Multiparticle reduced density matrix and a useful kind of entangled state for quantum teleportation

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(Received 16 October 2001; revised manuscript received 4 January 2002; published 17 April 2002)

It is pointed out that the possibility of teleporting an *arbitrary* unknown one-particle spin state is crucially connected with the maximal entanglement of the Einstein-Podolsky-Rosen pair, whose one-particle reduced density matrix is $\rho(i)=\frac{1}{2}\mathbf{I}_2$ $(i=1,2)$. It is shown that, to teleport an *arbitrary* k-particle spin state, one must prepare an ancillary N-particle $(N\geqslant 2k)$ entangled state, whose k-particle reduced density matrix has the form $(1/2^k)\mathbf{I}_{2^k}$ ($\mathbf{I}_{2^k}$ is the $2^k\times 2^k$ identity matrix). An alternative approach to constructing many-particle entangled states is developed by using $R_x(\pi)$, the collective rotation of π around a given axis (say, x axis). The entangled states constructed by using $R_x(\pi)$ operating on the basis of angular momentum uncoupling representation are just the GHZ states, which cannot be used for the teleportation of an arbitrary k $(\geqslant 2)$ particle spin state. The entangled states constructed by using $R_x(\pi)$ operating on the basis of angular momentum coupling representation turn out to be effective for the teleportation of an *arbitrary* multiparticle state. A formal extension of the scheme of Bennett *et al.* to deal with the teleportation of an arbitrary two (or more) particle spin state is discussed.

DOI: 10.1103/PhysRevA.65.052307 PACS number(s): 03.67.Hk, 03.65.Ta

I. INTRODUCTION

The first successful scheme for quantum teleportation was developed by Bennett *et al*. [1]. The idea behind teleportation is that a physical object is equivalent to the information needed to construct it [2], i.e., an event by which a physical object is transformed from one point in space to another without actual material transportation [3] . In classical physics there is no conceptual impediment to this, because a system can be scanned thoroughly without disturbing its state in a given location and using this information one can completely reconstruct it in a second site. However, this procedure is invalid for any real physical system, which ultimately consists of microscopic constituents obeying the principle of quantum mechanics. According to quantum mechanics, any attempt to gain information about an object, in general, always changes its state; i.e., a physical object in its original state is bound to be destroyed by the process of scanning, which has been demonstrated by Wooters and Zurek in the noncloning theorem of an unknown quantum state [4]. It is encouraging to note that the scheme of quantum teleportation of Bennett *et al.* has been realized in various conditions [5–9].

In the scheme for quantum teleportation, of Bennett *et al.* a key role is played by an ancillary Einstein-Podolsky-Rosen (EPR) pair of particles which will be initially shared by the sender (Alice) and the receiver (Bob). The peculiarity of an entangled state—the nonlocal correlation between two objects which was first pointed out by Einstein, Podolsky, and Rosen [10] (though the term is due to Schrödinger [11])—represents the quintessential distinction between quantum and classical physics. For a two-particle (spin-1/2) system, the Bell basis is widely used in quantum communication theory:

$$
\begin{aligned}
|\psi^-\rangle_{12}&=\frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2-|\downarrow\rangle_1|\uparrow\rangle_2),\\
|\psi^+\rangle_{12}&=\frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2+|\downarrow\rangle_1|\uparrow\rangle_2),\\
|\phi^-\rangle_{12}&=\frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\uparrow\rangle_2-|\downarrow\rangle_1|\downarrow\rangle_2),\\
|\phi^+\rangle_{12}&=\frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\uparrow\rangle_2+|\downarrow\rangle_1|\downarrow\rangle_2),
\end{aligned}
\tag{1}
$$

which may be considered as the simultaneous eigenstates of a complete set of commuting local two-body observables, i.e., any two of (S_z^2,S_x^2,S_y^2) [3] or, equivalently, $(\sigma_{1z}\sigma_{2z},\sigma_{1x}\sigma_{2x},\sigma_{1y}\sigma_{2y})$ [12].

Following the scheme of Bennett *et al.* [1], suppose the sender Alice has been given a single-particle in state $|\phi\rangle_A$ unknown to her,

$$|\phi\rangle_A=a|\uparrow\rangle_A+b|\downarrow\rangle_A=\begin{pmatrix}a\\ b\end{pmatrix}_A, \tag{2}$$

which is to be sent to Bob at a distant location. One may prepare an ancillary entangled two-particle state, e.g., $|\psi^-\rangle_{12}$. The three particles may be of different kinds. Then, particle 1 is sent to Bob and particle 2 to Alice. The complete state of the entire three-particle system is the product state $|\psi^-\rangle_{12}|\phi\rangle_A$, having neither classical correlation nor quantum entanglement between particle A and particles (1,2). An entanglement between particles A and 2 (or A and 1) is brought about in the next step—a joint Bell basis measurement. To describe this process, it is convenient to expand $|\psi^-\rangle_{12}|\phi\rangle_A$ in terms of the Bell basis for particles $(A,2)$:

TABLE I. The unitary transformations needed to extract a faithful replica of the one-particle spin state $|\phi\rangle_A$ to be sent.

	$\|\psi^-\rangle_{12}$	$\|\psi^+\rangle_{12}$	$\|\phi^-\rangle_{12}$	$\|\phi^+\rangle_{12}$
$\|\psi^-\rangle_{A2}$	$\mathbf{I}_2$	σ_z	$-\sigma_x$	$i\sigma_y$
$\|\psi^+\rangle_{A2}$	σ_z	$\mathbf{I}_2$	$-i\sigma_y$	σ_x
$\|\phi^-\rangle_{A2}$	$-\sigma_x$	$i\sigma_y$	$\mathbf{I}_2$	σ_z
$\|\phi^+\rangle_{A2}$	$-i\sigma_y$	σ_x	σ_z	$\mathbf{I}_2$

$$|\psi^-\rangle_{12}|\phi\rangle_A=\frac{1}{2}|\psi^-\rangle_{A2}\begin{pmatrix}a\\b\end{pmatrix}_1+\frac{1}{2}|\psi^+\rangle_{A2}\begin{pmatrix}a\\-b\end{pmatrix}_1+\frac{1}{2}|\phi^-\rangle_{A2}\begin{pmatrix}-b\\-a\end{pmatrix}_1+\frac{1}{2}|\phi^+\rangle_{A2}\begin{pmatrix}b\\-a\end{pmatrix}_1. \tag{3}$$

After Alice's measurement, Bob's particle 1 will have been projected onto one of the four states shown on the right hand side of Eq. (3). In the case of the Alice's first outcome $|\psi^-\rangle_{A2}$, Bob's state is

$$|\phi\rangle_1=\begin{pmatrix}a\\b\end{pmatrix}_1,$$

a replica of the original state $|\phi\rangle_A$. In the other three cases, Bob may apply a corresponding unitary transformation on his outcome to extract the original state information. That is, according to Alice's outcome which is sent to Bob through a classical channel, Bob may apply one of the following unitary operations on his outcome to extract a replica of the original state $|\phi\rangle_A$:

$$U_1=\mathbf{I}_2,\ \ U_2=\sigma_z,\ \ U_3=-\sigma_x,\ \ U_4=-i\sigma_y, \tag{4}$$

with $\mathbf{I}_2$ being the 2×2 identity matrix. The situation is similar if the prepared ancillary two-particle state is one of the other Bell bases, and the corresponding unitary transformations are summarized in Table I.

It should be emphasized that the possibility of teleporting an *arbitrary*, unknown one-particle spin state is crucially connected with the maximal entanglement of the prepared EPR pair, whose one-particle reduced density matrix is $\rho(i)=\frac{1}{2}\mathbf{I}_2$ $(i=1,2)$. To illustrate this point of view more clearly, let us try to perform the teleportation using other two kinds of basis usually adopted in angular momentum coupling theory.

