Motion-induced particle creation from a finite-temperature state

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We investigate the phenomenon of quantum radiation–i.e., the conversion of (virtual) quantum fluctuations into (real) particles induced by dynamical external conditions–for an initial thermal equilibrium state. For a resonantly vibrating cavity a rather strong enhancement of the number of generated particles (the dynamical Casimir effect) at finite temperatures is observed. Furthermore we derive the temperature corrections to the energy radiated by a single moving mirror and an oscillating bubble within a dielectric medium as well as the number of created particles within the Friedmann-Robertson-Walker universe. Possible implications and the relevance for experimental tests are addressed.

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I. INTRODUCTION

Motivated by previously obtained results [1], we generalize the canonical formalism adopted there towards the application to further scenarios and establish the derivations in more detail:

One of the main consequences of quantum theory is the existence of a nontrivial vacuum state. In contrast to the classical theory the quantum fields undergo fluctuations even in their state of lowest energy (the ground state)–the so-called vacuum fluctuations. These fluctuations have measurable consequences: For example, if the fields are constrained by the presence of external conditions, the energy associated to these fluctuations (the zero-point energy) may change owing to the imposed external conditions. As a result the quantum field may exert a force onto the external conditions in order to minimize its energy. The most prominent example for such a force is the Casimir [2] effect which predicts the attraction of two parallel perfectly conducting and neutral plates (i.e., mirrors) placed in the vacuum of the electromagnetic field. The prediction of this striking effect has been verified experimentally with relatively high accuracy [3,4].

A different–not less interesting–effect has not yet been rigorously verified in an experiment: The impact of the external conditions may also induce a conversion of the virtual quantum fluctuations of the field into real particles–the phenomenon of quantum radiation. As examples for such external conditions giving rise to the creation of particles we may consider moving mirrors, time-dependent dielectrics, or gravitational fields.

Various investigations have been devoted to this topic during the last decades, here we mention only some of the most important initial papers in chronological order: In 1970, Moore [5] presented the first explicit calculation of the quantum radiation on the basis of two $1+1$-dimensional moving mirrors. In this pioneering work he exploited the conformal invariance of the scalar field in $1+1$ dimensions. Based on this result, Fulling and Davies [6] presented a calculation of the radiation of a single moving mirror (again in $1+1$ dimensions) and pointed out the close analogy to the Hawking radiation. In 1982, Ford and Vilenkin [7] succeeded in developing a method for the calculation of the radiation generated by a moving mirror in higher dimensions, i.e., without exploiting the conformal invariance.

For the experimental verification of the phenomenon of quantum radiation, the scenario of a closed cavity might be most promising since one may exploit a resonance enhancement in this situation. The particle production inside a resonantly vibrating cavity has already been considered by several authors, see, e.g. [8–17] as well as [18,19] for reviews.

However, most of the investigations of quantum radiation are restricted to the vacuum state, i.e., to zero temperature. But in view of an experimental verification it is essential to study the finite-temperature effects. Realistic calculations of thermal effects on quantum radiation within the framework of quantum field theory of time-dependent systems at finite temperature are not yet available.

The remedy of this deficiency is the main intention of the present paper: In Sec. II, we set up the basic formalism for the quantum treatment of external conditions at finite temperatures. The developed methods are applied in Sec. III to the scenario of a trembling cavity. In Sec. IV, we focus on the resonance case and derive the number of created particles. Another scenario giving rise to quantum radiation–a dynamical dielectric medium–is considered in Sec. V. In Sec. VI, we demonstrate the flexibility of the canonical approach presented in Sec. II by calculating the finite-temperature corrections to the particle production in yet another example scenario–the Friedmann-Robertson-Walker universe. We shall close with a summary, some conclusions, a discussion, and an outlook.

Throughout this article natural units with

$$\hbar=c=G_{\rm N}=k_{\rm B}=\epsilon_0=\mu_0=1 \tag{1}$$

will be used. The signature of the Minkowski metric is chosen according to $g_{\mu\nu}=\text{diag}(+1,-1,-1,-1)$.

II. GENERAL FORMALISM

The objective is to investigate quantized bosonic fields obeying linear equations of motion under the influence of external conditions. At asymptotic times $|t|\uparrow\infty$ the external

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conditions, which are treated classically (not quantized), are assumed to approach a static $\hat{H}_0$-configuration, where the asymptotic Hamiltonian $\hat{H}_0$ can be diagonalized via a suitable particle definition. Initially the state of the quantum system is supposed to be described by thermal equilibrium at a given temperature T, which might be realized through the coupling to a corresponding heat bath. However, the coupling to the heat bath has to be switched off before the external conditions undergo dynamical changes in order to avoid relaxation processes (closed system). In general the time dependence of the external conditions causes the state of the quantum system to leave the initial thermal equilibrium. Accordingly, the calculation of the expectation value of relevant observables, e.g., the number of particles, before and after the dynamics may deviate. These differences can be interpreted as particles that are created or even annihilated by the dynamical external conditions.

A. Interaction picture

The following calculations are most suitably performed in the interaction representation. Accordingly, the dynamics of all operators $\hat{X}$ corresponding to observables are governed by the undisturbed Hamiltonian $\hat{H}_0$,

$$\frac{d\hat{X}}{dt}=i[\hat{H}_0,\hat{X}]+\left(\frac{\partial\hat{X}}{\partial t}\right)_{\rm expl}. \tag{2}$$

This Hamiltonian $\hat{H}_0$ describes the complete dynamics of the system at asymptotic times and can be diagonalized via a suitable particle definition

$$\hat{H}(|t|\uparrow\infty)=\hat{H}_0=\sum_I \omega_I\hat{N}_I+E_0=\omega_I\hat{N}_I+E_0, \tag{3}$$

where E_0 denotes the (divergent) zero-point energy. In most of the following formulas we make use of a generalized sum convention and drop the summation signs by declaring that one has to sum over all indices that do not occur at both sides of the equation. Equations with the same index appearing at both sides are valid for all possible values of this index.

The index I contains a complete set of quantum numbers labeling the different particle modes, e.g., $I=\{\boldsymbol{k}\}$ or $I=\{\omega,l,m\}$ etc. The particle energies are given by ω_I. For a thermodynamical consideration we have to describe the state of the field by the statistical operator $\hat{\varrho}$. In the interaction picture the time evolution of this density matrix is given by the von Neumann equation

$$\frac{d\hat{\varrho}}{dt}=-i[\hat{H}_1,\hat{\varrho}]. \tag{4}$$

The perturbation Hamiltonian $\hat{H}_1$ governs the influence of the variation of the external conditions upon the quantized field. Note, that this equation describes the time evolution of a closed quantum system, i.e., no measurements, etc., take place during the dynamics. It leads to a unitary time evolution operator and therefore also does not contain relaxation processes, etc. Measurements and relaxations would change the probabilities w_A of the statistical operator

$$\hat{\varrho}(t)=w_A|\Psi_A(t)\rangle\langle\Psi_A(t)|, \tag{5}$$

and can be incorporated into Eq. (4) via an explicit time derivative $(\partial\hat{\varrho}/\partial t)_{\rm expl}$.

B. Entropy

Without an explicit time-dependence $(\partial\hat{\varrho}/\partial t)_{\rm expl}$ Eq. (4) generates a unitary time evolution and hence, the microscopic entropy remains constant in time

$$S=-{\rm Tr}\{\hat{\varrho}\ln\hat{\varrho}\}={\rm const.} \tag{6}$$

Note, that a constant microscopic entropy arises also in classical mechanics where the time evolution is governed by the Liouville equation. By virtue of Liouville's theorem the total time derivative of the phase-space density ϱ vanishes and thus the classical microscopic entropy $\int d\Gamma\varrho\ln\varrho$ remains constant as well. But introducing the Boltzmann equation via averaging over multiparticle correlations, it is possible to define an effective entropy which increases in general (H theorem).

An analogous procedure can be performed in quantum theory: In practice, a complete knowledge about a given quantum system can never be achieved. Formally, this restriction defines a so-called observation level $\mathcal{G}=\{\hat{X}\}$ as a set of possibly relevant observables $\hat{X}$ (see [20]), where an averaging over all unknown and possibly irrelevant observables is understood. With respect to a given observation level $\mathcal{G}$ one can introduce an effective statistical operator $\hat{\varrho}_{\{\mathcal{G}\}}$ such that it yields the correct expectation values $\langle\hat{X}\rangle={\rm Tr}\{\hat{\varrho}_{\{\mathcal{G}\}}\hat{X}\}={\rm Tr}\{\hat{\varrho}\hat{X}\}$ for all operators $\hat{X}\in\mathcal{G}$. The effective statistical operator $\hat{\varrho}_{\{\mathcal{G}\}}$ averages over all irrelevant observables $\hat{X}\notin\mathcal{G}$ in order to maximize the effective entropy which is defined as $S_{\{\mathcal{G}\}}=-{\rm Tr}\{\hat{\varrho}_{\{\mathcal{G}\}}\ln\hat{\varrho}_{\{\mathcal{G}\}}\}$. This effective entropy $S_{\{\mathcal{G}\}}$ referring to a given observation level in general increases with time.

Introducing the internal energy $E=\langle:\hat{H}_0:\rangle$ as observation level the corresponding effective entropy S_E may increase under the influence of the dynamical external conditions reflecting the fact that particles have been created. The physical meaning of S_E, respectively, its change ΔS_E may become evident, if one assumes some energy-conserving relaxation process, e.g., mediated via a measurement after the dynamics has taken place, which thermalizes the system again at some higher, in principle, measurable temperature $T_E=T+\Delta T$. For a photon gas we find in the limit of high temperatures, respectively, of large volumes υ the following relations between the energy E, the effective entropy S_E, and the effective temperature T_E: $E=\upsilon T_E^4\pi^2/15$ and $S_E=\upsilon T_E^3 4\pi^4/45$. For small disturbances the relative increase of the effective temperature, the energy and the effective entropy behaves as $\Delta E/(4E)\approx\Delta T_E/T_E\approx\Delta S_E/(3S_E)$.

