

Robustness of nonclassical superposition states against decoherence

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We make a comparative study of quadrature squeezing, photon-number distribution, and Wigner function in a decayed quantum system. Specifically, for a field mode prepared initially the Schrödinger cat states interacting with a zero-temperature environment, we show that the rate of reduction of the nonclassical effects in this system is proportional to the occurrence of the decoherence process.

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I. INTRODUCTION

Decoherence process represents the transformation of superposition states into statistical mixture states, i.e., the off-diagonal elements of the system are suppressed. This can occur through, e.g., the interaction between system and environment. Actually, the decoherence process is important not only for understanding the quantum-classical transition [1], but also it may eventually be useful for applications that require keeping coherence in mesoscopic or macroscopic systems, such as quantum computation [2]. Furthermore, the decoherence is at the heart of the quantum theory of measurement [3].

On the other hand, superposition principle is at the heart of the quantum mechanics. It implies that probability densities of observable quantities usually exhibit interference effects instead of simply being added. The most significant examples reflecting the power of such a principle are the macroscopic quantum interference (Schrödinger cat) states [4], which exhibit various nonclassical effects, such as squeezing, sub-Poissonian statistics, and oscillations in photon-number distribution [5–7], even if their components are close to the classical ones [8], i.e., they are minimum uncertainty states and exhibit Poissonian distribution. These states can be defined as

$$|\alpha\rangle_\phi = A^{1/2} [|\alpha\rangle + \exp(i\phi)|-\alpha\rangle], \quad (1)$$

where $|\alpha\rangle$ is a coherent state with complex amplitude α , ϕ is a relative phase, and A is the normalization constant having the form

$$A = \frac{1}{2[1 + \exp(-2|\alpha|^2)\cos\phi]}. \quad (2)$$

Specifically, for $\phi=0$, π , and $\pi/2$ state (1) reduces to even coherent (ECS), odd coherent (OCS), and Yurke-Stoler (YSS) states, respectively. It is worth mentioning that there are two regimes controlling the behavior of the states (1), which are microscopic regime for small values of $|\alpha|$ (i.e., when the “distance” between the components of the cat is small) and macroscopic regime for large values of $|\alpha|$ [9]. In fact, these states are more nonclassical in the microscopic regime. In other words, the amount of nonclassical effects, such as the negative values in Wigner function and the os-

cillatory behavior in the photon-number distribution for these types of states are more pronounced in the microscopic regime than in the macroscopic regime. For more details about states (1), such as their generations and their properties when they are evolving in various optical systems, one can consult the paper [10], and references therein.

In this paper we study the relation between the decoherence process and the occurrence of the nonclassical effects in a decayed quantum system. More precisely, we compare development of nonclassical effects in both quadrature squeezing and photon-number distribution with the occurrence of interference pattern in the Wigner function. We perform such a comparison for the field mode prepared initially in the state (1) (described by the density matrix $\hat{\rho}$), which interacts with zero-temperature environment. The master equation in the Born-Markov approximation describing the system is [11]

$$\frac{d\hat{\rho}}{dt} = \frac{\gamma}{2} (2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}), \quad (3)$$

where γ is the decay constant and \hat{a} (\hat{a}^\dagger) is the annihilation (creation) operator designated to the mode of the field. The well-known time dependent solution for Eq. (3) is [12]

$$\hat{\rho}(t) = A \sum_{j,j'=1}^2 \exp(i\phi_{jj'}) \langle \alpha_j | \alpha_{j'} \rangle^{1-\mu} |\sqrt{\mu}\alpha_j\rangle \langle \sqrt{\mu}\alpha_{j'}|, \quad (4)$$

where $\alpha_1 = \alpha$, $\alpha_2 = -\alpha$, $\mu = \exp(-\tau)$, $\tau = t\gamma$ is the scaled decaying parameter and

$$\phi_{jj'} = \begin{cases} 0 & \text{for } j=j' \\ \phi & \text{for } j>j' \\ -\phi & \text{for } j<j'. \end{cases} \quad (5)$$

II. QUADRATURE SQUEEZING

As is well known squeezing is one of the most important phenomena in quantum optics because of its applications in various areas, e.g., in optics communication, quantum information theory, etc. [13]. Squeezed light can be measured by a homodyne detection where the signal is superimposed on a strong coherent beam of the local oscillator.

