum improvement is obtained for $\phi=\pi$, where the new fidelity is

$$F'=\frac{1}{2}\left(1+\cosh\left[\frac{(\gamma_1-\gamma_2)t}{2}\right]^{-1}\right). \tag{24}$$

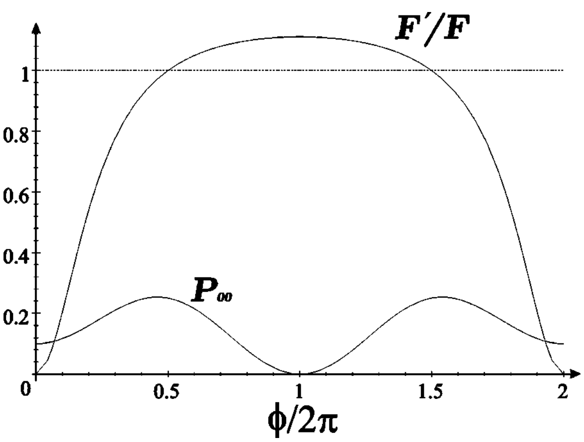

FIG. 5. Modified fidelity of the state of quantum fields from Eq. (23) as a function of adimensional time ϕ. Here we have assumed $\gamma_1=\gamma_2$ and $\gamma_1 t=0.1$.

Thus, the new fidelity $F'=1$ requires the same decay constant for both cavities. If the decay constants are approximately equal, with a difference close to 1%, the fidelity reaches a value of one with an error of the order 10^{-3}, for times long enough to observe loss of entanglement. However, the probability P_{00}, Eq. (22), of measuring both atoms in the lower level, vanishes.

In a specific measurement on the ancillas we can improve or reduce the entanglement in the field density matrix. Thus, we must disregard the entangled pairs in which the purification does not occur. However, we can also introduce a distillationlike process as a sequence of individual purification processes, where the final result will depend on an adequate choice of parameters. The choice is related to making more probable the case where purification occurs, even when it can be small. This sequence permits reaching values of fidelity close to unity. Finally, we can verify the effect of the purification process by sending an additional resonant atom through the second cavity as was discussed in Sec. II B.

III. PROTOCOL IN THE OPTICAL DOMAIN

The above described proposal can also be implemented in the optical domain of cavity QED, where the setup consists of a set of two optical cavities, each one containing at least two trapped ions. The ions allow both for establishing the entanglement between distant nodes of the network and for performing local operations in the purification process. Thus ions permit interactions with both the quantized mode of the cavity and the vibrational mode of two specific ions in a given cavity. This latter kind of interaction corresponds to the proposal for a quantum computer with cold trapped ions [18]. The schematic diagram for the relevant electronic transitions of ions in our case is given in Fig. 6. The relevant level structure that we consider includes three-level ions, $\{|c\rangle, |e_{1,0}\rangle, |g\rangle\}$, with a degenerate electric dipole forbidden transition $|e_{1,0}\rangle\rightarrow|g\rangle$. The classical driven fields Ω_1 and Ω_2 allow performing logic operations on ions, and Ω_c and a allow a Raman transition for creating entanglement between distant nodes, where a is the annihilation operator of the quantum cavity field mode. The transference of entanglement is performed via cavity leakage of photons. We shall assume that in one of the cavities there is a linear trap with n ions that are cooled to the fundamental state of the collective center of mass motion. The linear trap is located inside an optical cavity with a very narrow bandwidth, or very short decay time, typically of 10^{-7} s. In addition, as shown in Fig. 6, there exists a set of distant cavities containing at least

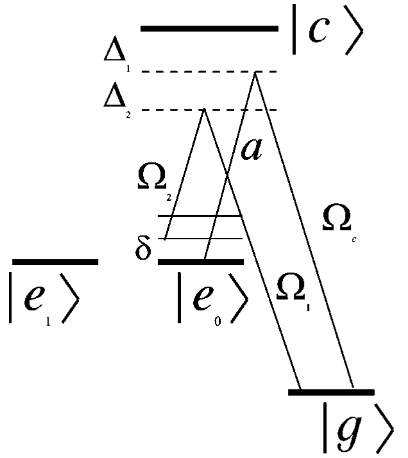

FIG. 6. Electronic structure of trapped ions in a linear trap inside the cavity.

two trapped ions with the same level structure mentioned above.