C. Time evolution

Within our approach the influence of the dynamics of the external conditions is represented by the perturbation Hamiltonian $\hat{H}_1(t)$ governing the time evolution of the statistical operator in Eq. (4). By means of the time-ordering operator $\mathcal{T}$ this equation can be integrated formally

$$\hat{\varrho}(t\uparrow\infty)=\hat{U}\hat{\varrho}(t\downarrow-\infty)\hat{U}^\dagger =\mathcal{T}\left[\exp\left(-i\int dt'\hat{H}_1(t')\right)\right]\hat{\varrho}(t\downarrow-\infty) \times\mathcal{T}^\dagger\left[\exp\left(i\int dt''\hat{H}_1(t'')\right)\right]. \tag{7}$$

The chronological operator $\mathcal{T}$ acting on two bosonic and self-adjoint operators $\hat{X}(t)$ and $\hat{Y}(t')$ is defined by

$$\mathcal{T}[\hat{X}(t)\hat{Y}(t')]=\hat{X}(t)\hat{Y}(t')\Theta(t-t')+\hat{Y}(t')\hat{X}(t)\Theta(t'-t), \tag{8}$$

and so on for more operators. Due to the Hermitian conjugation of the unitary time evolution operator $\hat{U}$ in Eq. (7), for which the position of all operators changes, it is convenient to introduce the antichronological operator $\mathcal{T}^\dagger$ (cf. [21]) as well

$$\mathcal{T}^\dagger[\hat{X}(t)\hat{Y}(t')]=(\mathcal{T}[\hat{X}(t)\hat{Y}(t')])^\dagger =\hat{Y}(t')\hat{X}(t)\Theta(t-t')+\hat{X}(t)\hat{Y}(t')\Theta(t'-t). \tag{9}$$

Combining both equations one obtains for two bosonic and self-adjoint operators

$$\{\hat{X}(t),\hat{Y}(t')\}=\mathcal{T}[\hat{X}(t)\hat{Y}(t')]+\mathcal{T}^\dagger[\hat{X}(t)\hat{Y}(t')] =\hat{Y}(t')\hat{X}(t)+\hat{X}(t)\hat{Y}(t'). \tag{10}$$

As we shall see below this property simplifies the calculation of the quadratic response of the number operator.

D. Canonical ensemble

As stated in Sec. II, we assume the quantum field to be initially at thermal equilibrium corresponding to some nonvanishing temperature $T=1/\beta>0$. For reasons of simplicity we restrict our further consideration to particles, that do not exhibit another conserved quantity than their energy.[1] This assumption is correct for photons, but not for charged pions, for instance. We consider only bosons. For that reason the chemical potential vanishes and the energy E is the only one observable that has a fixed expectation value. Minimizing the microscopic entropy with this constraint generates the canonical ensemble

$$\hat{\varrho}_0=\hat{\varrho}(t\downarrow-\infty)=\frac{\exp(-\beta\hat{H}_0)}{\mathrm{Tr}\{\exp(-\beta\hat{H}_0)\}}. \tag{11}$$

There are several examples in quantum field theory where the canonical ensemble is not capable of describing the thermal equilibrium correctly, usually in combination with infinite volumes. In order to also treat such cases, we assume the system to be confined into a finite volume where the canonical ensemble applies, calculate the expectation values (i.e. the trace), and consider the infinite volume limit afterwards.

E. Response theory

In general, the final expectation value of an observable

$$\langle\hat{X}\rangle=\mathrm{Tr}\{\hat{X}\hat{\varrho}(t\uparrow\infty)\}=\mathrm{Tr}\{\hat{X}\hat{U}\hat{\varrho}(t\downarrow-\infty)\hat{U}^\dagger\} \tag{12}$$

cannot be calculated explicitly for nontrivial interaction terms $\hat{H}_1$ owing to the complicated structure of the corresponding time-evolution operator $\hat{U}$. For that purpose one has usually to introduce some approximations. One possibility is given by the perturbation expansion with respect to powers of the disturbance $\hat{H}_1$. Assuming the perturbation Hamiltonian $\hat{H}_1$ to be small it is possible to expand the above expression in powers of $\hat{H}_1$. Neglecting all terms of third and higher order in $\hat{H}_1$, one obtains the quadratic response

$$\begin{aligned}\langle\hat{X}\rangle=&\mathrm{Tr}\{\hat{X}\hat{\varrho}_0\}+\mathrm{Tr}\left\{\hat{X}\left[\hat{\varrho}_0,i\int dt\hat{H}_1(t)\right]\right\}\\ &+\mathrm{Tr}\left\{\hat{X}\int dt\hat{H}_1(t)\hat{\varrho}_0\int dt'\hat{H}_1(t')\right\}\\ &-\frac{1}{2}\mathrm{Tr}\left\{\hat{X}\mathcal{T}\left[\int dt\hat{H}_1(t)\int dt'\hat{H}_1(t')\right]\hat{\varrho}_0\right\}\\ &-\frac{1}{2}\mathrm{Tr}\left\{\hat{X}\hat{\varrho}_0\mathcal{T}^\dagger\left[\int dt\hat{H}_1(t)\int dt'\hat{H}_1(t')\right]\right\}+O(\hat{H}_1^3).\end{aligned} \tag{13}$$

Focusing on the investigation of the particle production the relevant observable is the number operator $\hat{X}=\hat{N}_I$. Due to $[\hat{N}_I,\hat{\varrho}_0]=0$ the linear response vanishes and with the aid of Eq. (10) the quadratic response simplifies to

$$\begin{aligned}\langle\hat{N}_I\rangle=&\mathrm{Tr}\{\hat{N}_I\hat{\varrho}_0\}+\mathrm{Tr}\left\{\hat{N}_I\left[\int dt\hat{H}_1(t),\hat{\varrho}_0\right]\int dt\hat{H}_1(t)\right\}\\ &+O(\hat{H}_1^3)\\ =&\langle\hat{N}_I\rangle_0+\Delta N_I+O(\hat{H}_1^3).\end{aligned} \tag{14}$$

The first term $\langle\hat{N}_I\rangle_0$ of the above expression denotes the initial particle content in the canonical ensemble which is given by the Bose-Einstein distribution. The second term

[1] Otherwise, one may start with a grand canonical ensemble.

ΔN_I describes the particle creation or annihilation due to the presence of dynamical external conditions and will be calculated in the following.

F. Particle production

For quasifree quantum fields obeying linear equations of motion the perturbation Hamiltonian can be expressed–up to an irrelevant constant–as a bilinear form of the fields. Expanding the fields into the time-independent creation/annihilation operators representing particles in the asymptotic regions the disturbance $\hat{H}_1$ can be cast into the rather general form

$$\int dt\hat{H}_1(t)=\frac{1}{2}(\mathcal{S}_{JK}\hat{a}_J^\dagger\hat{a}_K^\dagger+\mathcal{S}_{JK}^*\hat{a}_J\hat{a}_K)+\mathcal{U}_{JK}\hat{a}_J^\dagger\hat{a}_K+\mathcal{C}. \tag{15}$$

The introduced matrices have to fulfill the conditions $\mathcal{S}_{JK}=\mathcal{S}_{KJ}$ and $\mathcal{U}_{JK}=\mathcal{U}_{KJ}^*$ because $\hat{H}_1$ is self-adjoint. The $\mathcal{S}$ term could be interpreted as a generator for a multimode squeezing operator and the $\mathcal{U}$ term as a hopping operator. Now the matrices $\mathcal{S}$ and $\mathcal{U}$ contain all information about the dynamical external conditions that are relevant for the quadratic response. Inserting the general form of the disturbance in Eq. (15) into Eq. (14) after evaluating the traces the quadratic response of the number operator takes the form

$$\Delta N_I=|\mathcal{S}_{IJ}|^2(1+\langle\hat{N}_J\rangle_0+\langle\hat{N}_I\rangle_0)+|\mathcal{U}_{IJ}|^2(\langle\hat{N}_J\rangle_0-\langle\hat{N}_I\rangle_0). \tag{16}$$

Note, that there is still a summation over the index J. One observes that merely the $\mathcal{S}$ term governs the production of particles and contains the vacuum contribution (first addend). The $\mathcal{U}$ term does not increase the total number of particles since it has the same structure as a classical master equation (e.g., used for the derivation of the H theorem), but it transforms particles from one mode into another and thereby also increases the energy. Investigating the high-temperature expansion of the Bose-Einstein distribution entering the equation above

$$\langle\hat{N}_I\rangle_0=\frac{1}{\exp(\beta\omega_I)-1}=\frac{1}{\beta\omega_I}-\frac{1}{2}+O(\beta), \tag{17}$$

one observes that the temperature-independent contributions (the term $-1/2$) cancel. The same occurs in the static Casimir effect [22–24] and may be interpreted as a consequence of the scale invariance in the classical limit. In the high-temperature limit the expectation value $\langle\hat{N}_I\rangle_0$ is linear in T. But special care is required for evaluating the number of created particles in Eq. (16) due to the remaining mode summation. Since the expansion in Eq. (17) has a finite radius of convergence its insertion into Eq. (16) may cause some problems in performing the mode summation. Therefore the resulting number of produced particles or the total radiated energy may possibly display another behavior as shown in Eq. (17). As we shall see later in Secs. III C and V A, the radiated energy may be proportional to even higher powers in T.

