Creation of large-photon-number path entanglement conditioned on photodetection

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Large-photon-number path entanglement is an important resource for enhanced precision measurements and quantum imaging. We present a general constructive protocol to create any large-photon-number path-entangled state based on the conditional detection of single photons. The influence of imperfect detectors is considered and an asymptotic scaling law is derived.

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It has been known for some time now that quantum metrology techniques allow for an improvement in precision imaging and measurements by exploiting entanglement. Examples of such improvements are in quantum lithography [1,2], quantum gyroscopy [3], entanglement-enhanced frequency metrology [4], and clock synchronization [5]. Experimental progress has been made in the demonstration of lithography [6], but according to our present understanding, full-scale implementations need sophisticated high-photon-number entangled states [2]. In particular, we need maximally entangled states of the form $|N::0\rangle \equiv (|N,0\rangle + |0,N\rangle)/\sqrt{2}$, where $|N\rangle$ are N-photon Fock states and $|0\rangle$ the vacuum. In general, we use the following notation:

$$|P::Q\rangle_{a,b}^{\varphi} \equiv \frac{1}{\sqrt{2}}(|P,Q\rangle_{a,b} + e^{i\varphi}|Q,P\rangle_{a,b}), \tag{1}$$

where a and b denote the two subsystems (modes), and φ is a relative phase. There have been several proposals to generate $|N::0\rangle$ states [7,8], but these typically need materials with large $\chi^{(3)}$ nonlinearities of the order of one. Such currently known nonlinearities are very small; typically they are of the order of $10^{-16}\ \mathrm{cm}^2\ \mathrm{s}^{-2}\ \mathrm{V}^{-2}$ [9].

In this paper, we show how to create entangled states of large photon number using only linear optics and photodetectors. In Sec. I, we will give a brief overview of the theory of parameter estimation; exploiting quantum entanglement to demonstrate the importance of the $|N::0\rangle$ states. Then, in Sec. II, we present a protocol to create $|N::0\rangle$ for any N. We show that it is generalizable to arbitrary N [10]. In Sec. III, we consider the case of imperfect detectors.

I. ENTANGLEMENT-ENHANCED PARAMETER ESTIMATION

In this section, we briefly describe the theory behind the various entanglement-enhanced imaging and measurement protocols. By using results from parameter estimation theory, we may easily derive the quantum-noise limits for uncorrelated measurements, where every sample is independent of every other, and for entanglement-enhanced experiments, where events are correlated. Quantum lithography, though technically not an estimation protocol, can also be described using this theory. The main purpose of this section is to demonstrate the importance of $|N::0\rangle$ states.

We start with the standard shot-noise limit. Consider an ensemble of N two-level systems in the state $|\varphi\rangle = (|0\rangle + e^{i\varphi}|1\rangle)/\sqrt{2}$, where $|0\rangle$ and $|1\rangle$ are arbitrary labels for the two levels. If $\hat{A} = |0\rangle\langle 1| + |1\rangle\langle 0|$, then the expectation value of $\hat{A}$ is given by

$$\langle\varphi|\hat{A}|\varphi\rangle = \cos\varphi. \tag{2}$$

When we repeat this experiment N times, we obtain

$$_N\langle\varphi|\ \ldots\ _1\langle\varphi|\left(\bigoplus_{k=1}^{N}\hat{A}^{(k)}\right)|\varphi\rangle_1\ \ldots\ |\varphi\rangle_N = N\cos\varphi. \tag{3}$$

Furthermore, it follows from the definition of $\hat{A}$ that $\hat{A}^2 = \mathbb{1}$ on the relevant subspace, and the variance of $\hat{A}$ given N samples is readily computed to be $(\Delta A)^2 = N(1-\cos^2\varphi) = N\sin^2\varphi$. According to estimation theory [11], we have

$$\Delta\varphi = \frac{\Delta A}{|d\langle\hat{A}\rangle/d\varphi|} = \frac{1}{\sqrt{N}}. \tag{4}$$

This is the standard variance in the parameter φ after N trials. In other words, the uncertainty in the phase is inversely proportional to the square root of the number of trials. This is the shot-noise limit.

