

# Characterization of distillable and activatable states using entanglement witnesses

B. Kraus,<sup>1</sup> M. Lewenstein,<sup>2</sup> and J. I. Cirac<sup>1</sup>

<sup>1</sup>*Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria*

<sup>2</sup>*Institut für Theoretische Physik, Universität Hannover, Hannover, Germany*

(Received 13 November 2001; published 10 April 2002)

We introduce a formalism that connects entanglement witnesses and the distillation and activation properties of a state. We apply this formalism to two cases: First, we rederive the results presented in Eggeling *et al.* (e-print quant-ph/0104095), namely, that one copy of any bipartite state with nonpositive partial transpose (NPPT) is either distillable, or activatable. Second, we show that there exist three-partite NPPT states, with the property that two copies can neither be distilled, nor activated.

DOI: 10.1103/PhysRevA.65.042327

PACS number(s): 03.67.Hk, 03.65.Ta, 03.65.Ca

## I. INTRODUCTION

Entanglement is one of the most fascinating features in quantum mechanics. It has been shown that maximally entangled states of two parties can be used in many applications of quantum information [1]. For instance one can teleport a state of a particle to another particle [2], which is spatially separated from it. It was also shown how to use an entangled state to send secret messages from one place to another [3]. In most of those proposals one needs pure maximally entangled states. In reality, however, the states which are produced in the laboratories are, due to the interaction with the environment, mixed.

It was shown by Bennett *et al.* [4], Deutsch *et al.* [5], and by Gisin [6], how to obtain, out of some copies of an entangled mixed state, pure maximally entangled states, using only local operations and classical communication [7,8]. This process is called distillation. Later on the Horodecki family proved that any entangled state of two qubits (two-level) systems can be distilled to a maximally entangled state [9]. They also showed that a necessary condition for distillability is that the partial transpose (3) of the density operator must be nonpositive semidefinite [10]. In higher dimensions, however, there exist states, fulfilling this necessary condition for distillation, but it is not possible to transform some copies of this state into a maximally entangled state [11,12]. Then there is still the possibility to distill some entanglement, if one allows Alice and Bob to share, in addition to their states, an entangled state, whose partial transpose is positive (PPTES) [13]. This process is called activation, since the entanglement contained in the copies of the original state is activated by a PPTES.

Recently, it has been shown [14] that one copy of any bipartite state can always be either distilled or activated by a PPTES [15]. We will show that this is not the case if we consider systems composed of more than two subsystems. That is, we give an example of a three-partite state, where one copy (even two copies) of it can neither be distilled, nor activated. We will introduce a formalism, which allows us to connect the problem of entanglement witnesses (EW's), which are observables that allow us to detect entangled states, to the problem of distillation and activation, for arbitrary states [16].

This paper is divided into four sections. In Sec. II, we

introduce our notation and summarize some known results concerning separability. There, we will also generalize the notion of entanglement witnesses (EW's) and some results concerning EW's [17,18] to more than two systems [17,19,20]. Then we review the results on distillation and activation of entanglement [9]. The technical details, concerning EW's, but also the notion of completely positive maps (CPM's) are written in an appendix. There we also reformulate the problem of distillation and activation of entanglement in terms of CPM's. The reason for that is that the main results of this paper can be understood without those technical details, but we need them to prove our statements. Section III is divided into two parts. In the first part we consider a density operator,  $\rho$  which describes the state of a system composed of two subsystems. For an arbitrary number, say  $N$ , of copies of this state we define an operator  $W_{\rho^{\otimes N}}$ . Then we show that those  $N$  copies of the state can be distilled if and only if (iff)  $W_{\rho^{\otimes N}}$  is not an EW, the entanglement of those  $N$  copies can be activated via a PPTES iff  $W_{\rho^{\otimes N}}$  is a special EW, namely, a nondecomposable EW (NDEW). Furthermore, if the entanglement of a state can be distilled, then the introduced formalism gives us a distillation protocol. If the entanglement can be activated, then we know which PPTES activates it. In the second part of Sec. III, we generalize those results to density operators describing more than two parties, by concentrating on three systems. Section IV contains two applications of the formalism developed. First we rederive the above-explained result presented in [14] in a simple manner. Second, we present an example of a three-partite state, whose partial transpose is not positive semidefinite, but, nevertheless one copy of the state can neither be distilled, nor activated. We then show that even if we consider two copies of some of those states the entanglement can neither be distilled, nor activated. Section V contains a summary of the results.