Here we investigate quadrature squeezing for the density

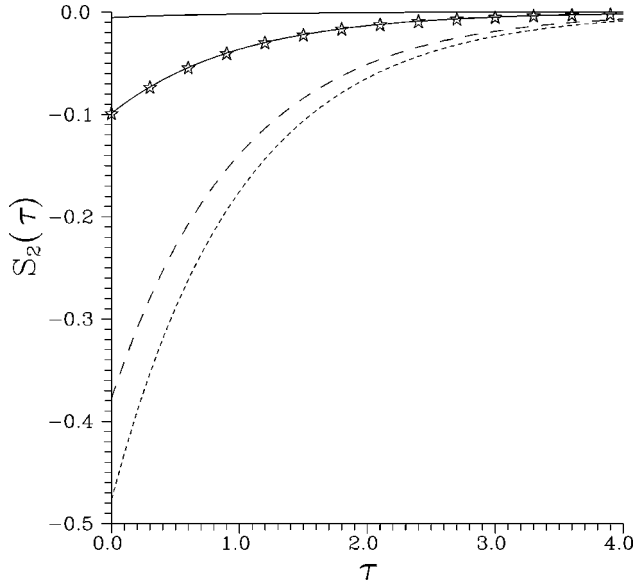


FIG. 1. Squeezing factor $S_2(\tau)$ of ECS case against the scaled decaying parameter τ for $\alpha=2$ (solid curve), 1.5 (star-centered curve), 1 (short-dashed curve), and 0.5 (long-dashed curve).

matrix (4). For this purpose we define the position and momentum operators, which are related to the conjugate electric and magnetic-field operators \hat{E} and \hat{H} of electromagnetic waves, as

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{Y} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger), \quad (6)$$

where $[\hat{X}, \hat{Y}] = i/2$; then the uncertainty relation reads $\langle(\Delta\hat{X})^2\rangle\langle(\Delta\hat{Y})^2\rangle \geq \frac{1}{16}$, where $\langle(\Delta\hat{X})^2\rangle = \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2$. Therefore, we can say that the mode is squeezed if $S_1(t) = 4\langle(\Delta\hat{X})^2\rangle - 1 < 0$ or $S_2(t) = 4\langle(\Delta\hat{Y})^2\rangle - 1 < 0$.

Now squeezing factors $S_j(t)$ for the system under consideration—restricting ourselves to ECS case—take the forms

$$S_1(t) = \mu S_1(0) = \frac{4\mu\alpha^2}{1 + \exp(-2\alpha^2)},$$

$$S_2(t) = \mu S_2(0) = \frac{-4\mu\alpha^2 \exp(-2\alpha^2)}{1 + \exp(-2\alpha^2)}, \quad (7)$$

where $S_j(0)$ are the initial squeezing factors. Here we have considered α to be real. From these expressions it is clear that the quantum fluctuation of the field decreases exponentially as a result of its interaction with environment. More precisely, $S_j(t)$ decay at the same rate as the intensity of the field [14]. Further, we see that squeezing exists provided that α and τ are finite. Of course, the origin of these nonclassical effects is in the interference between the components of the cat. One can also check that when τ is large enough, squeezing factors $S_j(t)$, $j=1,2$ tend to zero. In other words, the system tends to a steady state, which, in this case, is a pure state (vacuum state). In Fig. 1 we plot $S_2(\tau)$ for shown val-

ues of the parameters. Generally we can see that squeezing is more pronounced in the microscopic regime (when α is small). Furthermore, for $\alpha \geq 2$ the squeezing factor tends to zero regardless of the values of τ . In this case the system becomes a statistical mixture state or a vacuum state if the values of τ are small or large. This point will be clear when investigating the behavior of the Wigner function in Sec. IV.