For the description of angular momentum, usually two representations are adopted [13]: i.e., the angular momentum uncoupling and coupling representations. For a two-particle (spin-1/2) system, the basis of angular momentum uncoupling representation is the simultaneous eigenstate of the complete set of one-body observables (s_{1z},s_{2z}) and denoted by $|m_1\rangle_1|m_2\rangle_2$, $(m_1,m_2=\pm1/2)$ or, intuitively, denoted by

$$|\uparrow\rangle_1|\uparrow\rangle_2,\ \ |\downarrow\rangle_1|\downarrow\rangle_2,\ \ |\uparrow\rangle_1|\downarrow\rangle_2,\ \ |\downarrow\rangle_1|\uparrow\rangle_2, \tag{5}$$

whose one-particle reduced density matrix is

$$\rho(i)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad(i=1,2).$$

The basis of angular momentum coupling representation is the simultaneous eigenstate of (S^2,S_z), $S=s_1+s_2$, $S_z=s_{1z}+s_{2z}$, and denoted by $\chi_{SM}(1,2)$, $S=0$, $M=0$ (singlet state), and $S=1$, $M=0,\pm1$ (triplet states), which can be expanded in terms of $|m_1\rangle_1|m_2\rangle_2$:

$$\chi_{00}(1,2)=\frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2-|\downarrow\rangle_1|\uparrow\rangle_2),$$

$$\chi_{10}(1,2)=\frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2+|\downarrow\rangle_1|\uparrow\rangle_2),$$

$$\chi_{11}(1,2)=|\uparrow\rangle_1|\uparrow\rangle_2,\quad \chi_{1-1}(1,2)=|\downarrow\rangle_1|\downarrow\rangle_2. \tag{6}$$

It is noted that while $\chi_{11}(1,2)$ and $\chi_{1-1}(1,2)$ are separable, the other two $[\chi_{00}(1,2)$ and $\chi_{10}(1,2)]$ are entangled states, whose one-particle reduced density matrix is $\rho(i)=\frac{1}{2}\mathbf{I}_2$. This is easily understood because S_z is a one-body operator, whereas S^2 is a two-body operator.

Suppose the prepared ancillary two-particle state is one of the bases of angular momentum uncoupling representation; following the scheme of Bennett *et al.*, we have

$$|\uparrow\uparrow\rangle_{12}|\phi\rangle_A=|\uparrow\uparrow\rangle_{A2}\begin{pmatrix}a\\0\end{pmatrix}_1+|\downarrow\uparrow\rangle_{A2}\begin{pmatrix}b\\0\end{pmatrix}_1,$$

$$|\downarrow\downarrow\rangle_{12}|\phi\rangle_A=|\downarrow\downarrow\rangle_{A2}\begin{pmatrix}0\\b\end{pmatrix}_1+|\uparrow\downarrow\rangle_{A2}\begin{pmatrix}0\\a\end{pmatrix}_1,$$

$$|\uparrow\downarrow\rangle_{12}|\phi\rangle_A=|\downarrow\downarrow\rangle_{A2}\begin{pmatrix}b\\0\end{pmatrix}_1+|\uparrow\downarrow\rangle_{A2}\begin{pmatrix}a\\0\end{pmatrix}_1,$$

$$|\downarrow\uparrow\rangle_{12}|\phi\rangle_A=|\uparrow\uparrow\rangle_{A2}\begin{pmatrix}0\\a\end{pmatrix}_1+|\downarrow\uparrow\rangle_{A2}\begin{pmatrix}0\\b\end{pmatrix}_1. \tag{7}$$

Obviously, in this case there exists no unitary transformation to convert Bob's outcome into a replica of the original state $|\phi\rangle_A$; i.e., one cannot realize the teleportation of an arbitrary one-particle spin state. This is inherently connected with the fact that all four bases are disentangled states with one-particle density matrix

$$\rho(i)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad(i=1,2).$$

Next, if the prepared ancillary two-particle state is one of the bases of angular momentum coupling representation $\chi_{SM}(1,2)$, we have

$$\chi_{00}(1,2)|\phi\rangle_A=\frac{1}{2}\chi_{00}(A,2)\begin{pmatrix}a\\b\end{pmatrix}_1+\frac{1}{2}\chi_{10}(A,2)\begin{pmatrix}a\\-b\end{pmatrix}_1+\frac{1}{\sqrt{2}}\chi_{11}(A,2)\begin{pmatrix}0\\-a\end{pmatrix}_1+\frac{1}{\sqrt{2}}\chi_{1-1}(A,2)\times\begin{pmatrix}b\\0\end{pmatrix}_1,$$

$$\chi_{10}(1,2)|\phi\rangle_A=\frac{1}{2}\chi_{00}(A,2)\begin{pmatrix} a \\ -b \end{pmatrix}_1+\frac{1}{2}\chi_{10}(A,2)\begin{pmatrix} a \\ b \end{pmatrix}_1 +\frac{1}{\sqrt{2}}\chi_{11}(A,2)\begin{pmatrix} 0 \\ a \end{pmatrix}_1+\frac{1}{\sqrt{2}}\chi_{1-1}(A,2)\begin{pmatrix} b \\ 0 \end{pmatrix}_1,$$

$$\chi_{11}(1,2)|\phi\rangle_A=\frac{1}{\sqrt{2}}\chi_{00}(A,2)\begin{pmatrix} -b \\ 0 \end{pmatrix}_1+\frac{1}{\sqrt{2}}\chi_{10}(A,2)\begin{pmatrix} b \\ 0 \end{pmatrix}_1 +\chi_{11}(A,2)\begin{pmatrix} a \\ 0 \end{pmatrix}_1,$$

$$\chi_{1-1}(1,2)|\phi\rangle_A=\frac{1}{\sqrt{2}}\chi_{00}(A,2)\begin{pmatrix} 0 \\ a \end{pmatrix}_1+\frac{1}{\sqrt{2}}\chi_{10}(A,2)\begin{pmatrix} 0 \\ a \end{pmatrix}_1 +\chi_{1-1}(A,2)\begin{pmatrix} 0 \\ b \end{pmatrix}_1. \tag{8}$$

It is noted that only when the prepared two-particle state is $\chi_{00}(1,2)$ or $\chi_{10}(1,2)$ and Alice's outcome is $\chi_{00}(A,2)$ or $\chi_{10}(A,2)$ can one construct corresponding unitary transformation to convert Bob's outcome state into a replica of the original state $|\phi\rangle_A$. In the other two cases, the teleportation can not be realized. This is also connected to the fact that for the entangled states χ_{00} and χ_{10}, $\rho(i)=\frac{1}{2}\mathbf{I}_2$ $(i=1,2)$, but for the separable states $|11\rangle$ and $|1-1\rangle$,

$$\rho(i)=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (i=1,2).$$

To date, the teleportation of an arbitrary one-particle (spin-1/2) state has been discussed in detail and has been experimentally realized. The next step is to investigate the teleportation of *arbitrary* multiparticle information, which is a crucial ingredient for quantum communication and information [14]. As pointed out by Bennett *et al.* [1], since teleportation is a linear operation applied to the quantum state $|\phi\rangle$, it will work not only for pure states, but also with various kinds of mixed or entangled states. The key points for teleportation of a multiparticle information are to prepare a suitable ancillary many-particle entangled state and to make a suitable joint measurement. Usually, the Greenberger-Horne-Zeilinger (GHZ) states are used. The original motivation to prepare three-particle entanglements stems from the observation that three-particle entanglement leads to a conflict with local realism for nonstatistical predictions of quantum mechanics [15]. The incentive to produce GHZ states has been significantly increased by the advance of the field of quantum communication and quantum information processing [16]. The general form of GHZ states is discussed in Sec. II A, and in Sec. II B we will show that GHZ states are unable to teleport an *arbitrary* multiparticle $(k\geqslant 2)$ spin state, except states with only two components, which is intimately connected with the structure of multiparticle reduced density matrices (Sec. II C). It is emphasized that an N-particle $(N\geqslant 2k)$ spin-entangled state whose k-particle reduced density matrices have the form of $(1/2^k)\mathbf{I}_{2^k}$ can be used effectively to realize the teleportation of an arbitrary k-particle spin state. In Sec. III, an alternative approach to constructing many-particle entangled states is developed by using $R_x(\pi)$, the collective rotation of π around a given axis (say, x axis). The entangled states constructed by using $R_x(\pi)$ operating on the basis of angular momentum uncoupling representation are just the GHZ states, whereas those by using $R_x(\pi)$ operating on the basis of angular momentum coupling representation turn to be effective for the teleportation of an arbitrary multiparticle spin state. The structure of the two-particle reduced density matrices of the four-particle entangled states thus constructed is analyzed in detail. By using this kind of four-particle entangled states, the scheme of Bennett *et al.* is formally extended to deal with the teleportation of an arbitrary two-particle spin state in Sec. IV. The extension to teleport an arbitrary k-particle $(k>2)$ spin state is straightforward. A brief summary is given in Sec. V.

