

# Minimum-error discrimination between subsets of linearly dependent quantum states

Ulrike Herzog<sup>1</sup> and János A. Bergou<sup>2,3</sup>

<sup>1</sup>*Institut für Physik, Humboldt-Universität zu Berlin, Invalidenstrasse 110, D-10115 Berlin, Germany*

<sup>2</sup>*Department of Physics, Hunter College, City University of New York, 695 Park Avenue, New York, New York 10021*

<sup>3</sup>*Institute of Physics, Janus Pannonius University, H-7624 Pécs, Ifjúság útja 6, Hungary*

(Received 28 December 2001; published 9 May 2002)

A measurement strategy is developed for a different kind of hypothesis testing. It assigns, with minimum probability of error, the state of a quantum system to one or the other of two complementary subsets of a set of  $N$  given nonorthogonal quantum states occurring with given *a priori* probabilities. A general analytical solution is obtained for  $N$  states that are restricted to a two-dimensional subspace of the Hilbert space of the system. The result for the special case of three arbitrary but linearly dependent states is applied to a variety of sets of three states that are symmetric and equally probable. It is found that, in this case, the minimum-error probability for distinguishing one of the states from the other two is only about half as large as the minimum-error probability for distinguishing all three states individually.

DOI: 10.1103/PhysRevA.65.050305

PACS number(s): 03.67.-a, 03.65.Ta, 42.50.-p

## I. INTRODUCTION AND BASIC EQUATIONS

Due to their nonvanishing mutual overlaps, nonorthogonal quantum states cannot be perfectly distinguished. However, stimulated by the rapid developments in quantum information theory [1], the question as to how to discriminate between nonorthogonal states in an optimum way has gained renewed interest [2]. In particular, in quantum communication protocols several secure schemes have been suggested based on communicating via nonorthogonal quantum states. As a result, optimum discrimination between them became an inherent part of these schemes. For studying state discrimination, it is assumed that a quantum system is prepared in one of the  $N$  pure states,  $|\psi_k\rangle$ , that belongs to a given set of nonorthogonal states,  $\{|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_N\rangle\}$ , and that the *a priori* probabilities  $\eta_k$  for the preparation of either one of the states  $|\psi_k\rangle$  are also known. In order to devise an optimum state-discriminating measurement, strategies have been developed with respect to various criteria [2,3]. The earliest and simplest of these criteria is the requirement that the probability of getting a wrong result be as small as possible, with inconclusive results being forbidden and all states being individually distinguished. A minimum-error strategy of this kind has been developed for the case when only two states are given [4] and for *specific*  $N$  state problems [5–8], including  $N$  symmetric [6] and multiply symmetric [8] states. Recently the optimum strategy has also been found for three states exhibiting a mirror symmetry [9] but still no exact solution has been known for  $N > 2$  *arbitrary* states. Using the polarization states of a single photon, minimum-error discrimination has been experimentally realized for up to four symmetric nonorthogonal states [10].

In this paper we are concerned with a minimum-error strategy that involves  $N > 2$  *arbitrary* linearly dependent quantum states, by considering the following problem: We want to devise a measurement that allows us to decide, with the smallest possible error and without inconclusive answers, whether the actual state of the system belongs to the subset of states  $\{|\psi_1\rangle, \dots, |\psi_M\rangle\}$ , or to the complementary subset of the remaining states  $\{|\psi_{M+1}\rangle, \dots, |\psi_N\rangle\}$  with  $M < N$ . For

three given states the task reduces to distinguishing the state  $|\psi_1\rangle$  from the set of states  $\{|\psi_2\rangle, |\psi_3\rangle\}$  and can be referred to as quantum-state filtering with respect to the state  $|\psi_1\rangle$ . This task has recently been investigated for the particular optimization strategy that yields unambiguous discrimination at the expense of allowing inconclusive results to occur, the probability of which is minimized [11].