G. Correlations

The expectation value $\langle\hat{N}_I\rangle$ of the number operator after the dynamical period does not, in general, display the Bose-Einstein distribution because the field is no longer in the thermodynamic equilibrium. But for certain dynamics this expectation value could still behave as a thermal distribution corresponding to some effective temperature $T_{\rm eff}$. If particles are created during the dynamics this effective temperature will be larger than the initial temperature T. Such a phenomenon may occur even for a vanishing initial temperature $T=0$. This effect can be explained within the thermofield [25,26] formalism: Measuring only single-particle observables does not reveal the complete information about the quantum system under consideration. Formally, this restriction defines an observation level $\mathcal{G}$ [20] including all these single-particle observables (see the remarks in the previous section). The effective density-matrix $\hat{\varrho}_{\{\mathcal{G}\}}$ may indeed be equal to the statistical operator of a canonical ensemble corresponding to some effective temperature $T_{\rm eff}$. However, the real density-matrix $\hat{\varrho}$ cannot be equal to this effective statistical operator $\hat{\varrho}_{\{\mathcal{G}\}}$ because the microscopic entropy $S=-\mathrm{Tr}\{\hat{\varrho}\ln\hat{\varrho}\}$ is conserved during a unitary time evolution while the effective entropy $S_{\{\mathcal{G}\}}=-\mathrm{Tr}\{\hat{\varrho}_{\{\mathcal{G}\}}\ln\hat{\varrho}_{\{\mathcal{G}\}}\}$ has been increased for $T<T_{\rm eff}$. Within the investigation of only single-particle observables one can never distinguish between the two statistical operators $\hat{\varrho}$ and $\hat{\varrho}_{\{\mathcal{G}\}}$ and therefore one can never find out whether the measured temperature represents a real one (T) or an effective one ($T_{\rm eff}$). For this purpose it is necessary to consider many-particle observables. One suitable candidate is given by the two-particle correlation defined as

$$\mathcal{C}_{JK}=\langle\hat{N}_J\hat{N}_K\rangle-\langle\hat{N}_J\rangle\langle\hat{N}_K\rangle \quad \text{for} \quad J\neq K. \tag{18}$$

This quantity is particularly appropriate since it vanishes in the thermodynamic equilibrium

$$\langle\hat{N}_J\hat{N}_K\rangle_0=\langle\hat{N}_J\rangle_0\langle\hat{N}_K\rangle_0 \quad \text{for} \quad J\neq K. \tag{19}$$

The quadratic response of the correlation function can be evaluated as follows

$$\mathcal{C}_{JK}=\left\langle\int dt\hat{H}_1(t)\left[\hat{N}_J\hat{N}_K,\int dt\hat{H}_1(t)\right]\right\rangle_0-\langle\hat{N}_J\rangle_0\Delta N_K \\ -\langle\hat{N}_K\rangle_0\Delta N_J+O(\hat{H}_1^3). \tag{20}$$

As a simple example we may consider the vacuum case with $T=0$,

$$\mathcal{C}_{JK}=\langle 0|\int dt\hat{H}_1(t)\hat{N}_J\hat{N}_K\int dt\hat{H}_1(t)|0\rangle+O(\hat{H}_1^3), \tag{21}$$

where the correlation is positive (at least in lowest order in $\hat{H}_1$).

But at finite temperatures the correlation may also assume negative values: For example, in the case of a completely diagonal perturbation Hamiltonian we arrive at

$$\mathcal{C}_{JK} = -\langle \hat{N}_J \rangle_0 \Delta N_K - \langle \hat{N}_K \rangle_0 \Delta N_J + O(\hat{H}_1^3). \tag{22}$$

H. Bogoliubov transformation

It might be illuminating to consider the relation of the previous investigations to the formalism based on the Bogoliubov transformation. The initial and final creation and annihilation operators are connected through the Bogoliubov coefficients via

$$\hat{U}^\dagger \hat{a}_J \hat{U} = \alpha_{JK} \hat{a}_K + \beta_{JK} \hat{a}_K^\dagger . \tag{23}$$

Switching from the interaction picture to the Heisenberg representation the expectation value of the number operator

$$\langle \hat{N}_J \rangle = \mathrm{Tr}\{\hat{\varrho} \hat{N}_J\} = \mathrm{Tr}\{\hat{\varrho}_0 \hat{U}^\dagger \hat{N}_J \hat{U}\} = \langle \hat{U}^\dagger \hat{N}_J \hat{U} \rangle_0 \tag{24}$$

can be expressed in terms of the Bogoliubov coefficients

$$\langle \hat{N}_J \rangle = |\beta_{JK}|^2 + \langle \hat{N}_K \rangle_0 (|\alpha_{JK}|^2 + |\beta_{JK}|^2). \tag{25}$$

If we assume a completely diagonal Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ the associated time-evolution operator $\hat{U}$ factorizes

$$\hat{U} = \prod_I \hat{U}_I . \tag{26}$$

Accordingly, the Bogoliubov coefficients simplify to $\alpha_{JK} = \alpha_J \delta_{JK}$ and $\beta_{JK} = \beta_J \delta_{JK}$, respectively. Utilizing the unitary relation, which assumes for the diagonal coefficients the simple form $|\alpha_I|^2 = 1 + |\beta_I|^2$, we arrive at

$$\langle \hat{N}_I \rangle = \langle \hat{N}_I \rangle_0 + |\beta_I|^2 (1 + 2\langle \hat{N}_I \rangle_0). \tag{27}$$

Hence–for a diagonal Hamiltonian–the number of created particles at finite temperature is simply given by the corresponding vacuum expression times a thermal factor

$$\Delta N_\omega^T = \Delta N_\omega^{T=0} \left(1 + \frac{2}{\exp(\beta\omega) - 1} \right). \tag{28}$$

It should be mentioned here that this result is not restricted to a particular order perturbation theory–it holds for the general case of a diagonal Hamiltonian.

III. TREMBLING CAVITIES

Now we are in the position to apply the formalism presented in the previous section to a special system of a quantum field under the influence of dynamical external conditions. We consider a massless and neutral scalar field Φ confined in an arbitrary and weakly time-dependent domain $G(t)$ (a trembling cavity) with Dirichlet boundary conditions $\Phi = 0$ at $\partial G(t)$, see also Refs. [11] and [17]. The action generating the equation of motion is given by

$$\mathcal{A} = \frac{1}{2} \int dt \int_G d^3 r (\partial_\mu \Phi)(\partial^\mu \Phi) = \int dt L. \tag{29}$$

The Hamiltonian governing the dynamics of this system can be obtained in the following way: For every fixed time t we construct a complete and orthonormal set of *eigenfunctions* of the Laplacian inside the time-dependent domain $G(t)$ satisfying the required Dirichlet boundary conditions. Owing to the time dependence of the Dirichlet boundary conditions at $\partial G(t)$ this operator becomes time dependent as well. Hence also the proper *eigenfunctions* are time-dependent $f_I = f_I(t)$. We assume a finite domain $G(t)$ resulting in a purely discrete spectrum. The insertion of the expansion of the field

$$\Phi(t, \boldsymbol{r}) = \sum_I Q_I(t) f_I(t, \boldsymbol{r}) \tag{30}$$

into the Lagrangian in Eq. (29) reveals

$$L = \frac{1}{2}(\dot{Q}_I^2 - \omega_I^2(t) Q_I^2) + Q_I \mathcal{M}_{IJ}(t) \dot{Q}_J + \frac{1}{2} Q_I \mathcal{M}_{IJ}(t) \mathcal{M}_{JK}(t) Q_K . \tag{31}$$

Since the time derivative of the field Φ may also include the explicit time derivative of the *eigenfunctions*, we obtain an additional antisymmetric intermode coupling matrix

$$\mathcal{M}_{IJ}(t) = \int_{G(t)} d^3 r f_I \dot{f}_J . \tag{32}$$

Furthermore, the *eigenvalues* of the Laplace operator $\omega_I^2(t)$ are time dependent in general. By means of a Legendre transformation we obtain the Hamiltonian

$$H = \frac{1}{2}(P_I^2 - \omega_I^2(t) Q_I^2) + Q_I \mathcal{M}_{IJ}(t) P_J , \tag{33}$$

where P_J and Q_K are the canonical conjugated variables. In the following the undisturbed *eigenvalues* (frequencies) are denoted by $\omega_I^2(|t|\uparrow\infty) = \omega_I^2$ and their variation by $\Delta\omega_I^2(t) = \omega_I^2(t) - \omega_I^2$. After the canonical quantization and the separation of the undisturbed part

$$\hat{H}_0 = \frac{1}{2}(\hat{P}_J^2 + \omega_J^2 \hat{Q}_J^2) \tag{34}$$

the perturbation Hamiltonian may be cast into the form

$$\hat{H}_1(t) = \Delta\hat{E}(t) + \hat{W}(t) = \frac{1}{2} \hat{Q}_J^2(t) \Delta\omega_J^2(t) + \hat{Q}_J(t) \mathcal{M}_{JK}(t) \hat{P}_K(t). \tag{35}$$

The term including $\Delta\omega_J^2(t)$, i.e., $\Delta\hat{E}(t)$, arises from the change of the shape of the domain $G(t)$. This term $\Delta\hat{E}(t)$ is called squeezing contribution. The intermode coupling $\mathcal{M}_{JK}(t)$ contained in $\hat{W}(t)$–the velocity contribution–arises from the motion of the boundaries. Of course, the undisturbed Hamiltonian can be diagonalized through a particle definition $\hat{H}_0=\omega_J(\hat{N}_J+1/2)$ employing the usual creation/annihilation operators

$$\hat{a}_J=\sqrt{\frac{1}{2\omega_J}}(\omega_J\hat{Q}_J+i\hat{P}_J). \tag{36}$$

With the aid of the equations above it is now possible to calculate the expectation value of the number operator employing the results of the previous section. The evaluation of the trace $\mathrm{Tr}\{\cdots\}$ is most suitably performed in the basis of the $\hat{H}_0$ *eigen*kets. One obtains a nonvanishing trace only for those terms that contain the same number of creation and annihilation operators for every mode. Inserting the particular interaction Hamiltonian $\hat{H}_1(t)=\Delta\hat{E}(t)+\hat{W}(t)$ into Eq. (14) generates at a first glance four terms. However, owing to the antisymmetry of the matrix governing the intermode coupling $\mathcal{M}_{JJ}=0$ the mixing terms vanish

$$\mathrm{Tr}\{\hat{N}_I[\Delta\hat{E}(t),\hat{\varrho}_0]\hat{W}(t')\}=0,$$
$$\mathrm{Tr}\{\hat{N}_I[\hat{W}(t'),\hat{\varrho}_0]\Delta\hat{E}(t)\}=0. \tag{37}$$

Consequently the squeezing and the velocity contribution decouple on the level of the quadratic response

$$\Delta N_I=\mathrm{Tr}\left\{\hat{N}_I\left[\int dt\Delta\hat{E}(t),\hat{\varrho}_0\right]\int dt\Delta\hat{E}(t)\right\}+\mathrm{Tr}\left\{\hat{N}_I\left[\int dt\hat{W}(t),\hat{\varrho}_0\right]\int dt\hat{W}(t)\right\}=\Delta N_I^S+\Delta N_I^V. \tag{38}$$

Expressing $\hat{H}_1(t)=\Delta\hat{E}(t)+\hat{W}(t)$ with the aid of the matrices $\mathcal{S}$ and $\mathcal{U}$ as done in Eq. (15) one observes that the squeezing effect manifests in the diagonal elements of the matrices $\mathcal{S}$ and $\mathcal{U}$ while the velocity effect generates off-diagonal elements only.