## II. NOTATION AND REVIEW

The aim of this section is twofold. On the one hand we introduce our notation and on the other hand we summarize some known facts which we use to prove the main results of this paper. In the first subsection we recall the notion of separability. Then we generalize the results concerning entanglement witnesses [17,18] to more than two parties [20].

In the last section we recall the notion of distillation and activation of entanglement. As mentioned before, the technical details concerning EW's, the notion of CPM's, and the connection between CPM's and the distillation and activation problem can be found in the appendices.

Throughout this paper we denote by  $\{|1\rangle, \dots, |d\rangle\}$  the computational basis in  $\mathbb{C}^d$ . Whenever we consider two or more systems,  $A, B, \dots$ , we use the notation  $|i\rangle_A |j\rangle_B = |i, j\rangle_{AB}$  and if it is clear to which Hilbert spaces the states belong to then we omit the subscripts. For instance we write the (unnormalized) maximally entangled state in  $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$  as

$$|\Phi_d\rangle = \sum_{k=1}^d |k, k\rangle, \quad (1)$$

where  $d = \min\{d_1, d_2\}$ . In the following, the superscript  $T$  denotes the transposition in the computational basis. By  $\mathcal{B}(\mathcal{H})$  we denote the Hilbert space of bounded operators acting on the Hilbert space  $\mathcal{H}$ . Furthermore we denote by  $\mathbb{1}_n$  the identity matrix acting on a  $n$ -dimensional Hilbert space. Most of the paper will deal with composite systems. In this case the Hilbert spaces of the spatially separated systems will be denoted by  $\mathcal{H}_X$ , with  $\dim(\mathcal{H}_X) = d_X$ . We will only consider the nontrivial situations when  $d_X \geq 2$ . Whenever we consider the situation where Alice, Bob, Charly,  $\dots$ , have more than one particle we will denote them by  $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n, C_1, C_2, \dots, C_n, \dots$ , respectively. The corresponding total Hilbert space of each party is then  $\mathcal{H}_X = \mathcal{H}_{X_1} \otimes \dots \otimes \mathcal{H}_{X_n}$ , for  $X \in \{A, B, C, \dots\}$ . We will also use the notation  $\mathcal{H}_i = \mathcal{H}_{A_i} \otimes \dots \otimes \mathcal{H}_{X_i}$ , for  $i \in \{1, \dots, n\}$ . Capital letters as sub- or superscript will indicate on which system an operator acts on, e.g.,  $O^A$  denotes an operator acting on  $\mathcal{H}_A$ . For simplicity we will not normalize the states which we consider.

Let us now recall some facts concerning separability, entanglement witnesses, and distillation and activation properties. Throughout this section we consider a density operator,  $\rho$ , describing the state of several, say  $N$ , spatially separated systems. The Hilbert space on which  $\rho$  acts is  $\mathcal{H} = \mathcal{H}_A \otimes \dots \otimes \mathcal{H}_Z$ .