III. PHOTON-NUMBER DISTRIBUTION

Photon-number distribution $P(n)$ is an integral part of the modern description of light, which can be measured by photon detectors based on the photoelectric effect. Further, one of the most interesting nonclassical effects emerging from the superposition principle is the oscillatory behavior in $P(n)$. In general, such behavior is closely related to that of the Wigner function, however, this is a necessary but not sufficient condition. For example, the $P(n)$ of ECS, OCS, and YSS are completely different; whereas those of ECS and OCS exhibit pairwise oscillations in phase space (even number of photons can be observed for ECS and odd numbers for OCS), the distribution of YSS is a Poissonian even though the behavior of the Wigner function for these states is qualitatively similar. The second issue we want to address here is that in general the occurrence of squeezing in the quadrature variances does not need to be accompanied by oscillations in the $P(n)$ and vice versa. For instance, binomial states [15] can exhibit quadrature squeezing even though their $P(n)$ are close to Poissonian ones. In the same spirit, for ECS the oscillations in $P(n)$ are more pronounced when the “distance” between the basis of the cat increases, however, this is not the case for the quadrature squeezing, which is completely suppressed for $\alpha \geq 2$ (see Fig. 1).

Now we investigate the sensitivity of the $P(n)$ of the system under consideration to lossy mechanism. This quantity can be calculated easily [$P(n) = \langle n | \hat{\rho}(t) | n \rangle$] and one obtains

$$P(n) = 2A \frac{(\sqrt{\mu}\alpha)^{2n}}{n!} \exp(-\mu\alpha^2) \{1 + (-1)^n f(\alpha) \cos \phi\}, \quad (8)$$

where

$$f(\alpha) = \exp[-2\alpha^2(1-\mu)]. \quad (9)$$

By comparing the expression (7) with Eq. (8) we find that the dissipation is involved in two quantities by different ways and consequently the sensitivity of these quantities to lossy mechanism is completely different. As before, the origin of the nonclassical oscillations in the $P(n)$ lies in the interference in phase space. Further, in Eq. (8) the interference term is decaying by the factor $f(\alpha)$ [11] and thus its contribution is more pronounced—oscillatory behavior can occur in $P(n)$ —when α and μ are small. This situation is similar to that of the quadrature squeezing. We will discuss this point quantitatively in Sec. IV by investigating the behavior of the factor $f(\alpha)$. In Fig. 2 we plot the $P(n)$ for ECS against n for microscopic (a) and macroscopic (b) regimes,