The advantage of this implementation is twofold. First it also allows a purification protocol given in the Ref. [5] proposal, and second we can purify mixed states of more than two particles. In the Bennett proposal we must consider a set of n ions in each cavity. We establish entanglement between two ions by using time-reversal operations [33] or for unconditional teleportation of atomic states [26]. Thus, to get a modified density matrix, we perform bilateral operations on particles of partially entangled pairs, which must be followed by a measurement on one of the ions. Besides, this scheme also allows getting an entangled state from $m+1$ initial pairs. This last statement requires performing joint measurements on m particles, which can recognize generalized Bell's states. This can be performed by a sequence of controlled-NOT gates and individual measurements on ions.

A. Entanglement between distant pairs

In order to generate entangled states between distant particles, we assume that each ion in the linear trap in cavity C_1 can be addressed individually as is proposed in the original scheme of Ref. [18]. Let us consider a classical field Ω_c detuned in Δ_1 respect to the allowed transition $|c\rangle_i \rightarrow |g\rangle_i$ of the ith ion in the central cavity. The $|c\rangle_i \rightarrow |e_{1,0}\rangle_i$ dipole allowed transition is quantum mechanically described by creation and annihilation operators $a, a^\dagger$ and is assumed to be initially in the vacuum state. In the high detuning limit the upper level can be adiabatically eliminated such that a Raman configuration leads to an effective interaction term [27]:

$$V^I(t) = \frac{r^2(t)}{\Delta_s} |g\rangle_{ii}\langle g| + \Delta_s a^\dagger a |e_q\rangle_{ii}\langle e_q| + i r(t) \times (e^{-i\phi(t)} a^\dagger |e_q\rangle_{ii}\langle g| - e^{i\phi(t)} |g\rangle_{ii}\langle e_q| a), \quad (25)$$

where $q=1,0$ denotes orthogonal polarizations of the created photon; $r(t) = g\Omega_c(t)/2\delta$ (Rabi frequency of Raman transition) and $\Delta_s = g^2/\delta$ (cavity induced Stark shift). For simplicity, we have assumed a common detuning parameter between the pump fields and the cavity modes, and real coupling constants. Any additional frequency shifts can be included in the phase of the classical field $\phi(t)$. The time and intensity dependent terms correspond to dynamical shifts arising from the adiabatic elimination of the upper atomic level.

Thus, we obtain an effective anti-Jaynes-Cummings model. An important element in this process is the production of polarized photons, provided we drive the transition $|c\rangle_i \rightarrow |g\rangle_i$ with classical fields of a given polarization. Assume that the quantum field is initially in the vacuum state and the ion in the ground state, such that

$$|\psi\rangle_0 = |g\rangle_i |0\rangle_{C_1} |0\rangle_E, \quad (26)$$

where $|0\rangle_E$ represents the state of the environment initially in the vacuum. Driving the transition $|0\rangle_i \rightarrow |g\rangle_i$ during an appropriate time Δt, we have

$$|\psi\rangle_1 = (c_e |e_q\rangle_i |1_q\rangle_{C_1} + c_g |g\rangle_i |0\rangle_{C_1}) |0\rangle_E, \quad (27)$$

where $c_e = \cos\varphi$ and $c_g = \sin\varphi$. The photon is emitted by the cavity in a short time compared to relaxation scales of the ion because of the high cavity decay rate. The emission process leads to an emitted one photon wave packet such that

$$|\psi\rangle_2 = (c_e |e_q\rangle_i |1\rangle_E + c_g |g\rangle_i |0\rangle_E) |0\rangle_{C_1}, \quad (28)$$

where $|1\rangle_E$ denotes a one photon wave packet emitted by cavity C_1.