With the help of quantum entanglement we can improve this parameter estimation by a factor of $\sqrt{N}$. We will now employ the path-entangled input state $|N::0\rangle^{N\varphi}$, where $|N\rangle$ is a product collective state of the N qubits. The relative phase $e^{iN\varphi}$ can be obtained by a unitary evolution of one of the modes of $|\varphi_N\rangle \equiv |N::0\rangle^{\varphi}$. When we measure an observable $\hat{A}_N = |0,N\rangle\langle N,0| + |N,0\rangle\langle 0,N|$ we have

$$\langle\varphi_N|\hat{A}_N|\varphi_N\rangle = \cos(N\varphi). \tag{5}$$

Again, $\hat{A}_N^2 = \mathbb{1}$ on the relevant subspace, and

$$(\Delta A_N)^2 = 1 - \cos^2 N\varphi = \sin^2(N\varphi). \tag{6}$$

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Using Eq. (4) again, we obtain the so-called Heisenberg limit to the minimal detectable phase:

$$\Delta\varphi_H = \frac{\Delta A_N}{|d\langle \hat{A}_N\rangle/d\varphi|} = \frac{1}{N}. \tag{7}$$

Here, we see that the precision in φ is increased by a factor $\sqrt{N}$ over the standard noise limit, when we exploit quantum entanglement. As shown in Bollinger *et al*. [4], Eq. (7) is the optimal accuracy permitted by the Heisenberg uncertainty principle. In quantum lithography, one exploits the $\cos(N\varphi)$ behavior, exhibited by Eq. (5), to print closely spaced lines on a suitable substrate [1]. Gyroscopy and entanglement-enhanced frequency measurements [3,4] exploit the $\sqrt{N}$ increased precision. The physical interpretations of $\hat{A}_N$ and the phase φ might differ in the different protocols.

II. LARGE-PHOTON-NUMBER PATH ENTANGLEMENT

In this section, we discuss the creation of large-photon-number path entanglement using only linear optics and photodetectors. We first identify the practical difficulties of conditioning on nondetection, and then, instead, we introduce a generalizable scheme to create photon-number path entanglement based on actual detection.

A. Nondetection

Previously, we have shown that it is possible to create up to $|4,0\rangle + |0,4\rangle$ states with linear optics and projective measurements [10]. Subsequently, it was shown by Fiurášek [12] and Zou *et al*. [13] that, in principle, one can create any two-mode, entangled, photon-number eigenstate with linear optics and good Fock-state sources.

The difficulty with the Fiurášek-Zou protocols, however, is that they are based on *nondetection*. There are two problems with this approach: first, it means that the protocols are very sensitive to detector losses; second, there is a whole family of reasons why a detector will not register a photon (not necessarily connected to detector efficiencies). For example, the lasers might have been switched off, or the beams might be misaligned. In these cases there will be no detector counts. In such situations the outgoing state is not the required state but the vacuum.

More formally, let $|\Psi\rangle$ be the total state before any detection and $|0\rangle_d\langle 0|$ the projection operator associated with a nondetection in mode d (in this notation, $|n\rangle_d\langle n|$ would be associated with the detection of n photons). Furthermore, let $|\psi\rangle$ be the intended outgoing state associated with no photons in mode d. A perfect measurement of zero photons in mode d corresponds to a projection $|0\rangle_d\langle 0|$ that yields a state $|\psi\rangle\langle\psi|$. However, in practice the measurement will not be a simple projection operator, but a positive operator-valued measure $\hat{E}_0$ [14] given by

$$\hat{E}_0 \equiv \sum_{n=0}^{\infty} c_{0,n}|n\rangle\langle n|, \tag{8}$$

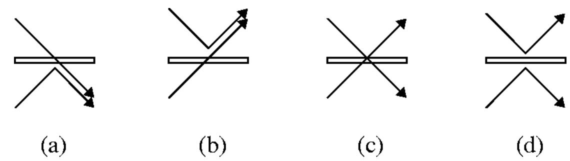

FIG. 1. Four possibilities exist when sending a $|1,1\rangle$ state through a beam splitter. The diagrams (c) and (d) lead to the same final state, but interfere destructively: (c) transmission-transmission $(i)(i) = -1$; (d) refection-reflection $(-1)(-1) = 1$.