### A. Separability

A state  $\rho$  is called fully separable if it can be prepared, using only local operations and classical communication (LOCC), out of a product state, e.g.,  $|0, \dots, 0\rangle$ . Equivalently, a state  $\rho$  is fully separable iff it can be written as

$$\rho = \sum_i p_i |a_i\rangle_A \langle a_i| \otimes \dots \otimes |z_i\rangle_Z \langle z_i|, \quad (2)$$

where  $p_i \geq 0$  and the  $|x_i\rangle$  belong to the Hilbert space of particle  $X$ . If  $\rho$  cannot be written as Eq. (2), it is inseparable (entangled). In what follows we simply call a state or a map separable if it is fully separable. It was shown by Peres [21] and the Horodecki family [22] that a density matrix  $\rho$  which describes the state of two qubits ( $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ ), or one qubit and a three-level system ( $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3$ ) is separable iff its partial transpose is positive semidefinite. Here, the partial

transpose of an Hermitian operator  $O$  with respect to system  $Y$  in the computational basis [23] is defined as

$$O^{T_Y} = \sum_{i,j=1}^{d_Y} {}_y \langle i | O | j \rangle_y |j\rangle_y \langle i|. \quad (3)$$

From the condition (2) it can be easily seen that a state  $\rho$  is separable iff  $\rho^{T_Y}$  is separable, for any system  $Y$ . Thus, it is clear that the partial transpose of a separable state is positive semidefinite. In the following we call a state  $\rho$ ,  $Y$ -PPT if its partial transpose with respect to system  $Y$  is a positive semidefinite operator. Otherwise we call it  $Y$ -NPPT. If a state has a positive (nonpositive) semidefinite partial transpose with respect to all systems then we call it simply PPT (NPPT). If we consider, for instance, only two systems, we have  $\rho^{T_B} = (\rho^{T_A})^T$  and therefore  $\rho$  is  $A$ -NPPT iff it is  $B$ -NPPT; we call such a state NPPT. Throughout the paper we use the fact that  $\text{tr}(\rho X^{T_A}) = \text{tr}(\rho^{T_A} X)$ .

Note that in higher dimensions ( $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^m$ ), where  $n + m > 5$ , or if the state describes a system composed of more than two parties, there exist entangled states whose partial transposes are positive semidefinite operators (PPTES) [24]. Thus, in this case the positivity of the partial transpose is no longer a sufficient, but only a necessary condition for separability. We will see later on that partial transposition also plays an important role in establishing the distillation and activation properties of a state.

### B. Entanglement witnesses

We call an operator,  $W = W^\dagger$ , acting on  $\mathcal{H}$  an  $N$ -partite entanglement witness (EW) (between the  $N$  parties) if the following properties are fulfilled:

(i)  $\langle a, \dots, z | W | a, \dots, z \rangle \geq 0 \quad \forall |a\rangle \in \mathcal{H}_A, \dots, |z\rangle \in \mathcal{H}_Z$ .

(ii)  $W$  is not positive, i.e.,  $W$  has at least one negative eigenvalue.

It can be easily seen that condition (i) ensures that  $\text{tr}(W\rho) \geq 0$  for any  $\rho$  separable. Thus, if for some density operator,  $\rho$ , and an EW,  $W$ ,  $\text{tr}(W\rho) < 0$ , then  $\rho$  must be entangled. In this case we say that  $W$  detects  $\rho$ . The important point concerning separability is that a state is entangled iff there exists an EW which detects it [22].

When talking about EW's one has to distinguish two different kinds. On the one hand, there are the so-called decomposable EW (DEW), which can be written as

$$W = O_0 + O_1^{T_A} + \dots + O_N^{T_Z}, \quad (4)$$

where the operators  $O_i$  are positive semidefinite [25]. It can be easily verified that those witnesses cannot detect any PPTES. On the other hand, nondecomposable EW (NDEW) cannot be written as Eq. (4) [26]. In [17] we showed that a two-partite EW,  $W$ , is a NDEW iff it detects a PPTES. This result can be easily generalized to an arbitrary number of parties and so we have

*Lemma 1.* An  $N$ -partite EW,  $W$ , is a NDEW iff it detects a PPTES.