Let us suppose that we have previously prepared another cavity C_j. In this cavity one ion is initially in the state $|e_q\rangle_j$, so that the state of the whole system is given by

$$|\psi\rangle_2 = (c_e |e_q\rangle_i |1_q\rangle_E + c_g |g\rangle_i |0\rangle_E) |e_q\rangle_j |0\rangle_{C_j}, \quad (29)$$

where we have omitted the vacuum state of the C_1 cavity because it is factorized. The same is done in the next step with factorized states. If the photon is absorbed by the C_j cavity, the state of the system changes to

$$|\psi\rangle_3 = (c_e |e_q\rangle_i |1_q\rangle_{C_j} + c_g |g\rangle_i |0\rangle_j) |e_q\rangle_j. \quad (30)$$

Now, a classical field to drive the ion Raman transition is turned on. Thus, after a time Δt, we get

$$|\psi\rangle_4 = (c_e |e_q\rangle_i |g\rangle_j + c_g |g\rangle_i |e_q\rangle_j). \quad (31)$$

For $\varphi = \pi/4$, we obtain an entangled state of two distant ions

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} (|e_q\rangle_i |g\rangle_j + |g\rangle_i |e_q\rangle_j). \quad (32)$$

We obtain in this way a perfectly correlated pair of two ions associated with two distant systems. We have assumed here a perfect generation of the correlated pair. Concerning the discussion of Sec. II A, a decoherence mechanism can produce loss of entanglement. In this case a possible mechanism can be a partial absorption of the one photon wave packet while traveling from one cavity to another. In this situation the state that we obtain can be more properly represented by a density operator ρ in a Werner state after a randomization process with local operations, namely,

$$\rho = F|\psi^+\rangle\langle\psi^+| + \frac{1-F}{3}(1-|\psi^+\rangle\langle\psi^+|), \tag{33}$$

where we assumed that the final state corresponds to a deviation from a maximally entangled state. The subsequent procedure continues in the same way as we have outlined in Sec. II C, where we considered the implementation of a controlled-NOT gate by introducing the auxiliary ancillary system for each separated node. In this case we consider a second ion in each cavity and realize the quantum gate by following the procedure we describe in the following section.

B. Local measurements

Local measurements on pairs of ions in the linear trap can be implemented by considering the individual addressing of ions in the central cavity by a pair of Raman fields Ω_{1i} detuned in $\Delta_1+\Delta_2$, and Ω_{2i} detuned in $\Delta_1+\Delta_2+\delta_i$ with respect to the lower ionic transitions $|c\rangle_i \rightarrow |g\rangle_i$ and $|c\rangle_i \rightarrow |e_{1,0}\rangle_i$. Eliminating adiabatically the excited level $|c\rangle_i$ and adjusting properly the detuning δ_i, it is possible to generate general dynamic evolution. It can be shown that by adjusting $\delta_i = -\nu_x$, the center of mass collective motion and the internal ionic levels evolve under the effective Hamiltonian

$$H_q = \hbar \frac{\Omega\eta}{2\sqrt{N}}(be^{i\phi}|e_q\rangle_{ii}\langle g| + |g\rangle_{ii}\langle e_q|b^\dagger e^{-i\phi}), \tag{34}$$

with η being the Lamb-Dicke parameter; b and $b^\dagger$ are the center of mass motion operators and $\Omega = \Omega_1\Omega_2/(\Delta_1+\Delta_2)^2$. For $\delta_i = 0$ (carrier transition) we obtain a single Hamiltonian

$$H_R = \hbar\frac{\Omega}{2}(e^{-i\phi}|e_q\rangle_{ii}\langle g| + |g\rangle_{ii}\langle e_q|e^{-i\phi}). \tag{35}$$

Choosing an interaction time between the laser pulses and the ion such that $t = k\pi\sqrt{N}/\Omega\eta$, the Hamiltonian structure H_q leads to the general evolution

$$U_i^{kq}(\phi) = \exp\left(-ik\frac{\pi}{2}(be^{i\phi}|e_q\rangle_{ii}\langle g| + |g\rangle_{ii}\langle e_q|b^\dagger e^{-i\phi})\right). \tag{36}$$

The Hamiltonian H_R leads to the following rotation operator:

$$R_i^k(\phi) = \exp\left[-ik\frac{\pi}{2}(|e_o\rangle_{ii}\langle g|e^{i\phi} + e^{-i\phi}|g\rangle_{ii}\langle e_o|)\right]. \tag{37}$$

In the case of $\phi = \pi/2$ the above gate corresponds to the Hadamard gate. According to Ref. [18], this physical picture allows implementing a controlled-NOT gate as

$$U_{\rm XOR} = R_{l_a}^{1/2}\left(-\frac{\pi}{2}\right) U_l^{1,0} U_{l_a}^{2,1} U_l^{1,0} R_{l_a}^{1/2}\left(\frac{\pi}{2}\right), \tag{38}$$