II. GHZ STATES AND TELEPORTATION OF MULTIPARTICLE INFORMATION

A. General form of GHZ states

For an N-particle (spin-1/2) system, the 2^N orthonormal GHZ states are expressed as

$$\frac{1}{\sqrt{2}}[|m_1,m_2,\ldots,m_N\rangle\pm|-m_1,-m_2,\ldots,-m_N\rangle], \tag{9}$$

where $|m_1,m_2,\cdots,m_N\rangle$ is the eigenstate of $(s_{1z},s_{2z},\ldots,s_{Nz})$ with eigenvalues $m_i=\pm 1/2$, $(i=1,2,\ldots,N)$. For $N=2$, the four GHZ states are just the well-known Bell basis. For $N=3$, the eight GHZ states may be considered as the simultaneous eigenstates of the complete set of commuting observables $(F_1=\sigma_{1y}\sigma_{2y}\sigma_{3x},$ $F_2=\sigma_{1y}\sigma_{2x}\sigma_{3y},F_3=\sigma_{1x}\sigma_{2y}\sigma_{3y})$ (see Table II) or a similar equivalent set [17]. Similarly, for $N=4$, the 16 GHZ states may be considered as the simultaneous eigenstates of $(G_1=\sigma_{1x}\sigma_{2x}\sigma_{3x}\sigma_{4x},G_2=\sigma_{1y}\sigma_{2y}\sigma_{3y}\sigma_{4y},$ $G_3=\sigma_{1x}\sigma_{2x}\sigma_{3y}\sigma_{4y},G_4=\sigma_{1x}\sigma_{2y}\sigma_{3x}\sigma_{4y})$ (see Table III).

B. Quantum teleportation of an arbitrary two-particle state

There have been developed some schemes for the teleportation of a multiparticle spin state, in which a $(k+1)$-particle

TABLE II. GHZ states of a three-particle system.

GHZ states	$\vert F_1F_2F_3\rangle$	GHZ states	$\vert F_1F_2F_3\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\uparrow\rangle+\vert\downarrow\downarrow\downarrow\rangle)$	$\vert---\rangle$	$\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\uparrow\rangle+\vert\downarrow\uparrow\downarrow\rangle)$	$\vert+-+\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\uparrow\rangle-\vert\downarrow\downarrow\downarrow\rangle)$	$\vert+++\rangle$	$\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\uparrow\rangle-\vert\downarrow\uparrow\downarrow\rangle)$	$\vert-+-\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\downarrow\rangle+\vert\downarrow\downarrow\uparrow\rangle)$	$\vert-++\rangle$	$\frac{1}{\sqrt{2}}(\vert\downarrow\uparrow\uparrow\rangle+\vert\uparrow\downarrow\downarrow\rangle)$	$\vert++-\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\downarrow\rangle-\vert\downarrow\downarrow\uparrow\rangle)$	$\vert+--\rangle$	$\frac{1}{\sqrt{2}}(\vert\downarrow\uparrow\uparrow\rangle-\vert\uparrow\downarrow\downarrow\rangle)$	$\vert--+\rangle$

GHZ state is used as the communication channel to teleport a k-particle catlike state [18]. For the $k=2$ case [19,20], the state to be sent is of the form

$$|\phi\rangle_{AB}=\alpha|\uparrow\uparrow\rangle_{AB}+\beta|\downarrow\downarrow\rangle_{AB} \quad \text{or} \quad \alpha|\uparrow\downarrow\rangle_{AB}+\beta|\downarrow\uparrow\rangle_{AB}, \tag{10}$$

which is usually called an EPR state. However, such a state is not an arbitrary two-particle spin state [see Eq. (11)]. It is noted that Bennett *et al.* [1] have given an important statement concerning the teleportation of an *arbitrary* state, ''reliable teleportation of an X-state particle requires a classical channel of $2\log_2(X)$ bits capacity.'' In particular, to realize the teleportation of an arbitrary two-particle spin state, one must prepare an ancillary four-particle entangled state. First, let us try to use a four-particle GHZ state to teleport an arbitrary two-particle state.

An arbitrary state of a two-particle (spin-1/2) system may be expanded in terms of the Bell basis:

$$|\Psi\rangle_{AB}=a|\psi^-\rangle_{AB}+b|\psi^+\rangle_{AB}+c|\phi^-\rangle_{AB}+d|\phi^+\rangle_{AB} = \begin{pmatrix} a\\ b\\ c\\ d \end{pmatrix}_{AB}. \tag{11}$$

One may prepare an ancillary four-particle GHZ state, e.g., $|++--\rangle_{1234}$. The complete state of the entire system is the pure state $|++--\rangle_{1234}|\Psi\rangle_{AB}$. Then particles (2,4) are sent to Alice and particles (1,3) to Bob. A joint measurement of GHZ basis for the four-particle system is made by Alice. To describe this process, it is convenient to expand $|++--\rangle_{1234}|\Psi\rangle_{AB}$ as follows:

$$\begin{aligned}
|++--\rangle_{1234}|\Psi\rangle_{AB}=&\frac{1}{2}|++--\rangle_{A2B4}\begin{pmatrix}0\\0\\c\\d\end{pmatrix}_{13}+\frac{1}{2}|--++\rangle_{A2B4}\begin{pmatrix}0\\0\\d\\c\end{pmatrix}_{13}+\frac{1}{2}|+++-\rangle_{A2B4}\begin{pmatrix}0\\0\\-c\\d\end{pmatrix}_{13}\\
&+\frac{1}{2}|---+\rangle_{A2B4}\begin{pmatrix}0\\0\\-d\\c\end{pmatrix}_{13}+\frac{1}{2}|+-+-\rangle_{A2B4}\begin{pmatrix}0\\0\\a\\b\end{pmatrix}_{13}+\frac{1}{2}|-+-+\rangle_{A2B4}\begin{pmatrix}0\\0\\b\\a\end{pmatrix}_{13}\\
&+\frac{1}{2}|+---\rangle_{A2B4}\begin{pmatrix}0\\0\\-a\\b\end{pmatrix}_{13}+\frac{1}{2}|-+++\rangle_{A2B4}\begin{pmatrix}0\\0\\b\\-a\end{pmatrix}_{13}.
\end{aligned} \tag{12}$$

Obviously, there exists no unitary transformation to convert Bob's outcome state for particles (1,3) into a replica of $|\Psi\rangle_{AB}$; i.e., it is impossible to teleport an arbitrary two-particle spin state using a four-particle GHZ state. We will show that, like the teleportation of an arbitrary one-particle spin state using a two-particle state, this impossibility is crucially connected to the *two-particle* reduced density matrix of GHZ states.