To treat our general minimum-error problem, we follow the standard lines and introduce two positive Hermitian quantum detection operators,  $\Pi_0$  and  $\Pi_1$  [2,4]. We define the operator  $\Pi_1$  by the property that  $\langle\psi_k|\Pi_1|\psi_k\rangle$  accounts for the probability to infer, from performing the measurement, the system to be in one of the states  $\{|\psi_1\rangle, \dots, |\psi_M\rangle\}$ , if it has been prepared in the state  $|\psi_k\rangle$ . Obviously, this inference is incorrect if  $k > M$ . Similarly, given again the preparation of the state  $|\psi_k\rangle$ , the quantity  $\langle\psi_k|\Pi_0|\psi_k\rangle$  denotes the probability for inferring the state of the system to belong to the subset of states  $\{|\psi_{M+1}\rangle, \dots, |\psi_N\rangle\}$ , which is an erroneous result if  $k \leq M$ . Clearly, the relation

$$\Pi_0 + \Pi_1 = \hat{1} \quad (1)$$

has to be obeyed, where  $\hat{1}$  is the unit operator. From the definition of the detection operators it follows that the probability to get a correct result reads

$$P^{M(N)} = \sum_{k=1}^M \eta_k \langle\psi_k|\Pi_1|\psi_k\rangle + \sum_{k=M+1}^N \eta_k \langle\psi_k|\Pi_0|\psi_k\rangle. \quad (2)$$

In order to devise the desired minimum-error measurement scheme, we have to determine the particular detection operators  $\Pi_0$  and  $\Pi_1$  that maximize the right-hand side (rhs) of Eq. (2) under the constraint (1). In general, the error-minimizing optimization problem is a highly nontrivial task.

## II. SOLUTION IN TWO DIMENSIONS

To enable simple analytical solutions, we restrict ourselves to the case when the  $N$  linearly dependent states span only a two-dimensional Hilbert space. We note that for

three linearly dependent states this is always the case. First, we show that in the two-dimensional case it is possible to represent the two detection operators  $\Pi_1$  and  $\Pi_0$  by two projection operators onto orthonormal states  $|\mu\rangle$  and  $|\nu\rangle$ , respectively. To see this, we start from the expression  $\Pi_1 = \lambda_1|v_1\rangle\langle v_1| + \lambda_2|v_2\rangle\langle v_2|$ , with  $|v_1\rangle$  and  $|v_2\rangle$  being the orthonormal eigenstates that belong to some non-negative eigenvalues  $\lambda_1$  and  $\lambda_2$ . Expanding a particular state  $|\psi\rangle$  as  $|\psi\rangle = \cos\beta|v_1\rangle + \sin\beta|v_2\rangle$ , where a possible relative phase factor has been included into the definition of  $|v_2\rangle$ , we arrive at

$$\langle\psi|\Pi_1|\psi\rangle = |\langle\mu|\psi\rangle|^2, \quad (3)$$

provided that we define  $|\mu\rangle = \sqrt{\lambda_1}|v_1\rangle \pm i\sqrt{\lambda_2}|v_2\rangle$ . With the help of the relation  $\hat{1} = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$ , we obtain in the same way the representation

$$\langle\psi|\Pi_0|\psi\rangle = \langle\psi|\hat{1} - \Pi_1|\psi\rangle = |\langle\nu|\psi\rangle|^2, \quad (4)$$

provided that  $|\nu\rangle = \sqrt{1-\lambda_1}|v_1\rangle \pm i\sqrt{1-\lambda_2}|v_2\rangle$ . Now we require that  $|\langle\mu|\psi_k\rangle|^2 + |\langle\nu|\psi_k\rangle|^2 = 1$  for an arbitrary state,  $|\psi_k\rangle = \cos\beta_k|v_1\rangle + e^{i\gamma_k}\sin\beta_k|v_2\rangle$ , which implies that  $|\mu\rangle\langle\mu| + |\nu\rangle\langle\nu| = \hat{1}$  has to be fulfilled. This only holds true when in the representations of  $|\mu\rangle$  and  $|\nu\rangle$  opposite signs are chosen and when in addition  $\lambda_2 = 1 - \lambda_1$ , leading to the orthonormality conditions  $\langle\mu|\mu\rangle = \langle\nu|\nu\rangle = 1$  and  $\langle\mu|\nu\rangle = 0$ . Therefore, in a two-dimensional Hilbert space the optimization problem posed by Eqs. (1) and (2) can be reduced to the problem of finding the specific normalized state  $|\mu\rangle$  that maximizes the expression

$$P^{M(N)} = \sum_{k=1}^M \eta_k |\langle\mu|\psi_k\rangle|^2 + \sum_{k=M+1}^N \eta_k (1 - |\langle\mu|\psi_k\rangle|^2), \quad (5)$$

which follows when  $\Pi_1 = |\mu\rangle\langle\mu|$  and  $\Pi_0 = \hat{1} - |\mu\rangle\langle\mu|$  are substituted into Eq. (2). Comparing this to the spectral representation of the detection operators, introduced before Eq. (3), we are led to identify  $|\mu\rangle$  with  $|v_1\rangle$  and  $|\nu\rangle$  with  $|v_2\rangle$  since the representation is unique. Then  $\lambda_1 = 1$  and  $\lambda_2 = 0$  follows. Once the optimum detection state is known, the maximum achievable probability of correctly assigning a quantum state to one of the two subsets, as well as the two detection operators necessary to perform the optimized measurement, are uniquely determined.