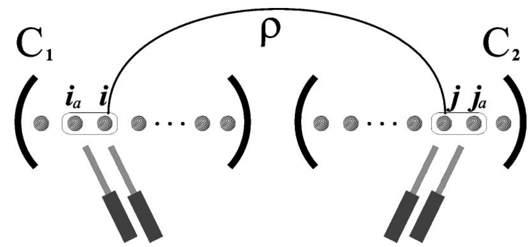

FIG. 7. Setup for purification protocol using trapped ions inside an optical cavity. The entanglement between ions i and j is purified by local operations on pairs (i,i_a) and (j,j_a).

where $l = i,j$ and $l_a = i_a, j_a$ are the control and the target qubits, respectively. Using the above gates, we can implement the purification circuit depicted in Fig. 4 and the corresponding setup using trapped ions as shown in Fig. 7, where the entanglement between ions i,j is purified by local operations on pairs (i,i_a) and (j,j_a).

IV. GENERATION OF *n*-PARTICLE MAXIMALLY ENTANGLED STATES

A system to generate and detect maximally entangled states (MES) of n particles is an important requirement in the experimental basis of multiparty communication by quantum networks. Of particular interest are several recent proposals [28–31] that have been suggested for making use of many-qubit entangled states in order to share quantum information among several parties. An example is the quasisimultaneous transfer of information to n clients using an n-qubit MES. As has been shown by Molotkov and Nazin [32], such a procedure not only implements key distribution but also allows one to check efficiently for the presence of an eavesdropper. Another interesting application is a controlled quantum communication network, which consists of a single *provider* (P) who is connected to the *clients* (C) via independent photonic channels. The provider creates a quantum communication channel with each one of the clients. In this case, the photonic channel is used to create entanglement between ions at provider and client.

In order to establish a quantum channel among clients, there exists a requirement on the provider: the capability of performing a joint measurement on particles at his place. This measurement permits swapping the initial entanglement among the provider and the clients to an entanglement among parties. Actually, the provider needs to perform a measurement for recognizing the maximally entangled states of an n-particle system, which are defined

$$|\Psi^{\pm}_{\rm mes}\rangle = \frac{1}{\sqrt{2}}(|0,s_2, \ldots, s_n\rangle \pm |1,1-s_2, \ldots, 1-s_n\rangle), \tag{39}$$

with $s_j = 0,1$. The state of the first particle can be arbitrarily chosen as 0 without loss of generality. These states form a basis for a Hilbert space of n two-level particles. Here, we consider a modified version of the purification protocol in the optical domain for creating multiparticle entanglement among distant particles. The provider consists of a cavity

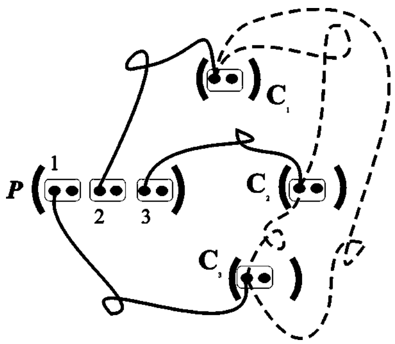

FIG. 8. Diagram for establishing multiparticles entanglement. Initially ions in the central cavity P are entangled with ions in cavities C_j. The initial entanglement is described with solid lines. After a collective measurement in cavity P on ions 1, 2, and 3 a maximally entangled state among cavities C_1, C_2, and C_3 is achieved.

with at least $2n$ trapped ions, the half of them are for establishing the set of n entangled pairs among ions in the main cavity and the distant cavities. The remainder ions at the provider are used for purification purposes. In the same way as the provider, each one of the clients has at least two ions. The joint measurement for recognizing MES is implemented using an adequate sequence of controlled-NOT gates, (XOR) gates [given by Eq. (38)], followed by measurement on individual ions at the provider. We assume that there are n pairs of maximally entangled states. Initial pairs are maximally entangled in the state (see Fig. 8)

$$|\psi^-_{k,k'}\rangle=\frac{1}{\sqrt{2}}(|0_k,1_{k'}\rangle-|1_k,0_{k'}\rangle), \tag{40}$$

where k and k' refer to the kth qubit at the *multiparticle communication provider* and the qubit at the k' party, respectively. The initial condition for the whole set of particles is given by