It should be noted that for a complete description of entanglement of a many-particle state, the entanglement between various subsystems must be taken into account. Let us assume an N-particle system is divided into two subsystems $A+B(N_A+N_B=N)$ and the corresponding reduced density matrices are denoted by $\rho(A)$ and $\rho(B)$. It can be shown that $\rho(A)$ and $\rho(B)$ have the same nonzero eigenvalues, but the numbers of zero eigenvalue can differ. Therefore, for a complete description of entanglement of an N-particle system, it is necessary to consider all the n-particle reduced density matrices, $n=1,2,\cdots,[N/2]$ ($[N/2]$ being the largest integer no larger than $N/2$). Thus, for a four-particle system, in addition to the one-particle reduced density matrix, the various two-particle reduced density matrices must be considered.

C. Reduced density matrix of GHZ states

Now, we will show that the impossibility of teleporting an arbitrary two-particle spin state by using the GHZ states of a four-particle system is crucially connected with the structure of the two-particle reduced density matrices.

(i) For all GHZ states, the one-particle reduced density matrix is of the same form

$$\rho(i)=\frac{1}{2}\mathbf{I}_2, \quad i=1,2,\ldots,N. \tag{13}$$

TABLE III. GHZ states of a four-particle system.

GHZ states	$\vert G_1\ G_2\ G_3\ G_4\rangle$	GHZ states	$\vert G_1\ G_2\ G_3\ G_4\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\uparrow\uparrow\rangle+\vert\downarrow\downarrow\downarrow\downarrow\rangle)$	$\vert+\ +\ -\ -\rangle$	$\frac{1}{\sqrt{2}}(\vert\downarrow\uparrow\uparrow\uparrow\rangle+\vert\uparrow\downarrow\downarrow\downarrow\rangle)$	$\vert+\ -\ -\ -\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\uparrow\uparrow\rangle-\vert\downarrow\downarrow\downarrow\downarrow\rangle)$	$\vert-\ -\ +\ +\rangle$	$\frac{1}{\sqrt{2}}(\vert\downarrow\uparrow\uparrow\uparrow\rangle-\vert\uparrow\downarrow\downarrow\downarrow\rangle)$	$\vert-\ +\ +\ +\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\uparrow\downarrow\rangle+\vert\downarrow\downarrow\downarrow\uparrow\rangle)$	$\vert+\ -\ +\ +\rangle$	$\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\uparrow\downarrow\rangle+\vert\downarrow\uparrow\downarrow\uparrow\rangle)$	$\vert+\ +\ +\ -\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\uparrow\downarrow\rangle-\vert\downarrow\downarrow\downarrow\uparrow\rangle)$	$\vert-\ +\ -\ -\rangle$	$\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\uparrow\downarrow\rangle-\vert\downarrow\uparrow\downarrow\uparrow\rangle)$	$\vert-\ -\ -\ +\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\downarrow\uparrow\rangle+\vert\downarrow\downarrow\uparrow\downarrow\rangle)$	$\vert+\ -\ +\ -\rangle$	$\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\downarrow\uparrow\rangle+\vert\downarrow\uparrow\uparrow\downarrow\rangle)$	$\vert+\ +\ +\ +\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\downarrow\uparrow\rangle-\vert\downarrow\downarrow\uparrow\downarrow\rangle)$	$\vert-\ +\ -\ +\rangle$	$\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\downarrow\uparrow\rangle-\vert\downarrow\uparrow\uparrow\downarrow\rangle)$	$\vert-\ -\ -\ -\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\uparrow\uparrow\rangle+\vert\downarrow\uparrow\downarrow\downarrow\rangle)$	$\vert+\ -\ -\ +\rangle$	$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\downarrow\downarrow\rangle+\vert\downarrow\downarrow\uparrow\uparrow\rangle)$	$\vert+\ +\ -\ +\rangle$
$\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\uparrow\uparrow\rangle-\vert\downarrow\uparrow\downarrow\downarrow\rangle)$	$\vert-\ +\ +\ -\rangle$	$\frac{1}{\sqrt{2}}(\vert\uparrow\uparrow\downarrow\downarrow\rangle-\vert\downarrow\downarrow\uparrow\uparrow\rangle)$	$\vert-\ -\ +\ -\rangle$

(ii) For $N>2$, the two-particle reduced density matrix is of the form

$$\rho(i,j)=\frac{1}{2}\begin{pmatrix}1&&&\\&1&&\\&&0&\\&&&0\end{pmatrix},\quad i\neq j,\quad i,j=1,2,\ldots,N. \tag{14}$$

(iii) Similarly, for $N>k$, the k-particle reduced density matrix is of the form

$$\rho(i_1,i_2,\ldots,i_k)=\frac{1}{2}\begin{pmatrix}1&&&&\\&1&&&\\&&0&&\\&&&\ddots&\\&&&&0\end{pmatrix},$$

$$i_1\neq i_2\neq\cdots\neq i_k,\quad i_1,i_2,\ldots,i_k=1,2,\ldots,N. \tag{15}$$

From the structure of these reduced density matrices, one can understand why the GHZ states are unable to teleport an arbitrary k-particle spin state, except for a special one with only two components in the 2^k-dimensional Hilbert space.

Therefore, to teleport an arbitrary k-particle spin state using an ancillary N-particle entangled state ($N\geqslant 2k$), the k-particle reduced density matrices should have the structure

$$\rho(i_1,i_2,\ldots,i_k)=\frac{1}{2^k}\mathbf{I}_{2^k},\quad i_1\neq i_2\neq\cdots\neq i_k,$$

$$i_1,i_2,\ldots,i_k=1,2,\ldots,N, \tag{16}$$

where $\mathbf{I}_{2^k}$ is the $2^k\times 2^k$ identity matrix. In the following we try to construct an alternative kind of many-particle entangled state meeting this requirement.

III. ALTERNATIVE APPROACH TO CONSTRUCTING ENTANGLED STATES

A. Signature

An effective way to construct entangled states is using a collective transformation of the system considered. We will show that a collective rotation of π around a given axis (say, x axis), $R_x(\pi)$, is a very useful candidate. The eigenvalue of $R_x(\pi)$, $r=e^{-i\pi\alpha}$, is referred to as a signature (α, the signature exponent) which is widely used in nuclear physics [21].

For an even-N-particle (spin-1/2) system, $R_x(\pi)^2=R_x(2\pi)=1$; thus, $r^2=1$ and $r=\pm 1$ ($\alpha=0$ or 1 Mod 2). For an odd-N-particle system, $R_x(\pi)^2=R_x(2\pi)=-1$; thus, $r^2=-1$, $r=\pm i$ ($\alpha=\mp 1/2$ Mod 2).

It is easy to show that the GHZ state (9) may be expressed as

$$\frac{1}{\sqrt{2}}[1+rR_x(\pi)]|m_1,m_2,\ldots,m_N\rangle,\quad N \text{ even},$$

$$\frac{1}{\sqrt{2}}[1-rR_x(\pi)]|m_1,m_2,\ldots,m_N\rangle,\quad N \text{ odd}, \tag{17}$$

which is the eigenstate of $R_x(\pi)$ with eigenvalue $r=e^{-i\pi\alpha}$; i.e., the GHZ states can be constructed by using $R_x(\pi)$ operating on the basis of angular momentum uncoupling representation. That is, the GHZ states for an N-particle (spin-1/2) system may be considered as the simultaneous eigenstates of the complete set of commuting local observables $(|s_{1z}|,|s_{2z}|,\ldots,|s_{Nz}|,R_x(\pi))$. In fact, under the operation

TABLE IV. Labeling of the Bell basis.