A. Squeezing

The diagonal form of the squeezing $\Delta\hat{E}(t)$ part of the perturbation Hamiltonian $\hat{H}_1(t)$ indicates the highly resonant character of this contribution. This fact simplifies the calculation of ΔN_I because only one mode–the mode I–survives in the trace. With the abbreviation

$$\mathcal{Q}_I^S=\frac{1}{2\omega_I}\int dt\Delta\omega_I^2(t)\exp(2i\omega_I t)=\frac{1}{2\omega_I}\widetilde{\Delta\omega_I^2}(2\omega_I), \tag{39}$$

the squeezing part $\mathcal{S}^S$ of the matrix $\mathcal{S}=\mathcal{S}^S+\mathcal{S}^V$ can be cast into the form

$$\mathcal{S}_{JK}^S=\mathcal{Q}_I^S\delta_{IJ}\delta_{IK}. \tag{40}$$

Of course, the squeezing part $\mathcal{U}^S$ of the matrix $\mathcal{U}=\mathcal{U}^S+\mathcal{U}^V$ is also strictly diagonal and hence does not contribute to ΔN_I [see Eq. (14)] with the result that only the $\mathcal{S}^S$ term is responsible for particle production. Therefore the squeezing contribution to the number of particles reads

$$\Delta N_I^S=|\mathcal{Q}_I^S|^2(1+2\langle\hat{N}_I\rangle_0)=|\mathcal{Q}_I^S|^2\left(1+\frac{2}{\exp(\beta\omega_I)-1}\right). \tag{41}$$

In accordance to the results of Sec. II H the particle production rate at temperature T equals the rate at zero temperature times the thermal distribution factor.

B. Velocity

Due to the more complicated structure of the velocity term, i.e., the off-diagonal elements, the calculation of the number of created particles involves an additional summation. Hence, the velocity contribution may not be cast into such a simple form as the squeezing term. The $\hat{W}$ part of the perturbation Hamiltonian can be expanded with the aid of the matrices

$$\mathcal{S}_{JK}^V=\frac{i}{2}\int dt\mathcal{M}_{JK}(t)\exp(i[\omega_J+\omega_K]t)\left(\sqrt{\frac{\omega_J}{\omega_K}}-\sqrt{\frac{\omega_K}{\omega_J}}\right), \tag{42}$$

and

$$\mathcal{U}_{JK}^V=\frac{i}{2}\int dt\mathcal{M}_{JK}(t)\exp(i[\omega_J-\omega_K]t)\left(\sqrt{\frac{\omega_J}{\omega_K}}+\sqrt{\frac{\omega_K}{\omega_J}}\right). \tag{43}$$

In this case both terms, the $\mathcal{S}^V$ and the $\mathcal{U}^V$ matrices contribute. The $\mathcal{U}^V$ term may even decrease the number of particles in a given mode, see Eq. (14). Nevertheless, the total energy increases.

C. Moving mirror

In order to illustrate the velocity effect we consider the most simple example of a single mirror in $1+1$ dimensions. This scenario has already been investigated by several authors, e.g., [6,7], at zero temperature. In this case the domain G takes the form $G(t)=[\eta(t),\infty]$ where $\eta(t)$ denotes the time-dependent position of the mirror with $\eta(t\downarrow-\infty)=\eta(t\uparrow+\infty)=0$. The index I can be identified with the wave-number $I\rightarrow k=\omega_I$ which assumes all positive real numbers. As the shape of the domain $G(t)$ does not change only the velocity-effect contributes. The intermode coupling matrix is given by (cf. [11,17])

$$\mathcal{M}_{IJ}(t)\rightarrow\mathcal{M}_{kk'}(t)=\dot{\eta}(t)\frac{2}{\pi}\mathcal{P}\left[\frac{kk'}{k^2-k'^2}\right], \tag{44}$$

where $\mathcal{P}$ denotes the principal value. The Fourier transform of a time-dependent function is denoted by a tilde: $\tilde{\eta}=\mathcal{F}\eta$

[see Eq. (39)]. Using this notation the matrices for the evaluation of the particle number read

$$\mathcal{S}_{kk'}=\frac{1}{\pi}\left(\sqrt{\frac{k}{k'}}-\sqrt{\frac{k'}{k}}\right)\frac{kk'}{k-k'}\tilde{\eta}(k+k') \tag{45}$$

and

$$\mathcal{U}_{kk'}=\frac{1}{\pi}\left(\sqrt{\frac{k}{k'}}+\sqrt{\frac{k'}{k}}\right)\frac{kk'}{k+k'}\tilde{\eta}(k-k'). \tag{46}$$

Inserting these expressions into Eq. (16) yields the following results for the expectation values of the number operator:

$$\Delta N_k=\int_0^\infty dk'\frac{kk'}{\pi^2}|\tilde{\eta}(k+k')|^2(1+\langle\hat{N}_{k'}\rangle_0+\langle\hat{N}_k\rangle_0)$$
$$+\int_0^\infty dk'\frac{kk'}{\pi^2}|\tilde{\eta}(k-k')|^2(\langle\hat{N}_{k'}\rangle_0-\langle\hat{N}_k\rangle_0). \tag{47}$$

Already for the most simple example the velocity contribution cannot be cast into a form being as simple as the squeezing term. But the formula above allows us to calculate the number of created particles within the quadratic response for arbitrary dynamics $\eta(t)$ and temperatures T. Deriving the total radiated energy from Eq. (47)

$$E=\int_0^\infty dk k\Delta N_k=\int dt\int dt'\,\eta(t)\,\eta(t')\mathcal{R}(t-t'), \tag{48}$$

where $\mathcal{R}$ denotes the quadratic response function, we obtain a more elucidative formula. For that purpose we have to perform integrations involving Bose-Einstein distribution functions entering in $\langle\hat{N}_k\rangle_0$. If we insert the usual expansion for those functions

$$\langle\hat{N}_k\rangle_0=\frac{1}{\exp(\beta k)-1}=\sum_{n=1}^{\infty}e^{-n\beta k} \tag{49}$$

the wave-number integration I_β^m leads to Hurwitz zeta functions

$$I_\beta^m=\sum_{n=1}^{\infty}\int_0^\infty dk\,k^m\exp[ik(t-t')-n\beta k]$$
$$=\sum_{n=1}^{\infty}\frac{m!}{(n\beta-i[t-t'])^{m+1}}=\frac{m!}{\beta^{m+1}}\zeta\left(m+1,1-i\frac{t-t'}{\beta}\right). \tag{50}$$

In terms of these functions the response function $\mathcal{R}=\mathcal{R}(\Delta t)=\mathcal{R}(t-t')$ yields after the k and k' integrations

$$\mathcal{R}=\frac{2}{\pi^2}\left(+\pi\delta^{(4)}(\Delta t)+\mathcal{P}\left[\frac{24i}{\Delta t^5}\right]\right)\frac{1}{24}$$
$$+\frac{1}{\pi^2}\left(-\pi\delta^{(2)}(\Delta t)-\mathcal{P}\left[\frac{2i}{\Delta t^3}\right]\right)\frac{\zeta(2,1-i\Delta t/\beta)}{\beta^2}$$
$$+\frac{2}{\pi^2}\left(-i\pi\delta^{(1)}(\Delta t)-\mathcal{P}\left[\frac{1}{\Delta t^2}\right]\right)\frac{\zeta(3,1-i\Delta t/\beta)}{\beta^3}$$
$$+\frac{1}{\pi^2}\left(-\pi\delta^{(2)}(\Delta t)+\mathcal{P}\left[\frac{2i}{\Delta t^3}\right]\right)\frac{\zeta(2,1-i\Delta t/\beta)}{\beta^2}$$
$$+\frac{2}{\pi^2}\left(-i\pi\delta^{(1)}(\Delta t)+\mathcal{P}\left[\frac{1}{\Delta t^2}\right]\right)\frac{\zeta(3,1-i\Delta t/\beta)}{\beta^3}. \tag{51}$$

The five terms above correspond directly to the five terms in Eq. (47). As one can easily check in Eq. (48), only the symmetric part $\mathcal{R}_{\rm sym}(\Delta t)=[\mathcal{R}(\Delta t)+\mathcal{R}(-\Delta t)]/2$ of the response function $\mathcal{R}(\Delta t)$ contributes to the radiated energy. Symmetrizing the response function a lot of cancellations occur and all divergent terms of the structure $1/\Delta t^n$ disappear. The resulting expression reads

$$\mathcal{R}_{\rm sym}(\Delta t)=\frac{1}{12\pi}\delta^{(4)}(\Delta t)-\frac{2}{\pi}\zeta(2)T^2\delta^{(2)}(\Delta t), \tag{52}$$

with the Riemann zeta-function $\zeta(n)$ that is related to the Hurwitz zeta function $\zeta(n,m)$ via $\zeta(n,1)=\zeta(n)$. Rewriting this expression into the total radiated energy yields

$$E=\frac{1}{12\pi}\int dt\,\ddot{\eta}^2(t)+\frac{\pi}{3}T^2\int dt\,\dot{\eta}^2(t). \tag{53}$$

The first term describes the vacuum contribution and was originally obtained by Fulling and Davies [6] using the conformal invariance of the scalar field in $1+1$ dimensions and has been later calculated by Ford and Vilenkin [7] via a more flexible method of perturbations of Green's functions. In both approaches the radiated energy was deduced by means of the point-splitting renormalization technique. The relevant contributions of the Green's functions used in Refs. [6,7] correspond to the vacuum part of the response function $\mathcal{R}(\Delta t)$ in our derivation.

The second term is a pure temperature effect and generalizes the vacuum results in Refs. [6,7] to the density matrix corresponding to the canonical ensemble. The relation between the finite-temperature correction to the radiated energy and the vacuum contribution is of the order $O(T^2\tau^2)$ where τ denotes a characteristic time scale of the dynamics of the mirror.