To solve the optimization problem, it is convenient to write the overlaps between the given states as

$$\langle\psi_k|\psi_l\rangle = A_{kl} = |A_{kl}|e^{i\alpha_{kl}}, \quad (6)$$

and to introduce the auxiliary state vector

$$|v\rangle = \frac{1}{\sqrt{1-|A_{12}|^2}}(|\psi_2\rangle - A_{12}|\psi_1\rangle). \quad (7)$$

For Eq. (5) to be valid, we have to assume that all  $N$  given states lie in a two-dimensional subspace, spanned by the states  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , or  $|\psi_1\rangle$  and  $|v\rangle$ , respectively. Since

$\langle v|v\rangle = 1$  and  $\langle\psi_1|v\rangle = 0$ , the states  $|\psi_1\rangle$  and  $|v\rangle$  provide a suitable orthonormal basis for representing any state,  $|\psi_k\rangle$ , as

$$|\psi_k\rangle = A_{1k}|\psi_1\rangle + e^{i\gamma_k}\sqrt{1-|A_{1k}|^2}|v\rangle, \quad (8)$$

with

$$e^{i\gamma_k} = \frac{A_{2k} - A_{21}A_{1k}}{\sqrt{1-|A_{12}|^2}\sqrt{1-|A_{1k}|^2}}. \quad (9)$$

The last equation can be verified by calculating the overlap  $\langle\psi_2|\psi_k\rangle$ , taking into account that  $\gamma_2 = 0$  because of the specific definition of the state  $|v\rangle$ . Similarly, we represent the detection state  $|\mu\rangle$  as

$$|\mu\rangle = \cos\varphi|\psi_1\rangle + e^{i\chi}\sin\varphi|v\rangle, \quad (10)$$

and obtain

$$\langle\mu|\psi_k\rangle = A_{1k}\cos\varphi + e^{i(\gamma_k-\chi)}\sqrt{1-|A_{1k}|^2}\sin\varphi. \quad (11)$$

Equation (10) accounts for all possible states in the two-dimensional Hilbert space of interest provided that both  $\varphi$  and  $\chi$  are variables in the interval  $[0, \pi)$ . The error-minimization problem is then reduced to finding those values of  $\varphi$  and  $\chi$  in Eq. (11) that maximize the probability  $P^{M(N)}(\varphi, \chi)$  in Eq. (5).

The solution to this optimization problem is straightforward. We begin by inserting Eq. (11) into Eq. (5) and, by making use of the fact that the *a priori* probabilities of the states fulfill the relation  $\sum_{k=1}^N \eta_k = 1$ , we readily arrive at

$$P^{M(N)} = \frac{1}{2} + R\cos(2\varphi) + |Q|\sin(2\varphi)\cos(\chi - \chi_Q), \quad (12)$$

where  $R$  and  $Q$  are defined as

$$R = \sum_{k=1}^M \eta_k \left( |A_{1k}|^2 - \frac{1}{2} \right) - \sum_{k=M+1}^N \eta_k \left( |A_{1k}|^2 - \frac{1}{2} \right), \quad (13)$$

and

$$Q = |Q|e^{i\chi_Q} = \sum_{k=1}^M \eta_k \frac{A_{2k}A_{k1} - A_{21}|A_{1k}|^2}{\sqrt{1-|A_{12}|^2}} - \sum_{k=M+1}^N \eta_k \frac{A_{2k}A_{k1} - A_{21}|A_{1k}|^2}{\sqrt{1-|A_{12}|^2}}. \quad (14)$$