Bell basis	$\|\|M\|,\alpha\rangle$	Usual labeling
$\frac{1}{\sqrt{2}}(\|\uparrow\rangle_1\|\downarrow\rangle_2-\|\downarrow\rangle_1\|\uparrow\rangle_2)$	$\|00\rangle_{12}$	$\|\psi^-\rangle_{12}$
$\frac{1}{\sqrt{2}}(\|\uparrow\rangle_1\|\downarrow\rangle_2+\|\downarrow\rangle_1\|\uparrow\rangle_2)$	$\|01\rangle_{12}$	$\|\psi^+\rangle_{12}$
$\frac{1}{\sqrt{2}}(\|\uparrow\rangle_1\|\uparrow\rangle_2-\|\downarrow\rangle_1\|\downarrow\rangle_2)$	$\|10\rangle_{12}$	$\|\phi^-\rangle_{12}$
$\frac{1}{\sqrt{2}}(\|\uparrow\rangle_1\|\uparrow\rangle_2+\|\downarrow\rangle_1\|\downarrow\rangle_2)$	$\|11\rangle_{12}$	$\|\phi^+\rangle_{12}$

$[1+rR_x(\pi)]$, any quantum state of an even-N system becomes an eigenstate of $R_x(\pi)$ with eigenvalue $r=\pm 1$. Similarly, under the operation $[1-rR_x(\pi)]$, any quantum state of an odd-N system becomes an eigenstate of $R_x(\pi)$ with eigenvalue $r=\pm i$.

Therefore, one may construct an alternative kind of entangled states by using $R_x(\pi)$ operating on the basis of angular momentum coupling representation, which is a simultaneous eigenstate of the complete set of commuting local observables (Λ,S^2,S_z), labeled by $\chi_{\lambda SM}$. $S^2\chi_{\lambda SM}=S(S+1)\chi_{\lambda SM}$, $S_z\chi_{\lambda SM}=M\chi_{\lambda SM}$ $(\hbar=1)$, and λ (an eigenvalue of Λ) is the other quantum number necessary for characterizing the quantum state, which depends on the order of angular momentum coupling. The alternative kind of orthonormal entangled states thus constructed is

$$\frac{1}{\sqrt{2(1+\delta_{M0})}}[1+rR_x(\pi)]\chi_{\lambda SM}, \quad N \text{ even}, \ r=\pm 1$$

$$(\alpha=0 \text{ or } 1 \text{ Mod } 2), \quad (18)$$

$$\frac{1}{\sqrt{2}}[1-rR_x(\pi)]\chi_{\lambda SM}, \quad N \text{ odd}, \ r=\pm i$$

$$(\alpha=\mp 1/2 \text{ Mod } 2). \quad (19)$$

It can be shown that under a suitable phase convention [22]

$$R_x(\pi)\chi_{\lambda SM}=(-1)^S\chi_{\lambda S-M}. \quad (20)$$

Thus, for $M=0$, $\chi_{\lambda S0}$ itself is an eigenstate of $R_x(\pi)$ with $r=(-1)^S$.

It is noted that, though $[S_z,R_x(\pi)]\neq 0$, we have $[S_z^2,R_x(\pi)]=0$. Thus, the states (18) and (19) are the simultaneous eigenstates of the complete set of commuting local observables $(\Lambda,S^2,|S_z|,R_x(\pi))$, and may be labeled as $|\lambda,S,|M|,\alpha\rangle$, whose physical meaning is very clear. It is noted that the property of entangled states thus constructed is quite different for even N and odd N. In this paper we focus on the even-N system.

B. Two-particle system, Bell basis

For a two-particle (spin-1/2) system, the basis of angular momentum coupling representation is given in Eq. (6). As mentioned above, the two basis with $M=0$, χ_{00} and χ_{10}, are themselves eigenstates of $R_x(\pi)$ with $r=\pm 1$ $(\alpha=0,1)$, whereas the other two χ_{11} and χ_{1-1} are not. According to Eq. (18), we may construct the spin-entangled state of a two-particle system which is just the well-known Bell basis:

$$\chi_{00}=|\psi^-\rangle_{12},$$

$$\chi_{10}=|\psi^+\rangle_{12},$$

$$\frac{1}{\sqrt{2}}(\chi_{11}-\chi_{1-1})=|\phi^-\rangle_{12},$$

$$\frac{1}{\sqrt{2}}(\chi_{11}+\chi_{1-1})=|\phi^+\rangle_{12}. \quad (21)$$

In fact, the four states are the simultaneous eigenstates of $(|S_z|,R_x(\pi))$, labeled $||M|,\alpha\rangle$, as shown in Table IV.

It is interesting to note that the entangled states constructed by using $R_x(\pi)$ in terms of the basis of angular momentum uncoupling representation [see Eq. (9)] and the basis of angular momentum coupling representation [see Eq. (21)] are the same for a two-particle system, i.e., the Bell basis. However, the situation is quite different for N (>2) particle systems (see below)

C. Four-particle system

The basis of a four-particle system in angular momentum coupling representation may be chosen as the simultaneous eigenstate of $(S_{12}^2,S_{34}^2,S^2,S_z)$, labeled by $\chi_{S_{12},S_{34},S,M}$, $S_{12}=s_1+s_2$, $S_{34}=s_3+s_4$, $S=S_{12}+S_{34}$:

$$S_{12}^2\chi_{S_{12},S_{34},S,M}=S_{12}(S_{12}+1)\chi_{S_{12},S_{34},S,M}, \quad S_{12}=0,1,$$

$$S_{34}^2\chi_{S_{12},S_{34},S,M}=S_{34}(S_{34}+1)\chi_{S_{12},S_{34},S,M}, \quad S_{34}=0,1,$$

$$S^2\chi_{S_{12},S_{34},S,M}=S(S+1)\chi_{S_{12},S_{34},S,M}, \quad S=0^2,1^3,2,$$

$$S_z\chi_{S_{12},S_{34},S,M}=M\chi_{S_{12},S_{34},S,M}, \quad |M|\leqslant S. \quad (22)$$

Among the 16 bases, the two bases with $M=\pm 2$ $(S=2)$ are separable, whereas the six bases with $M=0$ $(S=0^2,1^3,2)$ are entangled states with $\rho(i)=\frac{1}{2}\mathbf{I}_2$ $(i=1,2,3,4)$. The remaining eight bases with $M=\pm 1$ $(S=1^3,2)$ are of the intermediate-case with various forms of one-particle reduced density matrices.

The 16 four-particle entangled states constructed by Eq. (18) for a four-particle system are

TABLE V. The entangled states of a four-particle (spin-1/2) system, $|S_{12},S_{34},S,|M|,\alpha\rangle$.