IV. RESONANTLY VIBRATING CAVITY

Let us now investigate the finite-temperature effects on the dynamical Casimir effect in a resonantly vibrating cavity. In order to allow for an experimental verification the number of motion-induced particles should be as large as possible. One way to achieve this goal is to exploit the phenomenon of parametric resonance. It occurs in the case of periodically time-dependent perturbations characterized by some frequency ω. Within the quadratic response the number ΔN_I of created particles is proportional to the Fourier transform of the perturbation function [see Eq. (39)]. Assuming a harmonically oscillating disturbance the Fourier transform possesses a pronounced maximum at the resonance frequency. As a result particles with a mode frequency corresponding to ω will be produced predominantly. Obtaining large numbers may indicate that one has left the region, where second-order perturbation theory does apply.

A. Rotating-wave approximation

In the case of oscillating disturbances, however, it is possible to evaluate the time evolution operator $\hat{U}$ in all orders of $\hat{H}_1$ analytically employing yet another approximation, the so-called rotating-wave approximation (RWA, see, e.g., [10]). The main consequence of the RWA consists in the fact that it allows for the derivation of a time-independent effective Hamiltonian $\hat{H}_1^{\rm eff}$ after performing the integration over time

$$\int dt \hat{H}_1 \approx \mathbf{T} \hat{H}_1^{\rm eff}. \tag{54}$$

Let us assume that the explicit time dependence of the perturbation Hamiltonian $\hat{H}_1(t)$ possesses an oscillatory behavior like $\varepsilon \sin(2\omega t)$ during a sufficiently long time $\mathbf{T}$, such that the conditions $\omega \mathbf{T} \gg 1$, $\varepsilon \ll 1$ and $\varepsilon \omega \mathbf{T} = O[1]$ hold. Expanding the time evolution operator $\hat{U}$ into powers of ε and $\omega \mathbf{T}$ the RWA neglects all terms of order $O[\varepsilon^n (\omega \mathbf{T})^m]$ if $n > m$ holds. Since time integrations over oscillating functions result in smaller powers of $\mathbf{T}$ than the same integrations over time-independent quantities, within the RWA all terms including oscillations were omitted. Accordingly, only those terms, where the oscillations due to the time dependence of the operators (in the interaction picture governed by $\hat{H}_0$) and the explicit time-dependent disturbances cancel–i.e., which are in resonance $(n=m)$–contribute in the RWA. This approximation enables us to neglect the time-ordering $\mathcal{T}$ as well. The difference between the time-ordered and the original expression always contains commutators like $[\hat{H}_1(t), \hat{H}_1(t')]$. These quantities are always oscillating and therefore can be neglected within the RWA.

B. Fundamental resonance

For a harmonically vibrating cavity the effective Hamiltonian $\hat{H}_1^{\rm eff}$ can easily be calculated from the interaction operator in Eq. (35). Assuming harmonic time dependences $\sim \varepsilon \sin(2\omega t)$ or $\sim \varepsilon \cos(2\omega t)$ for both, the squeezing $[\Delta \omega_I^2(t)]$ and the velocity terms $[\mathcal{M}_{JK}(t)]$, only those terms will survive, which match the resonance conditions. These conditions are fulfilled if the oscillations of the operators, i.e., $\hat{Q}_J(t)$ and $\hat{P}_K(t)$, compensate the oscillations of the disturbances, i.e., $\Delta \omega_I^2(t)$ (squeezing) and $\mathcal{M}_{JK}(t)$ (velocity). For the squeezing term the resonance condition reads $\omega_I = \omega$ and for the velocity term $|\omega_J \pm \omega_K| = 2\omega$, respectively. Accordingly, the squeezing effect always creates particles with the frequency ω provided this cavity mode does exist. We restrict our further consideration to the situation, where the oscillation frequency ω corresponds to the lowest cavity mode, i.e., to the fundamental resonance

$$\omega = \min\{\omega_I\} = \omega_1 . \tag{55}$$

The fundamental resonance frequency ω_1 is determined by the characteristic size Λ of the cavity $\omega_1 \sim 1/\Lambda$. For the lowest mode $I=1$ the resonance condition for squeezing $\omega_I = \omega_1 = \omega$ is satisfied automatically. Although the condition for the $\mathcal{S}$ term of the velocity effect $|\omega_J + \omega_K| = 2\omega$ could be satisfied for $J=K=1$ (we assume a nondegenerate groundstate $I=1$), this term does not contribute since $\mathcal{M}_{JJ} = 0 = \mathcal{S}_{JJ}$. Whether the resonance condition for the $\mathcal{U}$ term of the velocity effect $|\omega_J - \omega_K| = 2\omega$ can be satisfied or not depends on the spectrum of the particular cavity under consideration. For a one-dimensional cavity the *eigenfrequencies* ω_I are proportional to integers and thus it can be satisfied leading to an additional velocity contribution. For most cases of higher-dimensional cavities this condition cannot be fulfilled. (See also the Appendix.) Thus the velocity effect does not contribute within the RWA (cf. Ref. [8]).

C. Squeezing operator

In the following calculations we assume a case for which only the squeezing term contributes (i.e., the rather general case). The effective Hamiltonian can be derived immediately from the only contributing $\Delta \omega_I^2$ terms

$$\begin{aligned} \int dt \hat{H}_1 &= \int dt \frac{\Delta \omega_I^2(t)}{4\omega_I} (\hat{a}_I^\dagger e^{i\omega_I t} + \hat{a}_I e^{-i\omega_I t})^2 \\ &= \frac{\omega_I \varepsilon}{2} \int_0^{\mathbf{T}} dt \sin(2\omega t) (\hat{a}_I^\dagger e^{i\omega_I t} + \hat{a}_I e^{-i\omega_I t})^2 \\ &\approx \frac{i\omega\varepsilon\mathbf{T}}{4} [(\hat{a}_1^\dagger)^2 - (\hat{a}_1)^2] = \mathbf{T} \hat{H}_1^{\rm eff}. \end{aligned} \tag{56}$$

Therefore the time evolution operator $\hat{U}$ coincides in the RWA with a squeezing operator $\hat{S}_1$ for the lowest mode $I = 1$,

$$\begin{aligned} \hat{U} &= \mathcal{T}\left[\exp\left(-i\int dt \hat{H}_1(t)\right)\right] \\ &\approx \exp\left(\frac{\omega\varepsilon\mathbf{T}}{4}[(\hat{a}_1^\dagger)^2 - (\hat{a}_1)^2]\right) = \hat{S}_1, \end{aligned} \tag{57}$$

with a squeezing parameter $\Xi = \omega \varepsilon \mathbf{T}/2$. This confirms the notion of the $\Delta \omega_I^2$ terms in the perturbation Hamiltonian (35) as squeezing contribution. Having derived a closed expression for the approximate time-evolution operator $\hat{U} \approx \hat{S}_1$ we are able to calculate the expectation value for the number operator to all orders in $\hat{H}_1^{\text{eff}}$ within the RWA

$$\langle \hat{N}_I \rangle \approx \text{Tr}\{\hat{\varrho}_0 \hat{S}_1^\dagger \hat{N}_I \hat{S}_1\} = \langle \hat{N}_I \rangle_0 + \delta_{I1} \sinh^2\left(\frac{\omega\varepsilon \mathbf{T}}{2}\right)(1+2\langle \hat{N}_1\rangle_0). \quad (58)$$

This nonperturbative result states that also at finite temperature the number of photons ΔN_1 created resonantly in the lowest cavity mode increases exponentially. The vacuum creation rate $\Delta N_1^{T=0} = \sinh^2(\omega\varepsilon\mathbf{T}/2)$ (see Ref. [8,9]) gets enhanced by a thermal distribution factor. Since the effective Hamiltonian becomes diagonal in the resonance case such a behavior is consistent with the results in Sec. II H.

D. Local quantities

So far we have considered merely the expectation values of global observables such as particle number and energy. But the canonical formalism developed here is also capable of investigating local quantities. As an example we may consider the two-point function

$$\langle \hat{\Phi}(\boldsymbol{r})\hat{\Phi}(\boldsymbol{r}')\rangle = \text{Tr}\{\hat{\varrho}\hat{\Phi}(\boldsymbol{r})\hat{\Phi}(\boldsymbol{r}')\}. \quad (59)$$

According to the results of the previous sections within the RWA the time evolution operator appears as a squeezing operator for the lowest mode $I=1$. Expanding the field $\hat{\Phi}(\boldsymbol{r})$ into the modes f_I yields

$$\langle \hat{\Phi}(\boldsymbol{r})\hat{\Phi}(\boldsymbol{r}')\rangle = \sum_{IJ} \text{Tr}\{\hat{\varrho}_0 \hat{S}_1^\dagger \hat{Q}_I \hat{Q}_J \hat{S}_1\} f_I(\boldsymbol{r}) f_J(\boldsymbol{r}'). \quad (60)$$

For $I \neq J$ the trace above vanishes and for $I=J\neq 1$ it coincides with the undisturbed (thermal) expression. Hence the only interesting case is $I=J=1$. In this situation the amplitudes $\hat{Q}_1$ are squeezed by the time-evolution operator $\hat{S}_1$. As a result the change of the correlation function induced by the dynamics of the cavity can be cast into the form

$$\begin{aligned}\Delta\langle \hat{\Phi}(\boldsymbol{r})\hat{\Phi}(\boldsymbol{r}')\rangle &= (e^{2\Xi}-1)\text{Tr}\{\hat{\varrho}_0 \hat{Q}_1^2\} f_1(\boldsymbol{r}) f_1(\boldsymbol{r}') \\ &= (e^{2\Xi}-1)(1+2\langle \hat{N}_1\rangle_0) f_1(\boldsymbol{r}) f_1(\boldsymbol{r}')/(2\omega_1) \\ &= (e^{2\Xi}-1)\left(1+\frac{2}{\exp(\beta\omega_1)-1}\right) \\ &\quad \times f_1(\boldsymbol{r}) f_1(\boldsymbol{r}')/(2\omega_1), \end{aligned} \quad (61)$$

where Ξ again denotes the squeezing parameter. In complete analogy one obtains the change of the correlation of the field momenta

$$\Delta\langle \hat{\Pi}(\boldsymbol{r})\hat{\Pi}(\boldsymbol{r}')\rangle = (e^{-2\Xi}-1)\left(1+\frac{2}{\exp(\beta\omega_1)-1}\right) \times f_1(\boldsymbol{r}) f_1(\boldsymbol{r}')\omega_1/2. \quad (62)$$

Note, that the canonical variables $\hat{Q}_I$ and the momenta $\hat{P}_I$ transform in an opposite way under squeezing: $\hat{Q}_I \to e^{\Xi}\hat{Q}_I$ whereas $\hat{P}_I \to e^{-\Xi}\hat{P}_I$.