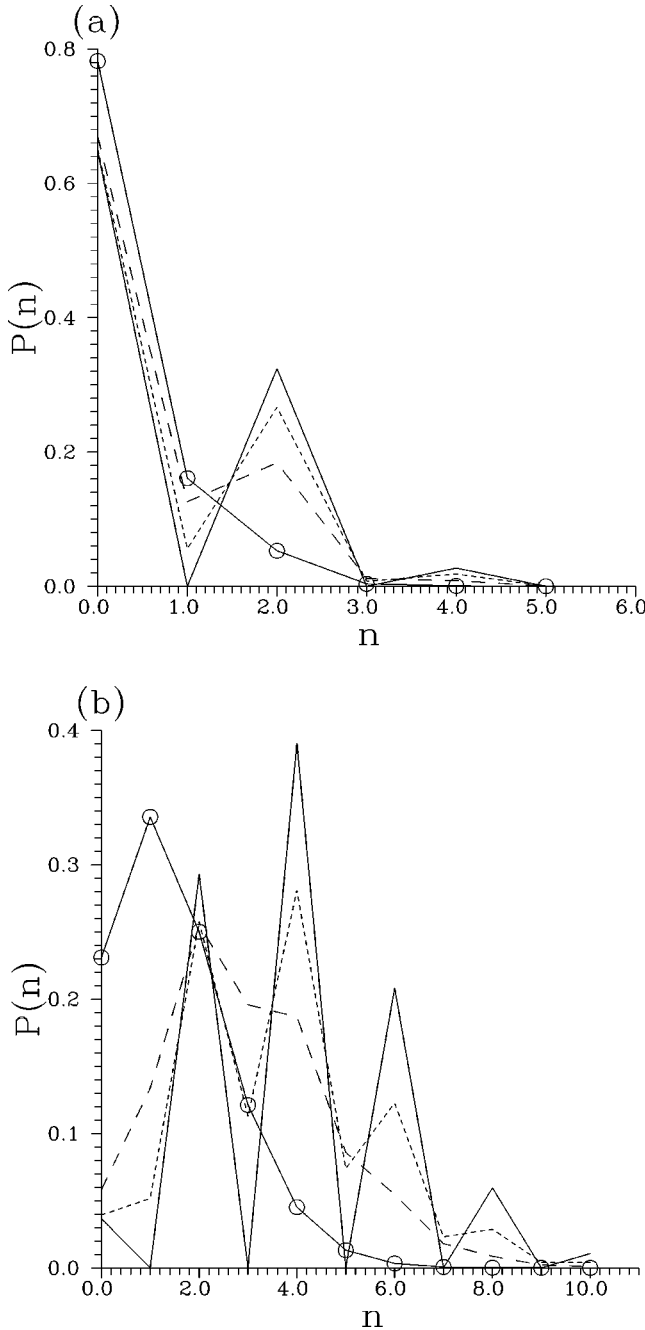


FIG. 2. $P(n)$ of ECS case against n for $\alpha=1$ (a), 2 (b) and for $\tau=0$ (solid curve), 0.1 (short-dashed curve), 0.3 (long-dashed curve), and 1 (circle-centered curve).

respectively, for given values of the parameters. By comparing the curves in Fig. 2(a) with those having the same values of τ in Fig. 2(b), one can conclude that the oscillations in $P(n)$ for macroscopic regime are suppressed faster than those for microscopic regime provided that τ is small. Also the comparison between the behavior of both the short-dashed curves in Fig. 1 and in Fig. 2(a) shows that the $P(n)$ is more sensitive to dissipation than the quadrature squeezing is. This is clear as one can observe that the oscillations in the $P(n)$ are completely suppressed, however, squeezing is still remarkable in the quadrature squeezing. The final remark is

that the behavior of the $P(n)$ when $\tau=0.3,1$ in Fig. 2 is close to that for the statistical mixture of coherent states under the influence of the decay mechanism.

IV. WIGNER FUNCTION

Wigner (W) function is one of the quasiprobability functions, which carries full information about the quantum system. This function is sensitive to the interference in phase space and can be realized in optical homodyne tomography [16]. Here we use this function to study the decoherence of the system under discussion. The definition of the decoherence has been given in the Introduction.

The W function can be defined as

$$W(\beta, t) = \frac{1}{\pi^2} \int d^2 \zeta \exp(\beta \zeta^* - \beta^* \zeta) C^{(w)}(\zeta, t), \quad (10)$$

where $C^{(w)}(\zeta, t)$ is the symmetrically ordered characteristic function having the form

$$C^{(w)}(\zeta, t) = \text{Tr}[\hat{\rho}(t) \exp(\hat{a}^\dagger \zeta - \hat{a} \zeta^*)], \quad (11)$$

where $\hat{\rho}(t)$ is the density matrix of the system, which for the system under consideration is given by Eq. (4). For the future purpose, we derive the W function following the same steps as in Ref. [8]. Thus we rewrite the W function in terms of the normally ordered moments of the creation and annihilation operators using the Baker-Hausdorff theorem. Therefore Eq. (10) takes the form

$$W(\beta, t) = \frac{1}{\pi^2} \sum_{n, m=0}^{\infty} \frac{\langle \hat{a}^{\dagger m}(t) \hat{a}^n(t) \rangle}{n! m!} I_{mn}, \quad (12)$$

where we have used the abbreviation

$$\begin{aligned} I_{mn} &= \int d^2 \zeta \exp\left(-\frac{1}{2}|\zeta|^2 + \zeta^* \beta - \zeta \beta^*\right) \zeta^m (-\zeta^*)^n \\ &\equiv (-1)^{n+m} \frac{\partial^{m+n}}{\partial \beta^{*m} \partial \beta^n} \\ &\quad \times \int d^2 \zeta \exp\left(-\frac{1}{2}|\zeta|^2 + \zeta^* \beta - \zeta \beta^*\right). \end{aligned} \quad (13)$$