No.	$\|S_{12},S_{34},S,\|M\|,\alpha\rangle$	Expansion in terms of $\|\|M\|\alpha\rangle_{12}\|\|M'\|\alpha'\rangle_{34}$	Expansion in terms of $\|\|M\|\alpha\rangle_{13}\|\|M'\|\alpha'\rangle_{24}$
1	$\|0,0,0,0,0\rangle$	$\|00\rangle_{12}\|00\rangle_{34}$	$\frac{1}{2}(\|11\rangle_{13}\|11\rangle_{24}-\|10\rangle_{13}\|10\rangle_{24}-\|01\rangle_{13}\|01\rangle_{24}+\|00\rangle_{13}\|00\rangle_{24})$
2	$\|0,1,1,0,1\rangle$	$\|00\rangle_{12}\|01\rangle_{34}$	$\frac{1}{2}(-\|11\rangle_{13}\|10\rangle_{24}+\|10\rangle_{13}\|11\rangle_{24}-\|01\rangle_{13}\|00\rangle_{24}+\|00\rangle_{13}\|01\rangle_{24})$
3	$\|0,1,1,1,0\rangle$	$\|00\rangle_{12}\|10\rangle_{34}$	$\frac{1}{2}(\|11\rangle_{13}\|01\rangle_{24}-\|10\rangle_{13}\|00\rangle_{24}-\|01\rangle_{13}\|11\rangle_{24}+\|00\rangle_{13}\|10\rangle_{24})$
4	$\|0,1,1,1,1\rangle$	$\|00\rangle_{12}\|11\rangle_{34}$	$\frac{1}{2}(-\|11\rangle_{13}\|00\rangle_{24}+\|10\rangle_{13}\|01\rangle_{24}-\|01\rangle_{13}\|10\rangle_{24}+\|00\rangle_{13}\|11\rangle_{24})$
5	$\|1,0,1,0,1\rangle$	$\|01\rangle_{12}\|00\rangle_{34}$	$\frac{1}{2}(-\|11\rangle_{13}\|10\rangle_{24}+\|10\rangle_{13}\|11\rangle_{24}+\|01\rangle_{13}\|00\rangle_{24}-\|00\rangle_{13}\|01\rangle_{24})$
6	$\|1,0,1,1,0\rangle$	$\|10\rangle_{12}\|00\rangle_{34}$	$\frac{1}{2}(\|11\rangle_{13}\|01\rangle_{24}+\|10\rangle_{13}\|00\rangle_{24}-\|01\rangle_{13}\|11\rangle_{24}-\|00\rangle_{13}\|10\rangle_{24})$
7	$\|1,0,1,1,1\rangle$	$\|11\rangle_{12}\|00\rangle_{34}$	$\frac{1}{2}(\|11\rangle_{13}\|00\rangle_{24}+\|10\rangle_{13}\|01\rangle_{24}-\|01\rangle_{13}\|10\rangle_{24}-\|00\rangle_{13}\|11\rangle_{24})$
8	$\|1,1,0,0,0\rangle$	$\frac{1}{\sqrt{3}}(\|11\rangle_{12}\|11\rangle_{34}-\|10\rangle_{12}\|10\rangle_{34}-\|01\rangle_{12}\|01\rangle_{34})$	$\frac{1}{2\sqrt{3}}(-\|11\rangle_{13}\|11\rangle_{24}+\|10\rangle_{13}\|10\rangle_{24}+\|01\rangle_{13}\|01\rangle_{24}+3\|00\rangle_{13}\|00\rangle_{24})$
9	$\|1,1,1,0,1\rangle$	$\frac{1}{\sqrt{2}}(-\|11\rangle_{12}\|10\rangle_{34}+\|10\rangle_{12}\|11\rangle_{34})$	$\frac{1}{\sqrt{2}}(\|01\rangle_{13}\|00\rangle_{24}+\|00\rangle_{13}\|01\rangle_{24})$
10	$\|1,1,1,1,0\rangle$	$\frac{1}{\sqrt{2}}(\|11\rangle_{12}\|01\rangle_{34}-\|01\rangle_{12}\|11\rangle_{34})$	$\frac{1}{\sqrt{2}}(\|10\rangle_{13}\|00\rangle_{24}+\|00\rangle_{13}\|10\rangle_{24})$
11	$\|1,1,1,1,1\rangle$	$\frac{1}{\sqrt{2}}(\|10\rangle_{12}\|01\rangle_{34}-\|01\rangle_{12}\|10\rangle_{34})$	$\frac{1}{\sqrt{2}}(\|11\rangle_{13}\|00\rangle_{24}+\|00\rangle_{13}\|11\rangle_{24})$
12	$\|1,1,2,0,2\rangle$	$\frac{1}{\sqrt{6}}(\|11\rangle_{12}\|11\rangle_{34}-\|10\rangle_{12}\|10\rangle_{34}+2\|01\rangle_{12}\|01\rangle_{34})$	$\frac{1}{\sqrt{6}}(\|11\rangle_{13}\|11\rangle_{24}-\|10\rangle_{13}\|10\rangle_{24}+2\|01\rangle_{13}\|01\rangle_{24})$
13	$\|1,1,2,1,1\rangle$	$\frac{1}{\sqrt{2}}(\|10\rangle_{12}\|01\rangle_{34}+\|01\rangle_{12}\|10\rangle_{34})$	$\frac{1}{\sqrt{2}}(\|10\rangle_{13}\|01\rangle_{24}+\|01\rangle_{13}\|10\rangle_{24})$
14	$\|1,1,2,1,2\rangle$	$\frac{1}{\sqrt{2}}(\|11\rangle_{12}\|01\rangle_{34}+\|01\rangle_{12}\|11\rangle_{34})$	$\frac{1}{\sqrt{2}}(\|11\rangle_{13}\|01\rangle_{24}+\|01\rangle_{13}\|11\rangle_{24})$
15	$\|1,1,2,2,1\rangle$	$\frac{1}{\sqrt{2}}(\|11\rangle_{12}\|10\rangle_{34}+\|10\rangle_{12}\|11\rangle_{34})$	$\frac{1}{\sqrt{2}}(\|11\rangle_{13}\|10\rangle_{24}+\|10\rangle_{13}\|11\rangle_{24})$
16	$\|1,1,2,2,2\rangle$	$\frac{1}{\sqrt{2}}(\|11\rangle_{12}\|11\rangle_{34}+\|10\rangle_{12}\|10\rangle_{34})$	$\frac{1}{\sqrt{2}}(\|11\rangle_{13}\|11\rangle_{24}+\|10\rangle_{13}\|10\rangle_{24})$

$$|S_{12},S_{34},S,|M|,\alpha\rangle=\frac{1}{\sqrt{2(1+\delta_{M0})}}\times[1+rR_x(\pi)]\chi_{S_{12},S_{34},S,M}, \tag{23}$$

which are the simultaneous eigenstates of the complete set of commuting observables $(S_{12}^2,S_{34}^2,S^2,|S_z|,R_x(\pi))$. The 16 bases are shown in Table V, in which also are given the expansions in terms of the direct product of the Bell basis for both subsystems $(1,2)\otimes(3,4)$ and $(1,3)\otimes(2,4)$. The relation between the two expansions can be easily established by using the $9j$ symbol in angular momentum coupling theory. It is important to find that, while all one-particle reduced density matrices are the same for both the 16 GHZ states and the states of Eq. (23), $\rho(i)=\frac{1}{2}\mathbf{I}_2$ $(i=1,2,3,4)$, the structure of two-particle reduced density matrices is quite different. The structure of the two-particle reduced density matrices for the 16 entangled states $|S_{12},S_{34},S,|M|,\alpha\rangle$ are as follows.

(A) For the former seven states with $S_{12}=0$ and/or $S_{34}=0$,

$$\rho(1,2)=\rho(3,4)=\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \tag{24}$$

which implies that there exists no entanglement between the subsystems (1,2) and (3,4). However,

$$\rho(1,3)=\rho(2,4)=\rho(1,4)=\rho(2,3)=\frac{1}{4}\mathbf{I}_4; \tag{25}$$

i.e., the seven states may be considered as maximally entangled states between two subsystems (1,3) and (2,4), which turns out to be essential for teleporting an arbitrary two-particle state (see Sec. IV).

(B) The latter nine states with $S_{12}=S_{34}=1$.

For the state $|1,1,2,0,2\rangle$,

$$\rho(i,j)=\frac{1}{6}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 4 & \\ & & & 0 \end{pmatrix}, \quad i\neq j, \quad i,j=1,2,3,4. \tag{26}$$

For the state $|1,1,0,0,0\rangle$,

$$\rho(1,2)=\rho(3,4)=\frac{1}{3}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \tag{27}$$

$$\rho(1,3)=\rho(2,4)=\rho(1,4)=\rho(2,3)=\frac{1}{12}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 9 \end{pmatrix}. \tag{28}$$

For the remaining seven, like all the GHZ states,

$$\rho(i,j)=\frac{1}{2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad i\neq j, \quad i,j=1,2,3,4. \tag{29}$$

The construction of entangled states for an N-particle system (N even, $N>4$) by Eq. (18) is similar, and it is found that there do exist some entangled states whose k-particle ($k=N/2$) reduced density matrices have the form $(1/2^k)\mathbf{I}_{2^k}$, which is essential for the teleportation of an arbitrary k-particle spin state.