These ingredients enable us to calculate the expectation values of the energy-momentum tensor $T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\partial_\rho\Phi\partial^\rho\Phi/2$. The expression for the change of the energy density reads

$$\Delta\langle \hat{T}_{00}\rangle = \frac{1}{4}\left(1+\frac{2}{\exp(\beta\omega_1)-1}\right) \times [(e^{2\Xi}-1)(\nabla f_1)^2/\omega_1 + (e^{-2\Xi}-1) f_1^2 \omega_1]. \quad (63)$$

The stress tensor T_{ij} consisting of purely spatial components of $T_{\mu\nu}$ can be calculated in a completely analog manner–one would obtain additional terms like $\partial_i f_1 \partial_j f_1$. This quantity can be used to deduce the mechanical properties, e.g., the pressure, of the radiation field inside the cavity. The expectation value of the energy flux density T_{0i} vanishes within the RWA, since the change of the energy is always one power of the vibration time $\mathbf{T}$ lower than the energy itself.

The above expression can be used to deduce the spatial distribution of the energy created by the dynamical perturbation of the cavity. Since the squeezing parameter Ξ is proportional to the vibration time $\mathbf{T}$, the first term at the right-hand side dominates for large time durations $\mathbf{T}$. In this situation the produced energy density behaves as $(\nabla f_1)^2$. For Dirichlet boundary conditions the *eigen*functions vanish at the boundary but their derivatives usually reach their maximum value there. In the center of the cavity the lowest *eigen*-function assumes its maximum and–consequently–its derivative vanishes. Ergo, the energy density is concentrated near the boundaries of the cavity in the case under consideration.

However, this assertion crucially depends on the imposed boundary conditions. For Neumann conditions the behavior is actually opposite.

E. Electromagnetic field

As we have observed above, the spatial distribution of the created energy density crucially depends on the imposed boundary conditions. In the case of the experimentally most relevant photon field, however, these conditions are more complicated than pure Dirichlet or Neumann-type ones. For the components of the dual field strength tensor $F^*_{\mu\nu}$ they may be expressed in the Lorentz covariant form

$$n^\mu F^*_{\mu\nu}|_{\Sigma(t)} = 0, \quad (64)$$

where n^μ denotes the (spacelike) unit vector orthogonal to the dynamical hypersurface of the boundary $\Sigma(t)$. Although the general formalism presented in Sec. II does also apply to the electromagnetic (EM) field we have restricted our considerations to a scalar field inside a trembling cavity so far.

Let us briefly discuss the main differences and common properties of the scalar and the EM field, respectively: As electromagnetism is a gauge field theory its quantization comes along with constraints (see Sec. V), which are absent for the scalar field. Due to the two polarizations (e.g., TE and TM, cf. [27]) the character of the boundary conditions for EM fields is more complicated and the derivation of its time-dependent instantaneous *eigen*modes within a dynamical cavity is less straightforward. The same holds for the explicit calculation of the velocity effect [overlap of the time-dependent *eigen*modes, cf. Eq. (32)].

However, the calculation of the squeezing contribution requires the knowledge of the (instantaneous) *eigen*frequencies only. These quantities are well known for rectangular, spherical, or cylindical cavities, see, e.g., [27]. Since for an appropriate resonantly vibrating cavity merely the squeezing effect is relevant, our results in Sec. IV can be transferred almost one-to-one to the EM field. In particular the thermal enhancement–one of the main results of our calculation, see Eq. (58)–carries over directly.

F. Discussion

In order to indicate the experimental relevance of the calculations above one may specify the characteristic parameters. For room temperature ≈ 290 K, which corresponds to thermal wavelengths of about 50 μm and considering a cavity of a typical size $\Lambda \approx 1$ cm one obtains a thermal factor $(1+2\langle \hat{N}_1\rangle_0)=O(10^3)$. As a consequence the number of photons ΔN_1 created by the dynamical Casimir effect (after the vibration time $\mathbf{T}$) at the given temperature T will be several orders of magnitude larger in comparison with the pure vacuum effect at $T=0$, see [1].

This enhancement occurring at finite temperatures could be exploited in experiments to verify the phenomenon of quantum radiation as long as back-reaction processes (and losses, etc.) can be neglected. Of course, one has also to take into account the number of photons $\langle \hat{N}_I\rangle_0$ present at the temperature T and their thermal variance

$$\sqrt{\sigma_0^2(N_I)}=\sqrt{\langle \hat{N}_I^2\rangle_0-\langle \hat{N}_I\rangle_0^2}=\sqrt{\langle \hat{N}_I\rangle_0+\langle \hat{N}_I\rangle_0^2}$$
$$=\langle \hat{N}_I\rangle_0\left[1+O\left(\frac{1}{N_I}\right)\right]. \tag{65}$$

The latter quantity reflects the statistical uncertainty when measuring the number of photons at a given temperature. In order to obtain a number of created particles ΔN_1 which is not much smaller than the corresponding thermal variance $\sqrt{\sigma_0^2(N_1)}$ one has to ensure conditions that will lead to a significant vacuum effect as well. This implies that the argument of the hyperbolic sine function in Eq. (58), i.e., the squeezing parameter $\Xi=\omega\varepsilon\mathbf{T}/2$ should be at least of order one. An estimate of the maximum value of the of the dimensionless amplitude of the resonance wall vibration $\varepsilon_{\max} < 10^{-8}$ is given in Ref. [8]. For a characteristic size of the cavity of about 1 cm corresponding to a fundamental frequency $\omega \approx 150$ GHz the squeezing parameter Ξ approaches one after several milliseconds. It still remains as a challenge whether or not the requirement $\Xi=O[1]$ could be achieved in a realistic experiment. But–provided an experimental device for generating a considerable vacuum contribution becomes feasible–there will be a strong enhancement of the dynamical Casimir effect at finite temperatures.

V. DIELECTRIC MEDIA

The previous sections were devoted to the investigation of a scalar field confined in a trembling cavity with Dirichlet boundary conditions simulating perfect conductors. Although the effect of quantum radiation has not yet been verified conclusively in an according experiment, a resonantly vibrating cavity is expected to provide one of the most promising scenarios for this aim. Of course, the assumption of perfectly conducting walls of the cavity is an idealization. One possible step towards a more realistic description is to consider a dielectric medium with a finite permittivity ϵ. Of course, one may also take into account the permeability, see, e.g., [28]. But for reasons of simplicity we restrict our further considerations to a purely dielectric medium.

The quantum radiation generated by a moving body with a finite refractive index was studied in Ref. [29], for example. More generally, one may consider a medium with an arbitrary changing permittivity $\epsilon(t,\boldsymbol{r})$ and a local velocity field $\boldsymbol{v}(t,\boldsymbol{r})$. Again these properties of the medium are treated classically, i.e., as an external background field. As the quantum field propagating in this background we consider the electromagnetic field. For nonrelativistic velocities of the medium the Lagrangian density governing the dynamics of the electromagnetic field is given by (see, e.g., [28,30])

$$\mathcal{L}=\frac{1}{2}(\epsilon\boldsymbol{E}^2-\boldsymbol{B}^2)+(\epsilon-1)\boldsymbol{v}\cdot(\boldsymbol{E}\times\boldsymbol{B}). \tag{66}$$

The particle definition for this vector field requires additional considerations. Since it is described by a gauge invariant theory, it possesses primary and secondary constraints, see, e.g., [31]. In Ref. [28] these gauge problems are solved by virtue of the reduction of variables. However, other procedures, e.g., the Dirac quantization, lead to the same results [32].

There have been various efforts to discover effects of quantum radiation for such dynamical dielectrics: One interesting idea goes back to Schwinger [33], who suggested to explain the phenomenon of sonoluminescence by this mechanism. Sonoluminescence means the conversion of sound into light. In an according experiment one generates sound waves in a liquid (water) in such a way that tiny oscillating bubbles emerge. Under appropriate conditions these bubbles emit light pulses, see, e.g., [34] and references therein. In spite of the considerable amount of work and the controversial discussions in order to clarify the relevance of quantum radiation with respect to sonoluminescence, see, e.g., [30,35,36] and also [19], there are still open questions, since the dynamics of the bubble and the behavior in its interior are not known sufficiently. We shall return to this point later on.

In view of the Lagrangian density in Eq. (66) the dynamical properties of the medium enter in two terms: In analogy to the cavity example we may distinguish between the squeezing effect due to a varying permittivity $\epsilon(t,\boldsymbol{r})$ and the velocity effect governed by $\boldsymbol{v}(t,\boldsymbol{r})$. In the following we consider situations where the squeezing term gives the dominant contribution (see Ref. [28], Sec. V) and neglect the velocity field ($\boldsymbol{v}=\boldsymbol{0}$). In contrast to the cavity example the squeezing term of the perturbation Hamiltonian will not be diagonal in general

$$\hat{H}_1(t)=\int d^3r\frac{1}{2}\left(\frac{1}{\epsilon(t,\boldsymbol{r})}-\frac{1}{\epsilon_\infty}\right)\hat{\boldsymbol{\Pi}}^2(t,\boldsymbol{r})$$
$$=\int d^3r\,\theta(t,\boldsymbol{r})\hat{\boldsymbol{\Pi}}^2(t,\boldsymbol{r}). \tag{67}$$

$\hat{\boldsymbol{\Pi}}=\hat{\boldsymbol{E}}$ denotes the canonical momentum density associated to the vector potential $\hat{\boldsymbol{A}}$, see, e.g., [28] and [17]. $\theta(t,\boldsymbol{r})$ symbolizes the deviation of the permittivity $\epsilon(t,\boldsymbol{r})$ from its asymptotic value $\epsilon_\infty=\epsilon(|t|\uparrow\infty,\boldsymbol{r})=\epsilon(t,|\boldsymbol{r}|\uparrow\infty)$.