**C. Distillation and activation of entanglement**

We consider the situation where an arbitrary number of parties share an arbitrary number of copies of the state  $\rho$ . Then, we call  $\rho$  fully distillable if the parties can, using LOCC, produce a maximally entangled state, shared among all the parties. Note that a PPT state can never be distilled, which can be easily seen as follows. If  $\rho$  is PPT then  $\rho^{\otimes N}$  is PPT for all  $N$ , and so is,  $\mathcal{E}[(\rho^{\otimes N})^{T_A}]$  for all  $N$  and  $\mathcal{E}$  separable, as mentioned in Appendix B. Thus, by LOCC one can never produce, out of a PPT state a maximally entangled state, which is NPPT. Note that, in the bipartite case with  $d_A=2$  and  $d_B$  arbitrary, it has been shown [11] that all NPPT states are distillable. However, for higher dimensions, there is a strong conjecture [11,12] that this is no longer true, i.e., that there exist undistillable NPPT states. In the case of more than two parties, the argumentation above implies that a state can only be distillable if all the partial transposes are non-positive semidefinite, i.e., if the state is NPPT.

If a state is not distillable, then it might be possible to activate its entanglement using a PPTES. That is, if the parties share, in addition to  $N$  copies of a state  $\rho$  a PPTES, then they might be able to produce a maximally entangled state, using LOCC. We call this process activation. Using the argumentation above, one immediately sees that only NPPT states are activatable.

Let us now, in order to be more specific, treat the case of two parties and the one of three independently. The next part of this section deals with the bipartite case. There we review the conditions which must be fulfilled for a state to be distillable or activatable. In the second part of this section we show how to generalize the results of two parties to three.

**1. Two parties**

In this scenario a state,  $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ , is called distillable if Alice and Bob can produce by LOCC a maximally entangled state (1), with  $d = \min(d_A, d_B)$ .

It has been shown [10] that the problem of distillation of a state  $\rho$  can be formulated in the following way:

*Lemma 2* [10]. A state  $\rho$  is distillable iff there exists a positive integer  $N$  and a state of the form

$$|\Psi\rangle = |e_1, f_1\rangle + |e_2, f_2\rangle, \tag{5}$$

such that

$$\langle \Psi | (\rho^{\otimes N})^{T_A} | \Psi \rangle < 0, \tag{6}$$

where  $\{e_1, e_2\}, \{f_1, f_2\}$  are two unnormalized orthogonal vectors in  $(\mathbb{C}^{d_A})^{\otimes N} [(\mathbb{C}^{d_B})^{\otimes N}]$ .

This condition simply means that iff there exists a  $\mathbb{C}^2 \otimes \mathbb{C}^2$  subspace, on which the projection of  $\rho^{\otimes N}$  is NPPT, then the state is distillable. This can be understood as follows: if there exists such a subspace then Alice and Bob can distill a maximally entangled state in  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . They can then use some of those distilled states to convert them, by LOCC, into a maximally entangled state in  $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ . On the other hand, a maximally entangled state in  $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$  can be converted into a maximally entangled state in  $\mathbb{C}^2 \otimes \mathbb{C}^2$ .

We call a state  $N$ -distillable if for this integer  $N$  condition (6) is fulfilled. Otherwise we call it  $N$ -undistillable. If a state is not distillable, then Alice and Bob might still be able to distill a maximally entangled state by LOCC, if they share, in addition to their copies of the state, a PPTES. We call this process activation.

We call a  $m$ -undistillable state  $\rho$  activatable if there exists a positive integer  $N \leq m$  and a PPTES  $\sigma$  such that  $\rho^{\otimes N} \otimes \sigma$  is one-distillable. Given an  $N$ -undistillable state  $\rho$ , we call it  $N$ -activatable if there exists a PPTES  $\sigma$  such that  $\rho^{\otimes N} \otimes \sigma$  is one-distillable.