IV. EXTENSION OF THE SCHEME OF BENNETT *et al.* TO THE TELEPORTATION OF AN ARBITRARY UNKNOWN TWO-PARTICLE STATE

Following the same line of scheme of Bennett *et al.*, we investigate the teleportation of an arbitrary, unknown two-particle state using the ancillary entangled states shown in Eq. (23). Assume Alice is asked to send an arbitrary (unknown to her) two-particle state $|\Psi\rangle_{AB}$ to Bob at a distant location:

$$|\Psi\rangle_{AB}=a|00\rangle_{AB}+b|01\rangle_{AB}+c|10\rangle_{AB}+d|11\rangle_{AB}$$
$$=\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}_{AB}, \quad |a|^2+|b|^2+|c|^2+|d|^2=1. \tag{30}$$

To teleport an *arbitrary* two-particle spin state, it is necessary to prepare an ancillary four-particle entangled state, of which some two-particle reduced density matrices are of the form $\frac{1}{4}\mathbf{I}_4$. Assume the prepared ancillary four-particle entangled state is, for example, $|0,0,0,0,0\rangle_{1234}$. Particles (2,4) are sent to Alice and particles (1,3) to Bob. Then, Alice makes a four-particle joint measurement of the complete set of commuting observables $(S_{A2},S_{B4},S,|M|,\alpha)$ which will project the state of particles $(A,2,B,4)$ onto one of the 16 bases [see Eq. (23)] and in the meantime the original state $|\Psi\rangle_{AB}$ is destroyed. The reentangling process may be expressed as

TABLE VI. Unitary transformations for the teleportation of an arbitrary two-particle spin state. The seven elements in the first column and the first row denote the input and outcome states.

	$\|0,0,0,0,0\rangle_{A2B4}$	$\|0,1,1,0,1\rangle_{A2B4}$	$\|0,1,1,1,0\rangle_{A2B4}$	$\|0,1,1,1,1\rangle_{A2B4}$	$\|1,0,1,0,1\rangle_{A2B4}$	$\|1,0,1,1,0\rangle_{A2B4}$	$\|1,0,1,1,1\rangle_{A2B4}$
$\|0,0,0,0,0\rangle_{1234}$	$\mathbf{I}_2(1)\otimes\mathbf{I}_2(3)$	$\mathbf{I}_2(1)\otimes\sigma_z(3)$	$-\mathbf{I}_2(1)\otimes\sigma_x(3)$	$-\mathbf{I}_2(1)\otimes i\sigma_y(3)$	$\sigma_z(1)\otimes\mathbf{I}_2(3)$	$-\sigma_x(1)\otimes\mathbf{I}_2(3)$	$-i\sigma_y(1)\otimes\mathbf{I}_2(3)$
$\|0,1,1,0,1\rangle_{1234}$	$\mathbf{I}_2(1)\otimes\sigma_z(3)$	$\mathbf{I}_2(1)\otimes\mathbf{I}_2(3)$	$\mathbf{I}_2(1)\otimes i\sigma_y(3)$	$\mathbf{I}_2(1)\otimes\sigma_x(3)$	$\sigma_z(1)\otimes\sigma_z(3)$	$-\sigma_x(1)\otimes\sigma_z(3)$	$-i\sigma_y(1)\otimes\sigma_z(3)$
$\|0,1,1,1,0\rangle_{1234}$	$-\mathbf{I}_2(1)\otimes\sigma_x(3)$	$-\mathbf{I}_2(1)\otimes i\sigma_y(3)$	$\mathbf{I}_2(1)\otimes\mathbf{I}_2(3)$	$\mathbf{I}_2(1)\otimes\sigma_z(3)$	$-\sigma_z(1)\otimes\sigma_x(3)$	$\sigma_x(1)\otimes\sigma_x(3)$	$i\sigma_y(1)\otimes\sigma_x(3)$
$\|0,1,1,1,1\rangle_{1234}$	$\mathbf{I}_2(1)\otimes i\sigma_y(3)$	$\mathbf{I}_2(1)\otimes\sigma_x(3)$	$\mathbf{I}_2(1)\otimes\sigma_z(3)$	$\mathbf{I}_2(1)\otimes\mathbf{I}_2(3)$	$i\sigma_z(1)\otimes\sigma_y(3)$	$-i\sigma_x(1)\otimes\sigma_y(3)$	$\sigma_y(1)\otimes\sigma_y(3)$
$\|1,0,1,0,1\rangle_{1234}$	$\sigma_z(1)\otimes\mathbf{I}_2(3)$	$\sigma_z(1)\otimes\sigma_z(3)$	$-\sigma_z(1)\otimes\sigma_x(3)$	$-i\sigma_z(1)\otimes\sigma_y(3)$	$\mathbf{I}_2(1)\otimes\mathbf{I}_2(3)$	$i\sigma_y(1)\otimes\mathbf{I}_2(3)$	$\sigma_x(1)\otimes\mathbf{I}_2(3)$
$\|1,0,1,1,0\rangle_{1234}$	$-\sigma_x(1)\otimes\mathbf{I}_2(3)$	$-\sigma_x(1)\otimes\sigma_z(3)$	$\sigma_x(1)\otimes\sigma_x(3)$	$i\sigma_x(1)\otimes\sigma_y(3)$	$-i\sigma_y(1)\otimes\mathbf{I}_2(3)$	$\mathbf{I}_2(1)\otimes\mathbf{I}_2(3)$	$\sigma_z(1)\otimes\mathbf{I}_2(3)$
$\|1,0,1,1,1\rangle_{1234}$	$i\sigma_y(1)\otimes\mathbf{I}_2(3)$	$i\sigma_y(1)\otimes\sigma_z(3)$	$-i\sigma_y(1)\otimes\sigma_x(3)$	$\sigma_y(1)\otimes\sigma_y(3)$	$\sigma_x(1)\otimes\mathbf{I}_2(3)$	$\sigma_z(1)\otimes\mathbf{I}_2(3)$	$\mathbf{I}_2(1)\otimes\mathbf{I}_2(3)$

$$|0,0,0,0,0\rangle_{1234}|\Psi\rangle_{AB}=\frac{1}{4}|0,0,0,0,0\rangle_{A2B4}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}_{13}+\frac{1}{4}|0,1,1,0,1\rangle\begin{pmatrix}-b\\-a\\d\\c\end{pmatrix}+\frac{1}{4}|0,1,1,1,0\rangle\begin{pmatrix}-c\\-d\\-a\\-b\end{pmatrix}+\frac{1}{4}|0,1,1,1,1\rangle\begin{pmatrix}-d\\-c\\b\\a\end{pmatrix}$$
$$+\frac{1}{4}|1,0,1,0,1\rangle\begin{pmatrix}b\\a\\d\\c\end{pmatrix}+\frac{1}{4}|1,0,1,1,0\rangle\begin{pmatrix}c\\-d\\a\\-b\end{pmatrix}+\frac{1}{4}|1,0,1,1,1\rangle\begin{pmatrix}d\\-c\\b\\-a\end{pmatrix}+\frac{1}{4\sqrt{3}}|1,1,0,0,0\rangle\begin{pmatrix}3a\\-b\\-c\\-d\end{pmatrix}$$
$$+\frac{\sqrt{2}}{4}|1,1,1,0,1\rangle\begin{pmatrix}b\\-a\\0\\0\end{pmatrix}+\frac{\sqrt{2}}{4}|1,1,1,1,0\rangle\begin{pmatrix}c\\0\\-a\\0\end{pmatrix}+\frac{\sqrt{2}}{4}|1,1,1,1,1\rangle\begin{pmatrix}d\\0\\0\\a\end{pmatrix}+\frac{1}{2\sqrt{6}}|1,1,2,0,2\rangle\begin{pmatrix}0\\-2b\\c\\d\end{pmatrix}$$
$$+\frac{\sqrt{2}}{4}|1,1,2,1,1\rangle\begin{pmatrix}0\\-c\\-b\\0\end{pmatrix}+\frac{\sqrt{2}}{4}|1,1,2,1,2\rangle\begin{pmatrix}0\\-d\\0\\b\end{pmatrix}+\frac{\sqrt{2}}{4}|1,1,2,2,1\rangle\begin{pmatrix}0\\0\\-d\\c\end{pmatrix}+\frac{\sqrt{2}}{4}|1,1,2,2,2\rangle\begin{pmatrix}0\\0\\-c\\d\end{pmatrix}. \tag{31}$$