where $c_{0,n} \geqslant 0$ and $\Sigma_k \hat{E}_k = \mathbb{1}$. This will lead to a different outgoing state

$$\hat{\rho}_{\text{out}} = \text{Tr}(\hat{E}_0|\Psi\rangle\langle\Psi|) = c_{0,0}|\psi\rangle\langle\psi| + (1-c_{0,0})\,\hat{\sigma}, \tag{9}$$

where $\hat{\sigma}$ is the density operator due to the noise. The fidelity of the outgoing state is then given by $F = \text{Tr}(\hat{\rho}_{\text{out}}|\psi\rangle\langle\psi|) = c_{0,0}$.

In general, there are many reasons why a detector might not record a photon. Many of these can be tested (for example, whether the equipment has been switched on or not), but never all of them. The crucial observation now is that all of the *untested* possibilities are going to contribute to $\hat{\sigma}$, and $c_{0,0}$ may become quite small (see also Ref. [15]).

This same argument can be applied to the detection of a single photon (i.e., projecting mode d onto $|1\rangle_d\langle 1|$). There will also be a noise contribution in the form of a density operator analogous to $\hat{\sigma}$. The difference is that there are many more reasons a detector will not record the presence of a photon than there are for detecting the photon. As a consequence, the fidelity of the outgoing state based on detection will be much larger than the fidelity of the state based on nondetection.

When we have a low-fidelity output state, we need to apply postselection. The output state, therefore, needs to be actually detected. As long as we do not have suitable quantum nondemolition measurement devices, the detection of the outgoing state generally precludes its further use in the intended application. We, therefore, need a production protocol that yields a high-fidelity output *state* (a notable exception is quantum lithography, where states of different photon numbers will not contribute to the imaging process [1]).

The question now is, what protocol allows us to create large-photon-number path entanglement conditioned upon photodetection? This is the subject of the rest of the paper.

B. Generating $|N::0\rangle$

Let us first briefly recall the case of a twofold coincidence at a beam splitter [10]. As shown in Fig. 1, when two indistinguishable photons enter a 50:50 beam splitter in both input modes, the phase relations will be such that the output modes will always be in the state $|2::0\rangle$. This is the operational mechanism of the Hong-Ou-Mandel (HOM) interferometer [16]. Labeling the input modes a and b, and the output modes c and d, the beam splitter can be characterized by the operator transformations

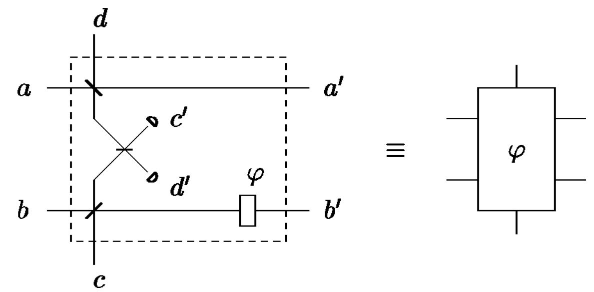

FIG. 2. The basic element of our large-photon-number path entanglement generator. The beam splitters split off two photons, which are subsequently detected in a twofold detector coincidence. There is an extra phase freedom φ in order to tune between several of such elements.

$$\hat{a}^\dagger \rightarrow (-\hat{c}^\dagger + i\hat{d}^\dagger)/\sqrt{2},$$

$$\hat{b}^\dagger \rightarrow (i\hat{c}^\dagger - \hat{d}^\dagger)/\sqrt{2}, \qquad (10)$$

and their Hermitian conjugates. It is now easily verified that $\hat{a}^\dagger \hat{b}^\dagger$ transforms into $(\hat{c}^\dagger)^2 + (\hat{d}^\dagger)^2$ up to an overall phase. There are no cross terms due to the reciprocal property of the symmetric beam splitter. However, this lack of cross terms will not generalize beyond $N=2$, since there are only a limited number of free parameters available to suppress high-N cross terms [17]. The critical property of the HOM interferometer, which we will use in our protocol, is that two photons from different input modes of the beam splitter cannot trigger a twofold detection coincidence at the output modes.

