Quantum arrival-time distributions from intensity functions

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The quantum time-of-arrival problem is discussed within the standard formulation of nonrelativistic quantum mechanics with parametric time. It is shown that a general class of arrival-time probability distributions results from the assumption that the arrival process of a quantum particle is similar in nature to other time-dependent arrival-type processes occurring, e.g., in population biology or queue theory. A simple but illustrative example related to the well-known Wigner discussion of the time-energy uncertainty relation is given and the numerical results obtained are compared with Kijowski's distribution [Rep. Math. Phys. **6**, 362 (1974)] of arrival times for a free quantum particle.

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All phenomena are described in physics as evolving in time, which is meant to be not only a formal evolution parameter, but also an operationally significant quantity, defined locally in the neighborhood of any event. One could even prove that the mere existence of a locally defined time is a necessary and sufficient condition for the possibility of physics as we know it (cf., e.g., [1]).

The time status in the standard nonrelativistic quantum mechanics is rather specific. Due to the well-known objections made by Pauli [2] in the early days of quantum mechanics, the incorporation of time as a dynamic variable, represented by a self-adjoint Hilbert space operator, has been basically abandoned. Instead, the notion of an ''external time'' similar to the Newtonian global time of classical mechanics has been adopted. In this formulation such quantities as expectation values or probabilities are well defined only at a given time instant and hence questions related to time intervals (e.g., ''when?,'' ''how long?'') cannot be easily answered.

Many attempts have been made to extend the standard formalism and define various time quantities which are tailored to a specific situation, such as the ''dwell time,'' ''traversal time,'' ''time of tunneling,'' etc. Recently, a particular interest has been observed in the seemingly simplest quantum time-of-arrival (TOA) problem, stimulated mainly through the papers by Muga, Brouard, and Macías [3] and Grot, Rovelli, and Tate [4].

Despite many research efforts in this field, as thoroughly summarized in a recent review by Muga and Leavens [5], there are still many controversies related to the quantum TOA problem, even in the simplest case of a free particle. Some of them seem to be related to differences in the formulation of the problem and additional implicit assumptions. For example, the question, ''When will a particle released at point A arrive at point B?'' assumes implicitly that it *will* happen and we only do not know when. In classical mechanics it is therefore assumed in this case that the particle moves along a definite trajectory connecting both points. Unfortunately, such assumptions cannot be generally transferred to the quantum realm.

The Bohm or Bohm-like quantum theories do postulate that a particle is a two-component pointlike object, with a well-defined position and velocity at each time instant, quite similar to the situation in classical mechanics. But, as shown by Deotto and Ghirardi [6], for systems with more than one spatial dimension there are infinitely many Bohm-like theories possible, which are inequivalent from the point of view of the trajectories followed by the particles. Therefore, the predictions made about the respective time quantities may also be different for different Bohm-like theories, as demonstrated recently by Finkelstein [7].

In the standard formulation of quantum mechanics, where particles are not supposed to move along definite trajectories, the definition of such time-interval-related quantities becomes even more controversial. Moreover, at least in the Copenhagen interpretation, all properties including the ''particle'' momentary position and the localization of the ''arrival point'' should be *measured* to be assumed as known.

Additional ambiguities for systems with more than one spatial dimension, similar to those found in the Bohmian and Bohmian-like theories, seem also to be present in conventional quantum theory [7]. Therefore, the results obtained up to now mainly for one-dimensional quantum systems cannot be generally extended to systems with a higher spatial dimensionality.

In the present paper we discuss the quantum TOA problem using the conventional notion of parametric time, labeling the consecutive quantum states during the system evolution. In contrast to the other approaches, we do not need to make any assumptions about the spatial dimensionality, the existence of ''virtual'' or real spatial trajectories, or appropriately tailored time operators. In our approach, the TOA probability distribution is obtained as an answer to the question, ''When will a ''particle,'' described by the state vector $|\psi_p(t)\rangle$ or, more generally, by a density operator $\hat{\rho}_p(t)$ at each time instant t beginning from t_0, be registered by a ''detector,'' represented by the state vector $|\psi_d\rangle$?'' We assume in the following that the time evolution of the particle state is Markovian, i.e., that the state at $t''\geq t'$ depends only on the state at time t', but the evolution is not necessarily a unitary (i.e., Hamiltonian) one. In general, due to interactions

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with the environment, the particle state will rather be given as a solution to an appropriate master equation. In any case, the sequence of consecutive particle states generated by the evolution law, labeled by the time parameter, gives an ''evolution path,'' which could be also viewed as a substitute for the classical trajectory.