For brevity, the labeling of each particle is omitted on the right-hand side (except the first term). It is seen that for the former seven (Nos. 1–7 in Table V) outcomes for particles (A,2,B,4), Bob may extract a faithful replica of the two-particle state $|\Psi\rangle_{AB}$ by using the following unitary transformations:

$$U_1=\mathbf{I}_4=\mathbf{I}_2(1)\otimes\mathbf{I}_2(3),$$

$$U_2=\begin{pmatrix}-\boldsymbol{\sigma}_x & 0\\ 0 & \boldsymbol{\sigma}_x\end{pmatrix}=\mathbf{I}_2(1)\otimes\sigma_z(3),\quad U_3=\begin{pmatrix}0 & -\mathbf{I}_2\\ -\mathbf{I}_2 & 0\end{pmatrix}=-\mathbf{I}_2(1)\otimes\sigma_x(3),$$

$$U_4=\begin{pmatrix}0 & \boldsymbol{\sigma}_x\\ -\boldsymbol{\sigma}_x & 0\end{pmatrix}=-\mathbf{I}_2(1)\otimes i\sigma_y(3),\quad U_5=\begin{pmatrix}\boldsymbol{\sigma}_x & 0\\ 0 & \boldsymbol{\sigma}_x\end{pmatrix}=\sigma_z(1)\otimes\mathbf{I}_2(3),$$

$$U_6=\begin{pmatrix}0 & \boldsymbol{\sigma}_z\\ \boldsymbol{\sigma}_z & 0\end{pmatrix}=-\sigma_x(1)\otimes\mathbf{I}_2(3),\quad U_7=\begin{pmatrix}0 & -i\boldsymbol{\sigma}_y\\ -i\boldsymbol{\sigma}_y & 0\end{pmatrix}=-i\sigma_y(1)\otimes\mathbf{I}_2(3), \tag{32}$$

which can be realized by simple *linear* and *local* operations. However, for the latter nine (Nos. 8–16 in Table V) outcomes, one cannot extract the replica of $|\Psi\rangle_{AB}$. Thus, the teleportation efficiency is 7/16.

For the other six states (Nos. 2–7) with $S_{12}=0$ or $S_{34}=0$, the situation is similar, and the corresponding unitary transformations are summarized in Table VI. It should be mentioned that for the former seven outcomes, Alice's joint measurement is, in fact, equivalent to a combination of two Bell basis projections for particles (A,2) and (B,4) (see Table V).

However, if the prepared ancillary four-particle state is any one of the remaining nine states (Nos. 8–16 in Table V) with $S_{12}=S_{34}=1$, one cannot realize the teleportation of an arbitrary unknown two-particle state $|\Psi\rangle_{AB}$. For example,

$$
\begin{aligned}
|1,1,0,0,0\rangle_{1234}|\Psi\rangle_{AB} = {} & \frac{1}{4\sqrt{3}}|0,0,0,0,0\rangle_{A2B4}\begin{pmatrix}3a\\-b\\-c\\-d\end{pmatrix}_{13} + \frac{1}{4\sqrt{3}}|0,1,1,0,1\rangle\begin{pmatrix}-3b\\a\\-d\\-c\end{pmatrix} + \frac{1}{4\sqrt{3}}|0,1,1,1,0\rangle\begin{pmatrix}-3c\\d\\a\\b\end{pmatrix}\\
& + \frac{1}{4\sqrt{3}}|0,1,1,1,1\rangle\begin{pmatrix}-3d\\c\\-b\\-a\end{pmatrix} + \frac{1}{4\sqrt{3}}|1,0,1,0,1\rangle\begin{pmatrix}3b\\-a\\-d\\-c\end{pmatrix} + \frac{1}{4\sqrt{3}}|1,0,1,1,0\rangle\begin{pmatrix}3c\\d\\-a\\b\end{pmatrix}\\
& + \frac{1}{4\sqrt{3}}|1,0,1,1,1\rangle\begin{pmatrix}3d\\c\\-b\\a\end{pmatrix} + \frac{1}{12}|1,1,0,0,0\rangle\begin{pmatrix}9a\\b\\c\\d\end{pmatrix} + \frac{1}{2\sqrt{6}}|1,1,1,0,1\rangle\begin{pmatrix}3b\\a\\0\\0\end{pmatrix} + \frac{1}{2\sqrt{6}}|1,1,1,1,0\rangle\begin{pmatrix}3c\\0\\a\\0\end{pmatrix}\\
& + \frac{1}{2\sqrt{6}}|1,1,1,1,1\rangle\begin{pmatrix}3d\\0\\0\\-a\end{pmatrix} + \frac{1}{6\sqrt{2}}|1,1,2,0,2\rangle\begin{pmatrix}0\\2b\\-c\\-d\end{pmatrix} + \frac{1}{2\sqrt{6}}|1,1,2,1,1\rangle\begin{pmatrix}0\\c\\b\\0\end{pmatrix} + \frac{1}{2\sqrt{6}}|1,1,2,1,2\rangle\begin{pmatrix}0\\d\\0\\-b\end{pmatrix}\\
& + \frac{1}{2\sqrt{6}}|1,1,2,2,1\rangle\begin{pmatrix}0\\0\\d\\-c\end{pmatrix} + \frac{1}{2\sqrt{6}}|1,1,2,2,2\rangle\begin{pmatrix}0\\0\\c\\-d\end{pmatrix}.
\end{aligned}
\tag{33}
$$

An extension to deal with the teleportation of an arbitrary k-particle $(k>2)$ spin state is straightforward.

V. SUMMARY

It is shown that, to teleport an arbitrary k-particle spin state, one must prepare an ancillary N-particle $(N\geqslant 2k)$ entangled state, whose k-particle reduced density matrix has the structure $(1/2^k)\mathbf{I}_{2^k}$. From this one can understand why it is impossible to teleport an *arbitrary* unknown k-particle $(k\geqslant 2)$ state using an ancillary GHZ state. An alternative kind of entangled states is constructed by invoking the collective rotation of π around x axis, $R_x(\pi)$, operating on the basis of an angular momentum coupling representation. The structure of the two-particle reduced density matrix of the alternative kind of four-particle entangled states is investigated. The scheme of Bennett *et al.* is extended to deal with the teleportation of an arbitrary two-particle spin state by using the four-particle entangled states thus constructed. The extension to teleport an arbitrary k (>2) particle spin state is straightforward. Of course, this is only a theoretical scheme; how to experimentally prepare a kind of ancillary entangled states and how to make a joint measurement of this kind of basis need further investigation.

What we have demonstrated above can be used to address any two-level system having the same algebra as a spin [23]; i.e., the levels $|e\rangle$ and $|g\rangle$ may be seen as the spin "up" and "down" states of a fictitious spin 1/2 along an arbitrary Oz direction, $s_x=1/2(|e\rangle\langle g|+|g\rangle\langle e|)$, $s_y=1/(2i)(|e\rangle\langle g|-|g\rangle\times\langle e|)$, $s_z=1/2(|e\rangle\langle e|-|g\rangle\langle g|)$. Moreover, this scheme for constructing entangled states can be extended to systems composed of particles with arbitrary spin j.

ACKNOWLEDGMENT

This work is supported by the Natural Science Foundation of China.

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