In this section, we first proceed with the general protocol for the creation of $|N::0\rangle_{ab}$, where N is even. The basic element of our protocol is depicted in Fig. 2. Two beam splitters split off photons from the main beams a and b. The reflected modes are then recombined in a 50:50 beam splitter, and the process is postselected on a twofold detector coincidence in the outgoing modes c' and d'. It is assumed initially that our detectors distinguish between one and more photons perfectly, but we consider the case of imperfect detectors below.

Since a twofold detector coincidence cannot be due to a single photon in both input beams, this procedure thus takes two photons from either mode a or b: $|N,N\rangle \rightarrow |N-2::N\rangle$. To complete the element we apply a phase shift to mode b, the value of which will be determined later. The protocol for making $|N::0\rangle$, with even N now requires us to create the input state $|N,N\rangle$ and stack $N/2$ of our basic elements. The output state, conditioned on an overall N-fold detector coincidence with suitable phase shifts, is then $|N::0\rangle$ (see Fig. 3).

To prove this statement, consider the two-photon detection of the basic element as $(\hat{a}^2 + e^{i\varphi}\hat{b}^2)|N,N\rangle$. We have to repeat this procedure $N/2$ times, yielding

$$\prod_{k=1}^{N/2} (\hat{a}^2 + e^{i\varphi_k}\hat{b}^2)|N,N\rangle. \qquad (11)$$

In order to obtain the N-photon path-entangled state, the polynomial in Eq. (11) should be $\hat{a}^N + \hat{b}^N$. This means, from the fundamental theorem of algebra [18], that the phase factors $\exp(i\varphi_k)$ are the $N/2$ roots of unity, that is,

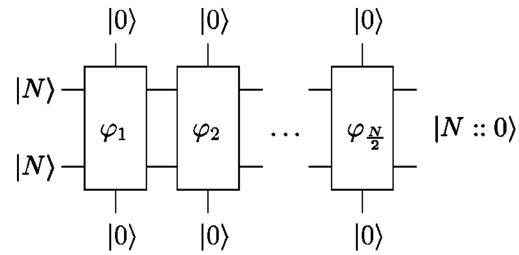

FIG. 3. Stacking the basic elements of Fig. 1 and setting the phase factors $\exp(i\varphi_k)$ to be the roots of unity, we create the state $|N,0\rangle + |0,N\rangle$ out of $|N,N\rangle$ conditioned on an N-fold detection coincidence.

$$\varphi_k = \frac{4\pi k}{N}. \qquad (12)$$

When N is very large, the probability that only two photons are reflected from the main beams becomes very small. This can be compensated by the use of weighted beam splitters: To split off two photons from an N-photon state most optimally, one should use a beam splitter with a transmission coefficient $(N-1)/N$. The probability of a successful state preparation event then scales asymptotically as $\sqrt{8\pi N}(1/4e)^N$ (see the Appendix for proof).

So far, we have only considered the detection of an even number of photons. However, for the general case, we also want to generate odd $|N::0\rangle$ states. The even case was straightforward, since it involved only two-photon detections that are naturally implemented as the detection of the two outgoing modes of a beam splitter. The odd case, however, requires single-photon detectors. If we allow for nondetection, this is also a straightforward task—we just condition on a single detection count in the two outgoing modes of the beam splitter. But nondetection is exactly what we want to avoid. In the following section, we investigate single-photon conditioning in the presence of polarization.