The diagonalization of the undisturbed Hamiltonian via a particle definition can be achieved with photons labeled by $I=\{\boldsymbol{k}\nu\}$ where $\boldsymbol{k}$ denotes the wave vector of the photon and ν counts the two possible polarizations. Within this basis the $\mathcal{S}$ and $\mathcal{U}$ matrices assume the form

$$\mathcal{S}_{\boldsymbol{k}\nu,\boldsymbol{k}'\nu'}=-\frac{\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}{\mho}(\boldsymbol{e}_{\boldsymbol{k}\nu}\cdot\boldsymbol{e}_{\boldsymbol{k}'\nu'})$$
$$\times\int d^4x\,\theta(\underline{x})\exp(i(\underline{k}+\underline{k}')\underline{x})$$
$$=-\frac{\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}{\mho}(\boldsymbol{e}_{\boldsymbol{k}\nu}\cdot\boldsymbol{e}_{\boldsymbol{k}'\nu'})\tilde{\theta}(\underline{k}+\underline{k}'), \tag{68}$$

where $\mho$ denotes the quantization volume, and

$$\mathcal{U}_{\boldsymbol{k}\nu,\boldsymbol{k}'\nu'}=\frac{\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}{\mho}(\boldsymbol{e}_{\boldsymbol{k}\nu}\cdot\boldsymbol{e}_{\boldsymbol{k}'\nu'})\tilde{\theta}(\underline{k}-\underline{k}'), \tag{69}$$

respectively. In complete analogy to the cavity we may calculate of the quadratic response of the number operator

$$\Delta N_{\boldsymbol{k}\nu}=\sum_{\boldsymbol{k}'\nu'}|\mathcal{S}_{\boldsymbol{k}\nu,\boldsymbol{k}'\nu'}|^2(1+\langle\hat{N}_{\boldsymbol{k}'\nu'}\rangle_0+\langle\hat{N}_{\boldsymbol{k}\nu}\rangle_0)$$
$$+\sum_{\boldsymbol{k}'\nu'}|\mathcal{U}_{\boldsymbol{k}\nu,\boldsymbol{k}'\nu'}|^2(\langle\hat{N}_{\boldsymbol{k}'\nu'}\rangle_0-\langle\hat{N}_{\boldsymbol{k}\nu}\rangle_0). \tag{70}$$

It is again possible to recognize the thermal corrections to the vacuum effect $\Sigma_{\boldsymbol{k}'\nu'}|\mathcal{S}_{\boldsymbol{k}\nu,\boldsymbol{k}'\nu'}|^2$. In Ref. [28] we gave a general proof that for massless and not self-interacting bosonic fields at zero temperature the spectral energy density $e(\omega)$ created by smooth and localized disturbances behaves as $e(\omega)\sim\omega^4$ for small ω. As one can easily check in the equation above this is no longer valid in general at finite temperatures due to the Boltzmann distribution function that becomes singular with $1/\omega$ for small ω.

A. Small R expansion

The structure of Eq. (70) is too complicated for a general discussion of the physical properties of the induced quantum radiation by means of simple expressions. For that purpose it is necessary to use some approximations. One possibility is to assume that the region where the permittivity $\epsilon(t,\boldsymbol{r})$ differs from its asymptotic value $\epsilon_\infty=\epsilon(|t|\uparrow\infty,\boldsymbol{r})=\epsilon(t,|\boldsymbol{r}|\uparrow\infty)$ is very small. This assumption can be used to expand the Fourier transform of the perturbation function $\theta(t,\boldsymbol{r})$ in powers of R, where R denotes a characteristic length scale of the disturbance

$$\tilde{\theta}(\underline{k})=\int dt\,e^{i\omega t}\int d^3r e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\theta(t,\boldsymbol{r})=\theta_0\int dt e^{i\omega t}V(t)+O(R^4)$$
$$=\theta_0\tilde{V}(\omega)+O(R^4). \tag{71}$$

For the lowest (volume $V\sim R^3$) term of this expansion it is possible to calculate the associated radiated energy in close analogy to the moving mirror example. But in the case under consideration the evaluations have to be accomplished in 3 +1 dimensions which leads to additional scale factors k^2 and k'^2 due to the d^3k and d^3k' integrations. This results in the occurrence of Hurwitz zeta functions of higher order $\zeta[4,1-i(t-t')/\beta]$ and $\zeta[5,1-i(t-t')/\beta]$ and therefore in higher powers of the temperature T. After some calculations the lowest-order terms in R and T of the total radiated energy yield

$$E=\left(\frac{\epsilon_\infty}{2\pi}\right)^3\frac{\theta_0^2}{3\times5\times7}\left(\int dt\,V^{\cdots 2}+\zeta(4)\frac{8!}{2!4!}T^4\int dt\ddot{V}^2\right). \tag{72}$$

In analogy to the moving mirror example in Sec. III C the first term describes the (lowest) vacuum contribution, which was already obtained in Ref. [28], whereas the second term represents the (lowest) temperature correction. But in contrast to the moving mirror in 1+1 dimensions the lowest thermal correction increases with T^4 in this scenario owing to the additional k,k' integrations.

If we would have the exact data for the oscillating bubble we were able to evaluate the number of photons generated by the quantum radiation and so we could quantify the contribution of this mechanism to the phenomenon of sonoluminescence–under the assumptions made. But as these data are not known yet with sufficient accuracy, this question remains unsolved at this stage.

In addition, within the presented formalism we are able to take into account a space-time-dependent permittivity ϵ–but not effects of relaxation, dispersion, and absorption (or amplification), etc., see also Sec. II. Since these effects might well be relevant for the scenario of sonoluminescence, this provides another limitation of the direct applicability of Eq. (72).

B. Large R limit

Now we consider the opposite situation and assume that the permittivity changes over very large volumes in the same way. In such a scenario the disturbance function θ becomes nearly position-independent $\theta(t,\boldsymbol{r})\approx\theta(t)$. In this limit it is also possible to simplify Eq. (70) since the d^3r integrations in Eqs. (68) and (69) produce $\delta^3(\boldsymbol{k}\pm\boldsymbol{k}')$ distributions and therefore the mode integrations break down. As a result, the expression for the quadratic response of the number operator obeys a structure similar to the squeezing term in the cavity example

$$\Delta N_{\boldsymbol{k}\nu}=|\mathcal{S}_{\boldsymbol{k}\nu,(-\boldsymbol{k})\nu}|^2(1+2\langle\hat{N}_{\boldsymbol{k}\nu}\rangle_0). \tag{73}$$

To establish the analogy once more we note that a perturbation like $\theta(t,\boldsymbol{r})=\varepsilon\sin(2\omega t)$ generates also a generalized squeezing operator

$$\hat{U}(\Xi)=\exp\left(\frac{\Xi}{2}\sum_{\nu,|\boldsymbol{k}|=\omega}(\hat{A}^\dagger_{\boldsymbol{k}\nu}\hat{A}^\dagger_{(-\boldsymbol{k})\nu}-\text{H.c.})\right), \tag{74}$$

similar to the resonantly vibrating cavity, see also [37]. However, there is a crucial difference between the medium and the cavity: In a closed cavity there exist only particles with special discrete frequencies (*eigen*values of the Laplace operator). In contrast, for the dielectric medium without boundary conditions all positive frequencies are occupied by photons. Hence one has to vibrate a (finite) cavity with a special (resonance) frequency in order to create particles while in a medium the frequency may be arbitrary.

Investigating the two-photon correlation $\mathcal{C}_{\boldsymbol{k}\nu,\boldsymbol{k}'\nu'}$ at zero temperature one observes that in this case the photons are most probably emitted back-to-back: $\mathcal{C}_{\boldsymbol{k}\nu,\boldsymbol{k}'\nu'}(T=0)\sim\delta(\boldsymbol{k}+\boldsymbol{k}')\delta_{\nu\nu'}$. At finite temperatures, the second term in Eq. (20) gives raise to an additional negative correlation which is isotropic, i.e., it does not depend on the directions of propagation of the two photons. However, the anisotropic and temperature-independent back-to-back correlation represents one possibility to distinguish between the photons arising from the quantum radiation and the purely thermal radiation. This observation (see [35]) might perhaps be used to clarify the origin of the photons (i.e., the underlying mechanism) within the phenomenon of sonoluminescence.

VI. FRIEDMANN-ROBERTSON-WALKER METRIC

In the previous sections we focused our attention on mirrors represented by Dirichlet boundary conditions and on dielectric media. Now we are going to apply the canonical formalism developed there to yet another scenario–where the gravitational field generates quantum radiation.

According to the commonly suggested scenario the cosmological evolution starts at a stage of high temperatures. It is generally believed that the back reaction of the cosmological particle production onto the gravitational sector yields a potentially significant contribution. Consequently, it will be important to calculate the temperature effects that could affect the cosmological dynamics at very early stages.

Let us consider the minimally coupled massless scalar field propagating in the conformally flat Friedmann-Robertson-Walker space-time; see, e.g., Ref. [38] for a related calculation at zero temperature.

The Friedmann-Robertson-Walker metric represents a solution of Einstein's equations for a homogeneous and isotropic distribution of matter and describes an expanding (or contracting) universe. Depending on the density ϱ of the matter (if we omit the cosmological constant) there exist three different branches of this solution: For densities ϱ exceeding the critical density ϱ_c one obtains the closed elliptical universe, which eventually recollapses. For $\varrho<\varrho_c$ one is led to the open hyperbolic universe and for $\varrho=\varrho_c$ we get the open conformally flat Friedmann-Robertson-Walker space-time. In contrast to the first case ($\varrho>\varrho_c$) the other scenarios ($\varrho\leqslant\varrho_c$) imply an eternally expanding universe.

In order to specify the correct value of the density ϱ one has to deal with the problem of the unknown dark matter. In view of the present status of the observations it might well be possible that the density ϱ is indeed close or equal to the critical value ϱ_c, which is connected to the Hubble constant. In any case, for small space-time domains the conformally flat Friedmann-Robertson-Walker space-time should be a good approximation. In terms of the conformal coordinates $(t,\boldsymbol{r})$ the corresponding metric is given by

$$ds^2=\Omega^2(t)(dt^2-d\boldsymbol{r}^2), \tag{75}$$

where $\Omega^2(t)$ denotes the scale factor governing the Hubble expansion. It describes the change of the measure of length and time scales during the cosmological evolution, e.g., inducing the cosmological redshift.