Carrying out the integration in Eq. (13) we obtain

$$I_{mn} = 2\pi (-1)^{n+m} \frac{\partial^{m+n}}{\partial \beta^{*m} \partial \beta^n} \exp(-2|\beta|^2). \quad (14)$$

After minor algebra and using the formula of Rodrigues' Laguerre polynomial, Eq. (14) reads

$$I_{mn} = 2^{n+1} \pi (-1)^m \beta^{*(n-m)} m! L_m^{n-m}(2|\beta|^2) \exp(-2|\beta|^2), \quad (15)$$

where $L_m^k(\cdot)$ are the associated Laguerre polynomials of order m .

On the other hand, the normally ordered expectation values $\langle \hat{a}^{\dagger m}(t) \hat{a}^n(t) \rangle$ associated with the density matrix (4) are given as [17]

$$\langle \hat{a}^{\dagger m}(t) \hat{a}^n(t) \rangle = A \sum_{j,j'=1}^2 \exp(i\phi_{jj'}) \langle \alpha_j | \alpha_{j'} \rangle \alpha_j^n \alpha_{j'}^{*m} \mu^{(m+n)/2}. \quad (16)$$

On substituting Eqs. (15) and (16) into Eq. (12) we arrive at

$$\begin{aligned} W(\beta, t) = & \frac{A \exp(-2|\beta|^2)}{\pi} \sum_{n,m=0}^{\infty} \left[\sum_{j,j'=1}^2 \exp(i\phi_{jj'}) \right. \\ & \left. \times \langle \alpha_j | \alpha_{j'} \rangle \alpha_j^n \alpha_{j'}^{*m} \right] \\ & \times \frac{(-1)^m 2^{n+1} \mu^{(n+m)/2}}{n! \beta^{*(m-n)}} L_m^{n-m}(2|\beta|^2). \quad (17) \end{aligned}$$

On using the generating function for Laguerre polynomials and the Taylor's expansion for the exponential function, Eq. (17) reduces to the following closed form:

$$\begin{aligned} W(\beta, t) = & \frac{2A}{\pi} [\exp(-2|\beta - \alpha\sqrt{\mu}|^2) + \exp(-2|\beta + \alpha\sqrt{\mu}|^2) \\ & + 2f(\alpha) \cos \phi \cos(4\alpha\sqrt{\mu} \text{Im}\beta) \exp(-2|\beta|^2)], \quad (18) \end{aligned}$$

where $f(\alpha)$ is given by Eq. (9).

In general, the W function of ECS, OCS, and YSS (at $t=0$) are consisting of two Gaussian bells corresponding to statistical mixture of individual composite states and interference fringes in between originating from the superposition between different components of the states. Actually, these fringes represent the signature of the nonclassical effects. For this reason several papers have been devoted to dealing with these fringes making them less or more pronounced by allowing the cat states to evolve in different quantum optical systems (e.g., see [10], and references therein). For the system under consideration we can easily conclude from Eq. (18) that as the interaction of the system with the environment is going on, the two Gaussian peaks of the statistical-mixture part move towards the origin and eventually merge into each other. This is quite obvious since the centers of the peaks are exponentially decaying function of time. Furthermore, the amplitude of the oscillatory term goes down by the factor $f(\alpha)$ similar to the $P(n)$. Such behavior can be explained as the flux of coherent energy transfers to the environment from the field and noise transfers to the field from the environment. More information about the system can be observed in Figs. 3(a) and 3(b) where we plot $W(\beta=x+iy)$ function for microscopic ($\alpha=1$) and macroscopic ($\alpha=2$) regimes, respectively. In both cases the scaled decaying parameter $\tau=0.3$. From Fig. 3(b) it is clear that the optical cavity field tends to an approximate statistical-mixture state, i.e., to a two-peak structure with negligible interference part.