**2. Three parties**

A three-partite state,  $\rho \in \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$  is fully distillable if Alice, Bob, and Charly can produce, using LOCC, out of an arbitrary number of copies of  $\rho$  a GHZ state [27]. Note, that this is possible iff (up to permutations) Alice and Bob can produce a state of the form (1) with  $d = d_{AB} = \min(d_A, d_B)$  and Bob and Charly can distill a state of the form (1) with  $d = d_{BC} = \min(d_B, d_C)$ . This can be easily understood using that two maximally entangled state, one in  $A$  and  $B$  and the other in  $B$  and  $C$  can always, by LOCC be combined to GHZ state,  $\sum_{i=1}^{\min(d_{AB}, d_{BC})} |iii\rangle$ . The other direction is also true, since, given two GHZ states one can, using LOCC, transform them into a maximally entangled state in  $A$  and  $B$  and one in  $B$  and  $C$ . Using this fact we only have to answer the question: ‘‘Can Alice, Bob, and Charly distill a maximally entangled state in  $A, B$  and one in  $B, C$ ?’’ Therefore, from now on we will only deal with the problem of bipartite entanglement distillation. We call a state,  $\rho$ ,  $BC$ -distillable if Alice, Bob, and Charly can produce, using LOCC, out of an arbitrary number of copies of  $\rho$  in  $BC$  a maximally entangled state, i.e., a state of the form (1) with  $d = \min(d_B, d_C)$ . Note that the best strategy for them to distill a maximally entangled state in  $B$  and  $C$  is that Alice measure a projector. The reason for this is that if they apply any other measurement, then they always reduce the entanglement of the outcoming state. We define  $AB$ - and  $AC$ -distillability analogously for the other cases.

If a state is undistillable we have, analogously to the bipartite case, the possibility to activate its entanglement using a PPTES. That is the parties share, in addition to some copies of their state a PPTES. Then they distill out of those states a maximally entangled state. If this is possible then we call the state activatable. Again, we have that it is activatable iff it is (up to permutations)  $AB$ -activatable and  $BC$ -activatable; that is the entanglement among all parties can be activated iff the entanglement between  $A$  and  $B$  and the one between  $B$  and  $C$  can be activated.

Let us now show under which conditions a state is  $XY$ -distillable, where  $X, Y \in \{A, B, C\}$ , on the example of  $BC$ -distillation.

*Lemma 2'*. A state  $\rho$  is  $BC$ -distillable iff there exists a positive integer  $N$  and a state of the form

$$|\Psi\rangle = |e_1, f_1\rangle + |e_2, f_2\rangle, \tag{7}$$

where  $\{e_1, e_2\}, \{f_1, f_2\}$  are two unnormalized orthogonal vectors in  $(\mathcal{H}_B)^{\otimes N}[(\mathcal{H}_C)^{\otimes N}]$ , and a state  $|h\rangle \in (\mathcal{H}_A)^{\otimes N}$  such that

$$\langle \Psi | \langle h | (\rho^{\otimes N})^{T_C} | h \rangle | \Psi \rangle < 0. \quad (8)$$

Note that condition (8) can only be fulfilled if  $\rho$  is  $B$ -NPPT as well as  $C$ -NPPT.

Analogous to Sec. II we call a state  $\rho$ ,  $N$ - $BC$ -distillable if for the integer  $N$ , condition (8) is fulfilled. We call  $\rho$   $N$  fully distillable if it is (up to permutations)  $N$ - $AB$ -distillable and  $N$ - $BC$ -distillable.

If a state is not distillable then we still have the possibility to activate its entanglement using a PPTES. Let us now characterize those states which are  $XY$  activatable on the example  $X=B$ ,  $Y=C$ .