The conditions for an extremum,  $\partial P^{M(N)}/\partial\varphi = 0$  and  $\partial P^{M(N)}/\partial\chi = 0$ , hold for  $\varphi = \varphi_e$  and  $\chi = \chi_e$ , with  $\sin(2\varphi_e) = |Q|/\sqrt{R^2 + |Q|^2}$ ,  $\cos(2\varphi_e) = R/\sqrt{R^2 + |Q|^2}$ , and  $\chi_e = \chi_Q$ , respectively. Note that  $\cos(2\varphi_e)$  and  $R$  have the same sign while  $\sin(2\varphi_e)$  is always positive. This choice of  $\chi_e$  and  $\varphi_e$  corresponds to the maximum of  $P^{M(N)}$  and, from Eq. (12), we obtain

$$P^{M(N)}(\varphi_e, \chi_e) = P_{\max}^{M(N)} = \frac{1}{2} + \sqrt{R^2 + |Q|^2}. \quad (15)$$

The corresponding detection state, onto which a projection has to be performed in a measurement scheme achieving the maximum probability, is determined by  $|\mu_e\rangle = \cos \varphi_e |\psi_1\rangle + e^{i\chi_e} \sin \varphi_e |v\rangle$ .

As applications of this general expression, we discuss two special cases. First, the solution can be cast to a considerably simpler form when the states are *real*. Real states have been considered before [12] in a different context. In this case the parameters of the optimum detection state,  $|\mu\rangle$ , can be calculated very easily. Both  $R$  and  $Q$  are real, yielding  $\chi_e = 0$  if  $Q \geq 0$  or  $\pi$  if  $Q < 0$ . The maximum probability of determining correctly to which of the two complementary subsets a state belongs is given in this case by Eq. (15) with

$$R = \sum_{k=1}^M \eta_k \left( A_{1k}^2 - \frac{1}{2} \right) - \sum_{k=M+1}^N \eta_k \left( A_{1k}^2 - \frac{1}{2} \right), \quad (16)$$

and

$$Q = \sum_{k=1}^M \eta_k A_{1k} \sqrt{1 - A_{1k}^2} - \sum_{k=M+1}^N \eta_k A_{1k} \sqrt{1 - A_{1k}^2}, \quad (17)$$

where, in the last step, we made use of the relation resulting from Eq. (9) with  $\gamma_k = 0$  and all the overlaps are assumed real.

As our second example, we consider the case of three *arbitrary* but linearly dependent states,  $N=3$ . Choosing  $M=1$  and taking  $\eta_1 + \eta_2 + \eta_3 = 1$  into account in Eq. (13), we readily obtain

$$R = \frac{1}{2} - \eta_2 |A_{12}|^2 - \eta_3 |A_{13}|^2. \quad (18)$$

The evaluation of  $|Q|$  is greatly facilitated if we notice that the first sum on the rhs of Eq. (14) has only one term and this term vanishes. A straightforward evaluation of the remaining two terms from the second sum yields

$$\begin{aligned} |Q|^2 &= \eta_2^2 |A_{12}|^2 (1 - |A_{12}|^2) + \eta_3^2 |A_{13}|^2 (1 - |A_{13}|^2) \\ &\quad + 2 \eta_2 \eta_3 (\text{Re } A_{12} A_{23} A_{31} - |A_{12}|^2 |A_{13}|^2). \end{aligned} \quad (19)$$

For this case the parameters of the optimum detection state  $|\mu\rangle$  can be seen to be  $\chi_e = \chi_Q$  and  $\tan(2\varphi_e) = |Q|/R$ , with  $|Q|$  and  $R$  substituted from the above equations. We do not give here a more explicit expression for  $\chi_e$  because it is slightly involved and enters only the detection states but not the final result for the maximum probability. Inserting the above values of  $|Q|$  and  $R$  into the general expression for the optimum probability finally gives

$$\begin{aligned} P_{\max}^{(1,2,3)} &= \frac{1}{2} + \frac{1}{2} \left[ 1 - 4 \sum_{k=2}^3 \eta_k (1 - \eta_k) |\langle \psi_1 | \psi_k \rangle|^2 \right. \\ &\quad \left. + 8 \eta_2 \eta_3 \text{Re}(\langle \psi_1 | \psi_2 \rangle \langle \psi_2 | \psi_3 \rangle \langle \psi_3 | \psi_1 \rangle) \right]^{1/2}. \end{aligned} \quad (20)$$

This expression describes the maximum attainable probability of correctly distinguishing the state  $|\psi_1\rangle$  from the set of states  $\{|\psi_2\rangle, |\psi_3\rangle\}$ . The minimum-error probability then follows as  $P_{\text{Error}}^{(1,2,3)} = 1 - P_{\max}^{(1,2,3)}$ . As expected, the result is independent of the individual phase factors of the given states, and for  $\eta_3 = 0$  it reduces to the pioneering formula [4] for minimum-error discrimination between only two nonorthogonal states.