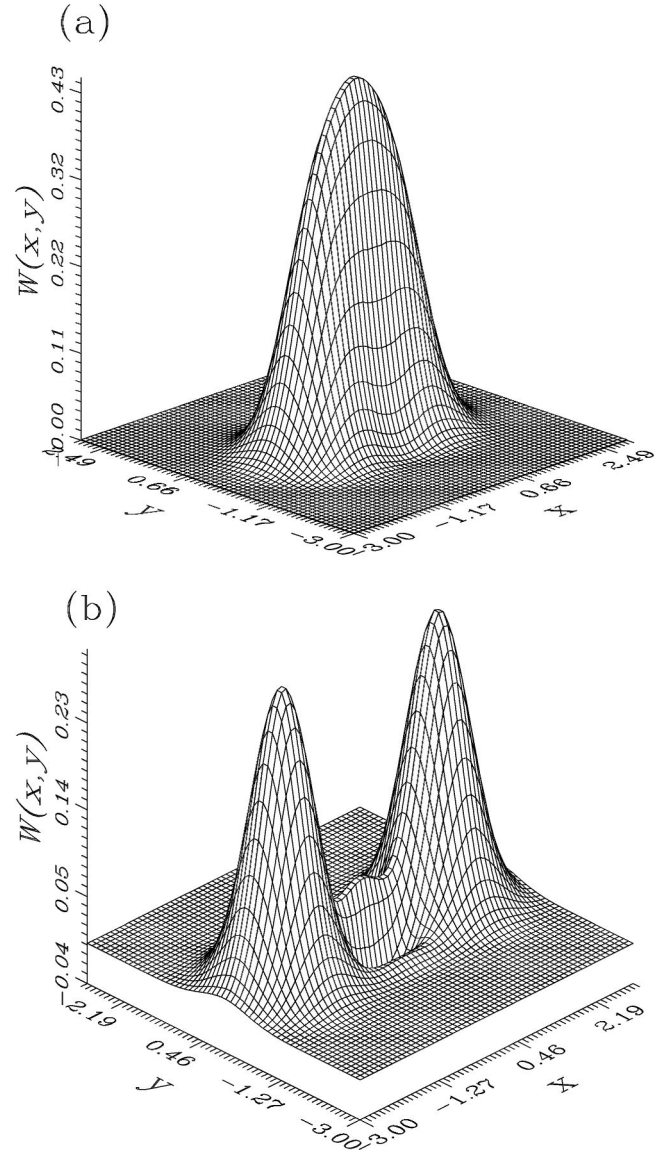


FIG. 3. The W function of ECS case for $(\alpha, \tau) = (1, 0.3)$ (a); $(\alpha, \tau) = (2, 0.3)$ (b).

Actually the suppression of the nonclassical interference pattern in the W function does not mean that the system reaches its equilibrium states [18]. Further for large interaction times the cavity field collapses to vacuum state irrespective of the type of the initial cat state. This can be checked from Eq. (18) as well as can be clearly seen in Figs. 8 and 9 in Ref. [8] (see curve 5 in these figures). This means that the superpositions of macroscopical cat states can be realized, but to have them surviving for some time the system must be completely isolated. Even a very slight interaction with the environment will very quickly reduce the superpositions to the corresponding statistical mixture states.

Now we turn our attention to the microscopic case [Fig. 3(a)]. From this figure one can observe that the noise ellipse related to squeezed states is similar to that of squeezed vacuum states. The origin of this behavior is in the competition between the diagonal and off-diagonal elements of the system. Actually, in the microscopic regime the contributions