C. Odd N and polarization degrees of freedom

The protocol presented in the preceding section generates only even-N path-entangled states $|N::0\rangle$. Furthermore, the photons are assumed to have the same polarization. In this section, we extend this scheme to odd N by using the extra degree of freedom of polarization.

The basic element for subtracting a photon from the main modes is shown in Fig. 4. Just as with the even case above, two beam splitters split off a portion of the main beams a and b. However, now they are recombined in a polarization beam splitter (PBS). The setup is chosen such that a photon originating from mode a will be transmitted in the PBS. Since the polarization of modes a and b are the same, a photon from mode b incident on the PBS would also be transmitted. However, we really want this photon to be reflected, so that it too ends up in the detector. In this way we erase the which-path information.

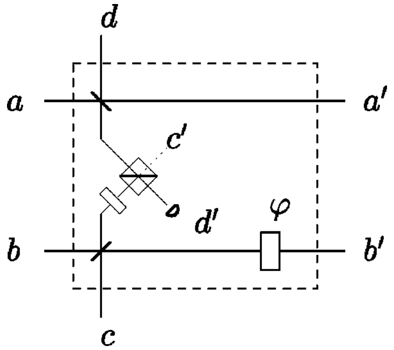

FIG. 4. This element is used to subtract a single photon from the two-mode entangled state. One photon, originating from either mode a or b, will either be transmitted in the polarization beam splitter to the detector (mode a), or it will undergo a $\pi/2$ polarization rotation and will be reflected to the detector (mode b). The second outgoing mode is empty. Rather than conditioning on a nondetection of the empty mode, we couple it to the environment.

One way to achieve this goal is to apply a $\pi/2$ polarization rotation to this mode. This will force the photon towards the detector. The secondary outgoing mode will now be empty, and as a consequence we will ignore it completely. That is, we do not need to condition this scheme on nondetection. When we stack these elements N times and use the input state $|N,N\rangle$, we create a general $|N::0\rangle$ state. The phase factors $\exp(i\varphi_k)$ are the $N/2$ roots of unity

$$\varphi_k = \frac{2\pi k}{N}. \tag{13}$$

Note that, as in Eq. (12), the phases span the $N/2$ roots of unity.

Let us elaborate a bit more on this distinction between nondetection and losing modes to the environment. In any experiment we trace out the (unwanted) coupling to the environment for the simple reason that we do not have control over all the interactions between our experimental setup and the rest of the universe. When this coupling is made small (i.e., the setup is isolated), this is a very good approximation. In the case of the secondary output mode c' of the PBS, the coupling to the environment consists of the loss of any photons in that mode. However, ideally there should not be any photons in that mode—this, therefore, constitutes a weak coupling.

The case of nondetection presupposes a sizeable portion of scattered photons in the outgoing beam, and aims to condition on the absence of these. We cannot trace over this mode, because the coupling to the environment is not weak—there are actually photons in that mode. This means that instead of tracing over the secondary mode, one needs to project it onto the vacuum $|0\rangle\langle 0|$, which is the source of the nondetection difficulties.

D. Nested protocols

The protocols presented so far are linear in the sense that we constructed a basic element as a two-mode gate that was repeated a number of times. This means that the number of detected photons increases linearly with the number of elements, and the efficiency (which is a product of the success rate of the different components), therefore, scales exponentially poorly.

For practical purposes it is important to find a scheme that scales logarithmically in the number of detectors, so that we only have polynomial efficiency deterioration. One protocol that looked promising exploited the unused input ports. We found that feeding both modes a,b *and* modes c,d in the basic element from Fig. 2 with states $|N::0\rangle$ yields the state

$$|\psi_{\rm out}\rangle = |2N-2::0\rangle, \tag{14}$$

based on a twofold detection coincidence. However, due to the fact that *two* $|N::0\rangle$ input states are required, the overall scaling was still exponentially poor.