### III. DISCUSSION

With respect to possible applications, the question arises how the maximum probability for getting a correct result in quantum-state filtering compares to the maximum probability for correctly discriminating, by means of a different measurement strategy, between all the given states individually. In the following we shall explore this question for a variety of symmetric states.

Let us investigate the set of three symmetric states

$$|\psi_k\rangle = \cos \beta |u_1\rangle + e^{i(2\pi/3)(k-1)} \sin \beta |u_2\rangle, \quad (21)$$

with  $k=1,2,3$ , and  $0 < \beta \leq \pi/4$ , which are assumed to occur with equal *a priori* probability. Here  $|u_1\rangle$  and  $|u_2\rangle$  denote any two orthonormal basis states. Obviously the states are linearly dependent and nonorthogonal. Due to their symmetry, the mutual overlaps are equal and we get  $4|A_{kl}|^2 = 4 - 3 \sin^2(2\beta)$  if  $k \neq l$ , where we again used the abbreviation  $A_{kl} = \langle \psi_k | \psi_l \rangle$ . Moreover, we obtain that  $8 \text{Re}(A_{12} A_{23} A_{31}) = 8 - 9 \sin^2(2\beta)$ . By substituting these expressions into Eq. (20) and taking into account that  $\eta_k = 1/3$ , we find the minimum-error probability for quantum state filtering with respect to the state  $|\psi_1\rangle$ ,

$$P_{\text{Error}}^{(1,2,3)}(\beta) = \frac{1}{6} [3 - \sqrt{1 + 3 \sin^2(2\beta)}]. \quad (22)$$

Because of the symmetry, the same expression holds for distinguishing any other state from the remaining two states. For comparison, we now consider individual discrimination between all three states. The general formula for minimum-error discrimination between  $N$  symmetric states, derived in Ref. [6], has been recently applied by one of us [13] to states of the form (21), yielding the maximum probability  $P_{\max}^{(1,2,3)} = \frac{1}{3} (|\sin \beta| + |\cos \beta|)^2$  for correctly distinguishing each state individually. From this result we obtain the minimum-error probability

$$P_{\text{Error}}^{(1,2,3)}(\beta) = 1 - P_{\max}^{(1,2,3)} = \frac{1}{3} [2 - \sin(2\beta)]. \quad (23)$$

The ratio  $P_{\text{Error}}^{(1,2,3)}(\beta)/P_{\max}^{(1,2,3)}(\beta)$  is found to vary between 0.5 for  $\beta=0$  or  $\pi/4$ , and the maximum value 0.56 for  $\beta \approx \pi/12$ . When  $\beta$  approaches zero, the physical difference between the states vanishes and the respective minimum-error probabilities, corresponding to random guessing, are twice as large as those for  $\beta = \pi/4$ , when both kinds of minimum-error probabilities take their smallest possible val-

ues. These values are equal to  $1/3$  when all three states are discriminated individually, and to  $1/6$  when only one of the states is distinguished.

The same values of the respective minimum-error probabilities also result for the set of equally probable real symmetric states  $|\psi_1\rangle=|u_1\rangle$ ,  $|\psi_2\rangle=-\frac{1}{2}(|u_1\rangle+\sqrt{3}|u_2\rangle)$ , and  $|\psi_3\rangle=-\frac{1}{2}(|u_1\rangle-\sqrt{3}|u_2\rangle)$  which are known as the trine states [10]. For the case that  $|u_1\rangle$  and  $|u_2\rangle$  refer to a single photon and represent horizontal and vertical linear polarization, respectively, these states have been used to verify experimentally the theoretical result  $1/3$  for the minimum-error probability in individual state discrimination [10]. On the other hand, from Eq. (20) with  $\eta_k=1/3$  we easily find that the minimum-error probability for distinguishing the state  $|\psi_1\rangle$  alone is only  $1/6$ . By using  $\tan(2\varphi_e)=|Q|/R$  and Eq. (10) with  $\chi=0$ , the proper projection state,  $|\mu_e\rangle$ , is found to be  $|u_1\rangle$ . Hence the corresponding quantum-state-filtering experiment for single photons could be performed with the help of a polarizing beam splitter that transmits the horizontal component and reflects the vertical one, or vice versa, as it is immediately expected in view of the symmetry of the problem.