$$|\Psi\rangle=\frac{1}{2^{n/2}}\otimes^n_{k=1}(|0_k,1_{k'}\rangle\pm|1_k,0_{k'}\rangle),$$

$$=\frac{1}{2^{n/2}}\sum_{j=0}^{2^{n-1}-1}(|\psi^-_{C_j}\rangle|\psi^+_{P_j}\rangle\pm|\psi^+_{C_j}\rangle|\psi^-_{P_j}\rangle), \tag{41}$$

where

$$|\psi^{\pm}_{C(P)_j}\rangle=\frac{1}{\sqrt{2}}(|0,j_2,\ldots,j_n\rangle\pm|1,1+j_2,\ldots,1+j_n\rangle) \tag{42}$$

with $j=2^0j_n+2^1j_{n-1}+\cdots+2^{n-1}j_2$.

Thus, if we perform a joint measurement on the particles at the *provider* that can detect states $|\psi^{\pm}_{C_j}\rangle$, we prepare the states $|\psi\mp_{P_j}\rangle$ at the clients. Then the relevant question here is how can we detect the MES at the provider. This question can be regarded as a maximally entangled state recognition of a n-qubit system. We shall perform this generalized Bell's states measurement using a disentanglement operation followed by one-particle measurements. The disentanglement process is given as

$$U_{\text{disent}}=H_1U_{\text{XOR}_{1,2}}U_{\text{XOR}_{2,3}}\cdots U_{\text{XOR}_{n-1,n}}, \tag{43}$$

where $U_{\text{XOR}_{k-1,k}}$ denotes a controlled-NOT gate between particles $k-1$ and k as source and target, respectively, and H is a Hadamard transformation. Now, we apply this operation to the global state in Eq. (41). In particular, the action of U_{disent} on one of the components of the provider state lets $|\xi^{\pm}_{P_j}\rangle=U_{\text{disent}}|\psi^{\pm}_{P_j}\rangle$, where

$$|\xi^{\pm}_{P_j}\rangle=|l_1\rangle\otimes^{n-1}_{k=0}|j_{n-k-1}+j_{n-k}\rangle. \tag{44}$$

Now, in all the ions we measure σ_z. The first measurement defines the sign of a client's state, i.e., $l=0$ $(l=1)$ corresponds to a + (−) sign, and the other will define the index j. The disentanglement and measurement processes consecutively are summarized as follows:

$$|\Psi\rangle\rightarrow U_{\text{disent}}|\Psi\rangle\overset{M}{\rightarrow}|\xi^{\mp}_{P_j}\rangle|\psi^{\pm}_{C_j}\rangle.$$

This scheme can also be used for preparing maximally entangled states. Of course, in the simplest case of initially factorized systems, maximally entangled states are obtained by applying an inverse sequence of unitary operations for state recognition. The most relevant feature of this procedure for generating multipartite maximally entangled states is that the quality of entanglement only depends on the degree of entanglement of initial pairs, so it does not require any purification protocol for many particles [25], which can be much more difficult than for two particles even when it is performed locally. In a recent work by Cinchetti and Twamley [34] the entanglement distribution between N users and a center was studied. They studied the case of using qubits as here, and also extend their analysis to the case of using qutrits instead of qubits. However, the problem related to the establishment of the maximally entangled qutrit states remains open, because it requires transmitting information stored in qutrits.

V. SUMMARY

We have studied the possibility of an experimental setup for a purification protocol based on POVM in cavities, at both the microwave and the optical domain. The conditional evolution of ancillas is obtained by a sequence of one and two qubit gates, which permits a conditional evolution needed for implementing the purification protocol. We find that the fidelity can increase for a broad range of parameters. It is also possible to introduce a distillationlike process as a sequence of individual purifications, where the final result will depend on adequate choice of numerical values of parameters. This sequence permits reaching values of fidelity close to unity. We think that the proposal in the microwave region can be experimentally implemented by using a setup similar to that used in the controlled quantum phase gate by Rauschenbeutel *et al.* [19] and more general noise conditions belong to the experimental setup for controlled entanglement of two-modes field inside a microwave cavity [22]. Thus, our proposal can also be extended to collective manipulation of partially entangled pairs by using two cavites with two-modes.