The final ''arrival position'' is represented in our approach by the detector state $|\psi_d\rangle$. Because we need only indicate that the particle was discovered by the detector or not, our detector is supposed to act as a two-state counter only, waiting to be triggered by the approaching particle. The act of detection marks the end of the part of the evolution path we are interested in while discussing the TOA problem.

Our approach was motivated by the observation that the quantum TOA problem is very close in nature to the ''time of death'' of a living being (cf., e.g., [8]) or the ''time of service'' when waiting in a queue (cf., e.g., [9]). These processes consist of a series of ''failure''-type events (e.g., ''I'm still alive,'' ''somebody else is being served'') which is terminated by a single ''success'' event. The probability of succeeding in the next time interval is usually not constant, as assumed in many decay processes discussed in physics, but varying in time. Moreover, the probability of succeeding at all may be sometimes less than 1 (e.g., ''unfortunately, we are successful in 70% of cases only''), which could reflect the detector nonideality. It is well known that similar phenomena can be commonly described through Poisson-type arrival processes (cf., e.g., [10,11]). Suggestions that the quantum arrival process is an inhomogeneous Poisson process have been made already in the context of event enhanced quantum theory [12,13] where some TOA models have also been discussed [14,15].

In order to proceed conveniently, let us find first the probability $\overline{P}(t)$ that our detector has not registered the particle during the whole time interval (t_0,t), assuming for the moment that the intensity function $\lambda(t)$ of the Poisson arrival process is already known. The intensity function here has a simple interpretation as the conditional probability density that the detection process will be completed in the next infinitesimal time interval, provided that it has *not* been completed yet until the time instant t. To keep the derivation simple, we will apply a formal discretization procedure here, dividing the interval (t_0,t) into n arbitrarily small parts, each of length $\Delta t=(t-t_0)/n$. The approximate probability of triggering the detector in the kth time interval would then be $\pi_k=\lambda(t_{k-1})\Delta t$, where $t_k=t_0+k\Delta t$.

Hence, the approximate probability $\overline{P}_n(t)$ that the detector will not be triggered at all during the whole time interval (t_0,t) is

$$\overline{P}_n(t)=\prod_{k=1}^{n}(1-\pi_k). \tag{1}$$

In the limit of infinite n we then get that

$$\overline{P}(t)=\lim_{n\to\infty}\overline{P}_n(t)=\exp\left\{-\int_{t_0}^{t}\lambda(t')dt'\right\}, \tag{2}$$

which leads directly to the expression for the time of arrival probability density $p(t)$ itself:

$$p(t)=\frac{d}{dt}[1-\overline{P}(t)]=\lambda(t)\overline{P}(t). \tag{3}$$

Therefore, the results obtained so far for the TOA probability density can be summarized in the following general expression:

$$p(t)=\lambda(t)\exp\left\{-\int_{t_0}^{t}\lambda(t')dt'\right\}. \tag{4}$$

One can easily check that $p(t)$ defined above has the formal properties that are usually expected from a time-of-arrival probability density (cf., e.g., [7,16]). Notice that if $\int_0^\infty\lambda(t)<\infty$ then the probability that the particle may not be detected at all is greater than zero, indicating the ''nonideality'' of the detection process.

The intensity function $\lambda(t)$ of the arrival process, which determines the time-of-arrival probability distribution, depends obviously on the particle ''evolution path,'' the detector state chosen, and the coupling between them. Notice that due to the assumed Markovian time evolution $\lambda(t)$ may be seen here as an ordinary detection probability density.