In conclusion, we remark that it is straightforward to generalize our basic Eq. (2) in order to account for discrimination between more than two subsets. However, since the detection operators always have to resolve the identity, they cannot be represented by projection operators onto orthogonal states if their number is larger than the dimensionality of the underlying Hilbert space. The measurement, therefore, would be a generalized [14] measurement in this case. The same applies if the number of detection operators is smaller than the number of dimensions of the Hilbert space, as it

happens if, e.g., Eq. (2) is applied to three linearly independent states. Finally it is interesting to relate our results to the Helstrom bound  $P_E=\frac{1}{2}[(1-\|w_1\rho_1-w_2\rho_2\|)]$  for the minimum-error probability of discriminating between two density operators  $\rho_1$  and  $\rho_2$  having the *a priori* probabilities  $w_1$  and  $w_2$ , respectively. Here the symbol  $\|\cdot\|$  denotes the trace norm  $\|\sigma\|\equiv\text{Tr}\sqrt{\sigma^\dagger\sigma}$ . After inserting  $w_1\rho_1=\sum_{k=1}^M\eta_k|\psi_k\rangle\langle\psi_k|$  and  $w_2\rho_2=\sum_{k=M+1}^N\eta_k|\psi_k\rangle\langle\psi_k|$ , the expressions ensuing from  $P_E$  for the cases we are interested in indeed confirm our results, without yielding the optimum detection operators, however.

To summarize, we derived the measurement strategy that minimizes the error probability for discriminating between two complementary subsets of a set of  $N$  nonorthogonal quantum states spanning a two-dimensional Hilbert space. The corresponding measurement is found to be a standard von-Neumann measurement, projecting onto two orthonormal states that have been determined in the paper. Assuming arbitrary *a priori* probabilities of the  $N$  linearly dependent nonorthogonal states, we obtained a general analytical expression for the minimum error probability or, equivalently, for the maximum probability of obtaining a correct result. As special cases of this general result, we gave explicit expressions for the case of  $N$  real states, Eq. (15), and for three arbitrary states, Eq. (20).

#### ACKNOWLEDGMENTS

J.B. wants to acknowledge the hospitality extended to him during his stay at the Humboldt-University in Berlin. The research of J.B. was also supported by the Office of Naval Research (Grant No. N00014-92J-1233) and by a grant from PSC-CUNY.

- 
- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Information* (Cambridge University Press, Cambridge, England, 2000).
  - [2] For an overview on quantum-state discrimination, see the review article A. Chefles, *Contemp. Phys.* **41**, 401 (2000).
  - [3] S. M. Barnett, C. R. Gilson, and M. Sasaki, *J. Phys. A* **34**, 6755 (2001).
  - [4] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
  - [5] H. P. Yuen, R. S. Kennedy, and M. Lax, *IEEE Trans. Inf. Theory* **21**, 125 (1975).
  - [6] M. Ban, K. Kurokawa, R. Momose, and O. Hirota, *Int. J. Theor. Phys.* **55**, 22 (1997).
  - [7] M. Sasaki, K. Kato, M. Izutsu, and O. Hirota, *Phys. Rev. A* **58**, 146 (1998).
  - [8] S. M. Barnett, *Phys. Rev. A* **64**, 030303 (2001).
  - [9] E. Andersson, S. M. Barnett, C. B. Gilson, and K. Hunter, *Phys. Rev. A* **65**, 052308 (2002).
  - [10] R. B. M. Clarke, V. M. Kendon, A. Chefles, S. M. Barnett, E. Rils, and M. Sasaki, *Phys. Rev. A* **64**, 012303 (2001).
  - [11] Y. Sun, J. A. Bergou, and M. Hillery, e-print quant-ph/0112051.
  - [12] M. Sasaki, S. M. Barnett, R. Josza, M. Osaki, and O. Hirota, *Phys. Rev. A* **59**, 3325 (1999).
  - [13] U. Herzog, *Fortschr. Phys.* **49**, 981 (2001).
  - [14] K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory* (Springer-Verlag, Berlin, 1983).