Note added. Recently we learned about a general proof of the quasidistillation of mixed rank two matrices given in Ref. [13].

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[1] A. Barenco, Contemp. Phys. **37**, 375 (1996).
[2] D.P. DiVincenzo, Science **270**, 255 (1995).
[3] C.H. Bennett, Phys. Today **48**, 24 (1995), and references cited therein.
[4] A. Einstein, B. Podolski, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[5] Charles H. Bennett, Gilles Brassard, Sandu Popescu, Benjamin Schumacher, John A. Smolin, and William K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[6] Charles H. Bennett, David P. Di Vicenzo, John A. Smolin, and William K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[7] W. Tittel, J. Brendel, B. Gisin, T. Herzog, H. Zbinden, and N. Gisin, Phys. Rev. A **57**, 3229 (1998).
[8] A. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[9] Lu-Ming Duan, G. Giedke, J.I. Cirac, and P. Zoller, Phys. Rev. A **62**, 032304 (2000).
[10] Jian-Wei Pan, Christoph Simon, Caslav Bruckner, and Anton Zeilenger, Nature (London) **410**, 1067 (2001).
[11] Nicolas Gisin, Phys. Lett. A **210**, 151 (1996).
[12] Micha Horodecki, Pawel Horodecki, and Ryszard Horodecki, Phys. Rev. A **60**, 1888 (1999).
[13] Frank Verstraete, Jeroen Dehaene, and Bart DeMoor, Phys. Rev. A **64**, 010101 (2001).
[14] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).
[15] L. Davidovich, N. Zagury, M. Brune, J.M. Raimond, and S. Haroche, Phys. Rev. A **50**, R895 (1994).
[16] J.M. Raimond, M. Brune, and S. Haroche, Phys. Rev. Lett. **79**, 1964 (1997).
[17] J.M. Raimond, E. Hagley, X. Maître, G. Nogues, C. Wunderlich, M. Brune, and S. Haroche, in *Mysteries, Puzzles, and Paradoxes in Quantum Mechanics*, edited by Rodolfo Bonifacio, AIP Conf. Proc. No. 461 (AIP, Woodbury, NY, 1999), p. 144.
[18] I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[19] A. Rauschenbeutel, G. Nogues, S. Osnaghi, P. Bertet, M. Brune, J.M. Raimond, and S. Haroche, Phys. Rev. Lett. **83**, 5166 (1999).
[20] Adrian Kent, Phys. Rev. Lett. **81**, 2839 (1998).
[21] N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **81**, 3279 (1998).
[22] A. Rauschenbeutel, P. Bertet, S. Osnaghi, G. Nogues, M. Brune, J.M. Raimond, and S. Haroche, Phys. Rev. A **64**, 050301(R) (2001).
[23] Scott Hill and Williams K. Wooters, Phys. Rev. Lett. **78**, 5022 (1997).
[24] Williams K. Wooters, Phys. Rev. Lett. **80**, 2245 (1998).
[25] M. Murao, M.B. Plenio, S. Popescu, V. Vedral, and P.L. Knight, Phys. Rev. A **57**, R4075 (1998).
[26] S. Lloyd, S.M. Shahriar, J.H. Shapiro, and P.R. Hemmer, Phys. Rev. Lett. **87**, 167903 (2001).
[27] C. Saavedra, K.M. Gheri, P. Thorma, J.I. Cirac, and P. Zoller, Phys. Rev. A **61**, 062311 (2000).
[28] S. Bose, V. Vedral, and P.L. Knight, Phys. Rev. A **60**, 194 (1999).
[29] S.J. van Enk, J.I. Cirac, and P. Zoller, Phys. Rev. Lett. **81**, 5932 (1997).
[30] M. Hillery, V. Bužek, and A. Berthiaume, e-print quant-ph/9806063.
[31] S. Bose, V. Vedral, and P.L. Knight, Phys. Rev. A **57**, 822 (1998).
[32] S.N. Molotkov and S.S. Nazin, JETP Lett. **63**, 687 (1996).
[33] J.I. Cirac, P. Zoller, H.J. Kimble, and H. Mabuchi, Phys. Rev. Lett. **78**, 3221 (1997); T. Pellizzari, *ibid.* **79**, 5242 (1997).
[34] M. Cinchetti and J. Twamley, Phys. Rev. A **63**, 052310 (2001).