To give a simple but illustrative example, let us assume that the intensity function is given as

$$\lambda(t)=\lambda_0|\langle\psi_d|\psi_p(t)\rangle|^2, \tag{5}$$

where $\lambda_0\geqslant 0$ is a constant multiplier. This choice reflects the interpretation of $\lambda(t)dt$ as the probability of successful detection within the next infinitesimal time interval dt, which is set here as proportional to the instantaneous transition probability evaluated according to the probabilistic postulates of conventional quantum mechanics. One of the advantages of this intensity function is that it could be directly translated to the Weyl-Wigner-Moyal or other well-behaving phase-space representation of quantum mechanics (see, e.g., [17] and references therein). On the other hand, non-normalizable detector states $|\psi_d\rangle$ are preferred here, because otherwise $\overline{P}(\infty)>0$ and we will have to take into account the detector imperfectness.

Searching for an ''ideal'' arrival probability distribution, with absolute precision of the position localization, one may further assume that the detector state is a position eigenstate placed at $x=x_0$, i.e., that $|\psi_d\rangle=|x_0\rangle$, which gives

$$\lambda(t)=\lambda_0|\psi_p(x_0,t)|^2, \tag{6}$$

i.e., the intensity function $\lambda(t)$ is in this case proportional to the probability density for the *presence* of the particle at x_0 at the time instant t.

The ideal TOA probability density Eq. (4) becomes then

$$p_{x_0}(t)=\lambda_0|\psi_p(x_0,t)|^2 \times\exp\left\{-\lambda_0\int_{t_0}^{t}|\psi_p(x_0,t')|^2dt'\right\}. \tag{7}$$

This result may be directly compared with the probability density $p_u(t) \propto |\langle u|\psi(t)\rangle|^2$ postulated in 1972 by Wigner [18] to get the time-energy uncertainty relation, where $|u\rangle$ denotes "any state vector" ([18], pp. 240–241; see also a recent discussion in [19]). With the position eigenvector $|x_0\rangle$ substituted for $|u\rangle$, it becomes exactly $|\psi(x_0,t)|^2$ and it coincides with our intensity function Eq. (6).

It can be easily seen that for small values of the elapsed time $t-t_0$ the originally proposed Wigner density and Eq. (7) should remain in quite good agreement. But it is also evident that for larger t the Wigner density needs a damping correction term, which is provided by the exponential factor in our Eq. (7).

In order to get a better grasp of the introduced TOA probability density, let us consider a Gaussian one-dimensional wave packet "particle" described by

$$\psi(x,t)=\frac{1}{(\sqrt{2\pi L^2})^{1/4}}\frac{1}{\sqrt{1+i\hbar t/2mL^2}}\times\exp\left\{-\frac{(x-p_0t/m)^2}{4L^2(1+i\hbar t/2mL^2)}+\frac{ip_0x}{\hbar}-\frac{ip_0^2t}{2m\hbar}\right\}, \tag{8}$$

which is the solution of the free particle Schrödinger equation with the initial state

$$\psi_0(x)=\frac{1}{(\sqrt{2\pi L^2})^{1/4}}\exp\left\{-\frac{x^2}{4L^2}+\frac{ip_0x}{\hbar}\right\}, \tag{9}$$

i.e., a minimum uncertainty wave packet with $\langle\hat{q}\rangle=0$, $\langle\hat{p}\rangle=p_0$, $\Delta q_0=L$, and $\Delta p_0=\hbar/2L$, released at $t=0$ [20]. The Wigner density $|\psi(x_0,t)|^2$ in this case may be obtained easily in a simple analytical form:

$$|\psi(x_0,t)|^2=\frac{1}{\sqrt{2\pi\sigma_x^2(t)}}\exp\left\{-\frac{(x_0-p_0t/m)^2}{2\sigma_x^2(t)}\right\}, \tag{10}$$

where

$$\sigma_x^2(t)=L^2\left[1+\left(\frac{\hbar}{2mL^2}\right)^2t^2\right]. \tag{11}$$