These scaling considerations are important for practical implementations of entanglement-enhanced precision measurements, because an increase in the required resources (photons) might outweigh the benefit of gaining a $\sqrt{N}$ precision improvement. Since the scaling of the resources depends critically on the details of the protocol employed, it is not clear from these general considerations what the overall behavior of a given network will be.

III. IMPERFECT DETECTORS

There are several sources of errors for a detector. It might fail to signal that a photon was present, a case in which we speak of a deteriorated efficiency. Alternatively, it might signal the detection of a photon, even though no photon was actually present. This is called a dark count. Since we only consider schemes that operate in short time windows, these dark counts can be neglected. Finally, the detector might not be able to distinguish between one or more photons. Such a detector does not have single-photon resolution [19].

We can see immediately that imperfect detection efficiency is going to affect the scaling law. In particular, the asymptotic scaling will behave as

$$\pi_N = \sqrt{8\pi N}\left(\frac{\eta}{4e}\right)^N, \tag{15}$$

where π_N is the (asymptotic) probability of creating the state based on N detected photons and η is the detector efficiency. That is, the protocol scales exponentially poorly with the detector efficiency, as expected. Here we have taken identical detectors throughout the scheme.

When we use detectors with a single-photon resolution but limited efficiency, two photons can easily be mistaken for a single photon. That is, one of them might not be detected. When the occurrence of a two-photon state is very unlikely, this is not so much of a problem, but when it is likely, the output state will be significantly degraded. Unfortunately, in our protocol the beam splitter strips off two photons on average, which means that it is quite likely that more than two photons end up in the detector. This way, our scheme has become a protocol conditioned on the nondetection of two-photon states, which is exactly what we wanted to avoid.

However, there is a way to mend this drawback: when we increase the transmittivity of the beam splitter, the probability of having more than two photons in the detectors will decrease. Therefore, at the cost of a lower production rate (i.e., low efficiency), we can maintain high-quality $|N::0\rangle$ states (high fidelity). This adjustment is not possible in nondetection schemes.

IV. CONCLUSIONS

In this paper we have demonstrated a general, detection-based protocol to create $|N::0\rangle$ states for use in entanglement-enhanced parameter estimation. Existing protocols are less practical because they either require $\chi^{(3)}$ nonlinearities near unity [7,8], or they condition on nondetection [12,13]. Currently, $\chi^{(3)}$ nonlinearities are very small [9], and we argued that nondetection schemes are problematic in their experimental implementation.

We have shown that one can indeed create arbitrary $|N,0\rangle+|0,N\rangle$ states using only linear optics and conditioned on single-photon detections. For the case of odd N we needed to invoke the extra freedom of polarization. The protocol presented here is the generalization of our previous work [10], which was succesful in creating path-entangled states up to $|4,0\rangle+|0,4\rangle$.

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APPENDIX: ASYMPTOTIC SCALING LAW

We will now prove the asymptotic scaling law for even $|N,0\rangle+|0,N\rangle$ states. Let the transmission and reflection coefficients of the beam splitter be given by t and r, with $t+r=1$. The probability of reflecting k out of N photons is then given by $p_k(N)=\binom{N}{k}t^{N-k}r^k$, from which it immediately follows that $\Sigma_k p_k(N)=1$. The event where two photons are reflected from one beam splitter and none from the other then occurs with probability $2p_0(N)p_2(N)$, where the factor 2 takes into account the fact that we do not know from which mode the two photons originate. Furthermore, we are post-selecting on twofold coincidences, which means that we have an extra factor of $\frac{1}{2}$ that incorporates the reduced probability that the two photons branch off at the beam splitter. By maximizing the expression $t^{2N-2}(1-t)^2$ we found that the optimal transmission coefficient is given by $t=(N-1)/N$. The total probability of finding a twofold detector coincidence is then given by

$$p_{\text{twofold}}=2\left(\frac{1}{4}\right)^N\frac{N!}{N^N}. \tag{A1}$$

Using Sterling's formula $N!\approx\sqrt{2\pi N}e^{-N}N^N$, we find that for large N the protocol scales as $\sqrt{8\pi N}(1/4e)^N$.

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