However, the following calculations will become easier if we introduce a slightly different time coordinate $t\to\tau$ with

$$d\tau=\Omega^{-2}dt, \tag{76}$$

see also Ref. [38]. In terms of the time coordinate τ the metric can be cast into the form

$$ds^2=\Omega^6(\tau)d\tau^2-\Omega^2(\tau)d\boldsymbol{r}^2. \tag{77}$$

The action of a minimally coupled massless scalar field propagating in this particular curved space-time reads

$$\mathcal{A}=\frac{1}{2}\int d^4x\sqrt{-g}\,\partial_\mu\Phi g^{\mu\nu}\partial_\nu\Phi$$
$$=\frac{1}{2}\int d\tau\int d^3r[\dot{\Phi}^2-\Omega^4(\nabla\Phi)^2]. \tag{78}$$

As the advantage of the time coordinate τ we observe the cancellation of the scale factor in front of the $\dot{\Phi}^2$ term. Consequently the equation of motion assumes the simple form

$$\frac{\partial^2\Phi}{\partial\tau^2}=\Omega^4(\tau)\nabla^2\Phi. \tag{79}$$

After the usual canonical quantization procedure the Hamiltonian can be cast into the form

$$\hat{H}(\tau)=\frac{1}{2}\int d^3r[\hat{\Pi}^2+\Omega^4(\tau)(\nabla\hat{\Phi})^2]. \tag{80}$$

One observes a close similarity to the large R limit in Sec. V B. As we shall see below, this similarity holds also for the number of created particles.

In complete analogy to Sec. III it is possible to diagonalize this time-dependent Hamiltonian by an expansion of the field $\hat{\Phi}$ into a complete set of orthogonal *eigen*functions of the Laplace operator. Owing to the spatial homogeneity and isotropy of the space time there is an ambiguity concerning the selection of such a basis set. Here we choose the *eigen*functions to be completely time independent, $f_I = f_I(\boldsymbol{r})$. As a consequence, adopting this expansion in Eq. (30), i.e.,

$$\hat{\Phi}(t,\boldsymbol{r}) = \sum_I \hat{Q}_I(t) f_I(\boldsymbol{r}),$$

the resulting modes $\hat{Q}_I(t)$ do not obey any intermode interaction due to the spatial integration and the orthogonality and time independence of the *eigen*functions f_I. Hence, the time-dependent Hamiltonian is diagonal

$$\hat{H}(\tau) = \frac{1}{2} \sum_I [\hat{P}_I^2(\tau) + \Omega^4(\tau)\omega_I^2 \hat{Q}_I^2(\tau)], \tag{81}$$

where $-\omega_I^2$ denote the time-independent *eigen*values of the Laplacian. Now we may use the outcome of Sec. II H and we arrive at

$$\Delta N_\omega^T = \Delta N_\omega^{T=0}\left(1 + \frac{2}{\exp(\beta\omega) - 1}\right). \tag{82}$$

It should be mentioned here that the particle number above is–strictly speaking–merely a formal quantity since it does not describe particles in a well-defined and unique sense. The Friedmann-Robertson-Walker space-time is not time-translationally invariant and thus does not possess a corresponding Killing vector. Ergo, the definition of energy necessitates additional considerations. It is not possible to define a physical reasonable *and* conserved energy. Of course, this fact is consistent with the permanent particle creation. Hence, the interpretation of the above quantity ΔN_ω is not obvious–at zero as well as at finite temperatures, see also Ref. [39]. But here we are mainly interested in the influence of finite (initial) temperatures. Fortunately, the finite-temperature effects factorize out and the problems mentioned above concern the prefactor $\Delta N_\omega^{T=0}$ only. In summary we may draw the conclusion that–putting aside the problem of the interpretation of the vacuum term $\Delta N_\omega^{T=0}$–the particle creation in the Friedmann-Robertson-Walker space-time at finite (initial) temperatures gets strongly enhanced by a thermal factor in analogy to the resonantly vibrating cavity.

Note that the phenomenon of particle creation (quantum radiation) in the conformally flat Friedmann-Robertson-Walker space-time can be observed merely for fields which are not conformally invariant. As counterexamples we may quote the massless scalar field in $1+1$ dimensions, the massless and conformally coupled scalar field in $3+1$ dimensions (see also Ref. [38]), and–last but not least–the electromagnetic field in $3+1$ dimensions. Obviously the absence of any mass terms is essential for the conformal invariance. For these conformally invariant fields the equation of motion does not lead to any mixing of positive and negative frequency solutions within the conformally flat Friedmann-Robertson-Walker metric and thus no particles are created.

In view of the above observation one might object that the results of this section are almost irrelevant, since $\Delta N_\omega^{T=0} = 0$ holds for all physical reasonable fields. However–in spite of the fact that no fundamental scalar field has been definitely observed yet–there are scalar fields widely believed to exist (e.g., Higgs, inflaton, quintessence). Since our formalism is restricted to free and independently evolving fields it can only be applied after the cosmological period in which these fields decouple from the thermal bath of the remaining particles/fields.[2]

Hence, the temperature one has to insert in Eq. (82) is exactly the temperature at which this phase transition took place. In order to evaluate the importance of the finite-temperature effects in Eq. (82) one has to compare the typical frequencies of the created particles with the decoupling temperature. For a very rough estimate one may assume that particles with frequencies of the same order of magnitude as the Hubble expansion parameter are produced predominantly. But after the Planck era the temperature of the universe is considerably larger than the Hubble expansion parameter (both in energy units)–except for the period of inflation, where the temperature drops drastically. So one would expect the finite-temperature effects in Eq. (82) to be important if the scalar field under consideration does not just decouple during inflation.

Unfortunately we cannot give a more quantitative estimate of Eq. (82) and its consequences (e.g., the back reaction, see also [40]) here since the necessary data are not explicitly known yet.

VII. SUMMARY

Calculating the number of particles created by dynamical external conditions we found that for the case of a completely diagonal Hamiltonian the number of produced particle at finite temperature equals the analog quantity at zero temperature times a thermal factor.

Focusing on the scenario of a resonantly vibrating cavity we were able to derive within the RWA the effective perturbation Hamiltonian which turned out to be diagonal. As a consequence we observe an enhancement of the dynamical Casimir effect at finite temperatures as described above.

In contrast to this nonperturbative result the finite temperature corrections to the energy radiated by a single moving mirror in $1+1$ dimensions was calculated within response theory.

In close analogy we derived the energy of the photons generated by a bubble with an oscillating radius within a dielectric medium.

[2]This decoupling can be understood in complete analogy to the phase transition of recombination $e^- + p^+ \rightarrow H$ where the photon field became effectively free and now can be observed as the cosmic microwave background. In addition, the restriction of the formalism to free fields allows for treating only small fluctuations around the minimum of the (possibly nonlinear) potential.

Finally we investigated the particle production within the Friedmann-Robertson-Walker universe at finite temperatures–where the Hamiltonian can again be cast in a diagonal form.

VIII. CONCLUSION

As a main result of this paper we have presented a theoretical description of quantum radiation at finite temperatures by generalizing the canonical approach developed earlier [11,17,28,41,42]. The major advantage of this formalism is its generality and flexibility.

Depending on the characteristic scales associated to the perturbation the effects of finite temperatures represent potentially significant contributions to the quantum radiation and hence should be taken into account for realistic estimations of this striking effect.

This observation may be interpreted in the following way: Not only the vacuum fluctuations but also the thermal excitations are converted into real particles by the influence of the dynamical external conditions.

IX. DISCUSSION

For perhaps the most interesting scenario in view of a possible experimental verification of the dynamical Casimir effect–the resonantly vibrating cavity–we specified some relevant parameters in Sec. IV F.

But as it became evident in Sec. V, a cavity filled with a dielectric medium with a resonantly oscillating permittivity $\epsilon(t)=\epsilon_\infty+\Delta\epsilon\sin(2\omega t)$ generates quite similar effects. Identifying the dimensionless amplitude ε of the vibration of the cavity in the first case with the relative change of the permittivity $\Delta\epsilon$ in the second case the set of relevant parameters is completely equivalent in both situations.

It turns out that the thermal factor enhancing the dynamical Casimir effect is of order 10^3. But this enormous amplification should be contrasted to the thermal variance of the number of particles present initially which may complicate the measurement of the number of produced particles. For the relevant temperature regions both terms are of the same order of magnitude.

As a consequence the finite-temperature effects do not necessarily generate difficulties concerning the experimental verification of the dynamical Casimir effect. From our point of view there is no need to perform an experiment at low temperatures–which might be much more involved than one at room temperature.

X. OUTLOOK

There are several possibilities to extend the presented calculations to more general scenarios: First, one may drop the idealization of a perfectly conducting cavity and take losses into account [43]. In analogy, it would be interesting to study the effects of a nontrivial dispersion relation of a dielectric medium in view of the phenomenon of quantum radiation. This might be especially interesting for the investigation of the dielectric black hole analogs; see, e.g., [44]. The investigation of the phenomenon of quantum radiation for interacting fields obeying nonlinear equations of motion is rather challenging.

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APPENDIX

In the previous publication [1] we studied the possibility of satisfying the resonance condition for the velocity term (see Sec. IV B) and stated: *For most cases of higher-dimensional cavities, e.g., a cubic one, this condition cannot be fulfilled.* Whereas the main statement is correct, the example of the *cubic* cavity is particularly unfortunate–since in this special case the resonance condition *can* be fulfilled, see also [9]. For the rather general case of a rectangular (or cylindric) cavity with transcendental ratios of the cavity dimensions the resonance conditions for the squeezing and the velocity term cannot be matched simultaneously. With regard to an experimental verification of the dynamical Casimir effect it appears to be reasonable to utilize such a cavity with the *eigen*frequencies being well separated from the velocity resonance conditions–instead of e.g., a cubic one–since the occurrence of the velocity term lowers the resonant particle creation rate in general.

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