of the statistical-mixture components are located close to the origin of the phase space. Furthermore, the comparison between the behavior of quadrature squeezing, photon-number distribution, and W function [i.e., the comparison of Fig. 1, Fig. 2, and Figs. 3(a) and 3(b) for the specified values of the parameters] shows that the occurrence of the nonclassical effects and decoherence phenomenon are qualitatively on the same level. More precisely, the more the system decoheres, the more the nonclassical effects decrease. This conclusion is completely different from that in Ref. [8]. The reason for this difference is that when the authors of Ref. [8] compare the decay of the interference part in phase space based on W function (Fig. 6) with the behavior of the quadrature squeezing (Figs. 8 and 9) they chose the field amplitude $\alpha=2$, for which squeezing does not exist. Therefore both developments cannot be compared and thus they arrived at misleading conclusions. Furthermore, they explained their results by using a series form for the W function [see (17); further in Eq. (5.13) of Ref. [8] there is a misprint in this expression, where the square root should not be in its denominator) and concluded that “it is clearly seen that the Wigner function always decays faster than the second-order squeezing.” Actually, this discussion is not persuasive because the expansion contains the terms of both the mixture and the interference components symmetrically.

We conclude by giving a quantitative analysis of the factor $f(\alpha)$ in Fig. 4. Such analysis can give insight into the occurrence (or nonoccurrence) of the decoherence process in regard to the values of α and the interaction time. As is clear from Eq. (9), $f(\alpha)$ exponentially decays whenever α increases provided that $\mu \neq 1$ (i.e., $\tau \neq 0$) and has its minimum value at $\mu = 0$ (i.e., τ is very large). Actually, Fig. 4, even if it is relatively simple, can give the smallest values of α for which the system can be completely decohered for certain values of the interaction time. For instance, for $\tau=0.1, 0.3, 0.8$ the corresponding smallest values are $\alpha=5, 3, 2$ for which the system is completely decohered. In this case the density matrix describing the system has typically the form $\hat{\rho}_\tau(t) = \frac{1}{2} [|\alpha_t\rangle\langle\alpha_t| + |-\alpha_t\rangle\langle-\alpha_t|]$ where $\alpha_t = \alpha\sqrt{\mu}$ [7]. It is clear that these results agree with the fact that the nonclassical effects occur in the microscopic regime. Finally, it is worth mentioning that the decoherence in the present system can be overcome by including amplifying media in the cavity [19].

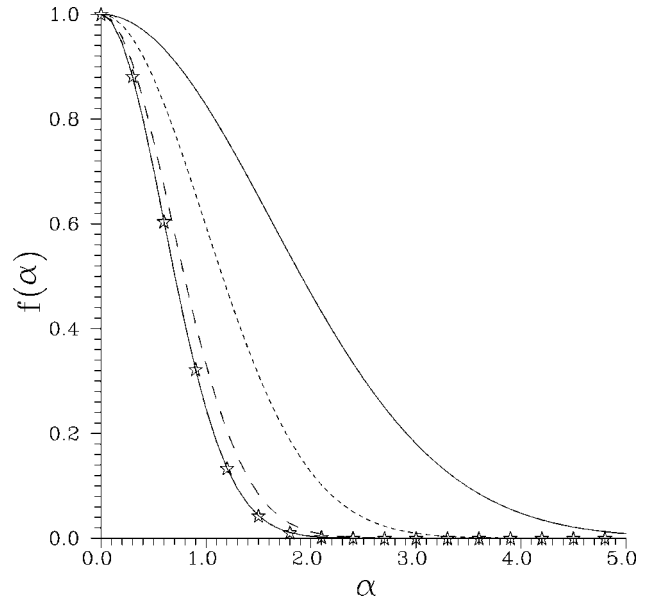


FIG. 4. The function $f(\alpha)$ against α for $\tau=0.1$ (solid curve), 0.3 (short-dashed curve), 0.8 (long-dashed curve), and 1.2 (star-centered curve).

In conclusion, we have shown that the sensitivity of quadrature squeezing and W function to lossy mechanism is on the same level. This is not a surprising result since the W function is built on the complementarity of the canonical operators [20]. On the other hand, the $P(n)$ is more sensitive to dissipation than the quadrature squeezing. Furthermore, the decoherence process is more visible in the macroscopic regime. Thus in a more realistic situation the generation and detection of a macroscopic superposition state is very difficult due to the unavoidable coupling with environment and the consequent dissipation [21]. Finally in the view of the quantities studied here, the nonclassical superposition states cannot be saved from decoherence.

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