The resulting TOA probability density $p_{x_0}(t)$, calculated according to Eq. (7) for the arrival position at $x_0=5$ a.u., normalized to unity ($\lambda_0=\sqrt{\pi}$), is plotted in Fig. 1, together with the Kijowski TOA density [5,16] for positive momenta,

$$\Pi_K[t;x_0;\psi_0]=\left|\int_0^\infty dp\left(\frac{p}{mh}\right)^{1/2}\langle p|\psi_0\rangle\times\exp\left[-\frac{-ip^2t}{2m\hbar}\right]\exp\left[\frac{ipx_0}{\hbar}\right]\right|^2, \tag{12}$$

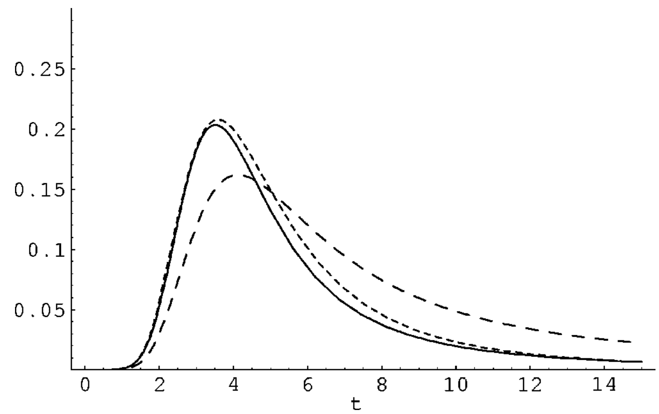

FIG. 1. The ideal TOA densities for arrival at $x_0=5$ a.u. compared: the continuous line is the TOA probability density calculated according to Eq. (7), while the short-dashed and the long-dashed lines correspond to the Kijowski and Wigner densities, respectively (see the text for details). All quantities are in atomic units.

calculated from the initial state Eq. (9), and the original Wigner density $|\psi(x_0,t)|^2$ [Eq. (10)]. For simplicity, all calculations were performed assuming $\hbar=L=m=p_0=1$.

It could be seen that our calculated TOA probability density remains in good agreement with the Kijowski density, regarded as the closest quantum object to an ideal classical arrival-time distribution [5].

On the other hand, it is amazing how similar the original Wigner time distribution is to the other distributions depicted in Fig. 1, but for obvious reasons, it is usually rejected as a valid *quantum* TOA probability density. Nevertheless, in the classical limit $\hbar\to 0$ the time dependence drops out of σ_x^2 in Eq. (10) and the quantity $\{|\psi(x_0,t)|^2\}_{\hbar=0}$ becomes a perfect candidate for a *classical* TOA density. But it can then reflect only the imprecision of the initial particle position at $t=0$ "transported" to the arrival position, because other parameters are held fixed. In the ideal case of precise localization, when $L\to 0$, we may recover the well-known classical picture with a moving point particle, and the intensity function Eq. (6) and the TOA probability density Eq. (7) are then both proportional to $\delta(x_0-p_0t/m)$, as they should be.

Recently, Marchewka and Schuss, in their Feynman trajectory studies of a quantum particle impinging on an absorbing wall [21–23], found expressions for the survival probability that are similar to our Eq. (2). It seems therefore plausible that the arrival process discussed above could be seen as a "common denominator" for several more specific TOA models, each supplying its own recipe for the appropriate intensity function. Knowing already a particular TOA probability density $p(t)$, it is possible, at least in principle, to obtain the corresponding intensity function

$$\lambda(t)=p(t)\Bigg/\left[1-\int_0^t dt'p(t')\right], \tag{13}$$

because of Eq. (4) and the following relation:

$$\exp\left\{-\int_{t_0}^t\lambda(t')dt'\right\}=1-\int_{t_0}^t dt'p(t'). \tag{14}$$

Moreover, the Poisson intensity function itself can also be directly estimated from experimental data in computationally efficient ways (see, e.g., [24]), which may facilitate the comparison of theoretical predictions with experimental results.

As a final remark, notice that the intensity function obtained from a given probability density via Eq. (13) cannot always be cast into the simple form given by Eq. (5) or Eq. (6) involving physically relevant states. This is evident, e.g., in the case of $p(t)=\lambda_0 \exp[-\lambda_0 t]$, where $\lambda(t)=\lambda_0=\text{const}$, and it indicates that in general more elaborate forms of intensity functions may be necessary.

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