We call a state  $\rho$ , which is not  $m$ - $BC$ -distillable,  $BC$ -activatable if there exists an integer  $N \leq m$  and a PPTES  $\sigma$  such that  $\rho^{\otimes N} \otimes \sigma$  is 1- $BC$ -distillable. Given an  $N$ - $BC$ -undistillable state  $\rho$ , we call it  $N$ - $BC$ -activatable if there exists a PPTES,  $\sigma$ , such that  $\rho^{\otimes N} \otimes \sigma$  is 1- $BC$ -distillable. We call  $\rho$   $N$  fully activatable if it is (up to permutations)  $N$ - $AB$ -activatable and  $N$ - $BC$ -activatable.

### III. CHARACTERIZATION OF DISTILLABLE AND ACTIVATABLE STATES USING ENTANGLEMENT WITNESSES

In this section we show that there exists a connection between EW's and the distillation and activation properties of states. We will define an operator which allows us to answer the questions, if  $N$  copies of a state are distillable, or if not, if their entanglement can be activated using a PPTES. Using this formalism it is easy to rederive the result [14] that any bipartite NPPT is either one-distillable, or one-activatable. One the other hand, it allows us to prove that there exists three-partite NPPT state which are neither two-distillable, nor two-activatable.

This section is divided into two parts. In the first part we show this connection for the bipartite case, whereas in the second part we extend those results to three parties. Both parts have the same structure; first we define the operator which allows us to draw the connection between EW's and the distillation and activation properties of a state. Then we state our main results of the paper. In the last part of each section we prove those results. The reader who is not interested in the proofs can skip Secs. III A 2 and III B 2.

#### A. Two parties

Let us denote by  $X$  an arbitrary positive operator acting on  $\mathcal{H}_2 = \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$ . We define

$$W_X = P_{A_1, B_1} \otimes X_{A_2, B_2}^{T_{A_2}} \quad (9)$$

where  $P_{A_1, B_1}$ , acting on  $\mathcal{H}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2$  is the projector onto the maximally entangled state (1), with  $d=2$ . Now, we will show that  $W_\rho$  allows us to answer the questions, if the entanglement of this state can be distilled, or, if not, if it can be

activated using a PPTES. In particular, we show that, depending on whether  $W_{\rho^{\otimes N}}$  is not an EW, a NDEW, or a DEW, the corresponding state  $\rho$  is  $N$ -distillable or  $N$ -activatable, or neither  $N$ -distillable nor  $N$ -activatable, respectively. This is stated by the following theorems and corollaries, which will be proven below.

#### 1. Main results

*Theorem 1.* A state  $\rho$  is  $N$ -distillable iff  $W_{\rho^{\otimes N}}$  is not an EW [it does not fulfill (i)].

*Theorem 2.* A state  $\rho$  which is  $N$ -undistillable, is  $N$ -activatable iff  $W_{\rho^{\otimes N}}$  is a NDEW.

Those theorems state that EW's do not only allow us to determine whether a state is entangled or not, but also characterize the distillability properties of a state. We have that  $W_\rho$  is not an EW iff  $\rho$  is one-distillable (Theorem 1). Now, iff  $\rho$  is one-undistillable it can be either one-activatable or not. In this case  $W_\rho$  is an EW (Theorem 1), which can be either decomposable or nondecomposable. Then the entanglement of  $\rho$  can be activated via a PPTES iff  $W_\rho$  is a NDEW (Theorem 2). It is worth mentioning that using these results, it is not only possible to know if the entanglement of a state is distillable, or if it can be activated but it can be seen by the proofs that they provide us with a distillation protocol. That is, given a state which can be distilled, then the separable state, which is "detected" by the witness,  $W_\rho$  corresponds to a LOCC, which distills a maximally entangled state (see Appendix B and [28]). On the other hand, the PPTES, which activates the entanglement is easily determined by the state, which is detected by the NDEW,  $W_\rho$ . Those theorems also imply the following:

*Corollary 1.* A state  $\rho$  is neither  $N$ -distillable nor  $N$ -activatable iff  $W_{\rho^{\otimes N}}$  is decomposable.

Note that the results above are not only a way of rewriting the problems, but they really allow for insight into the problems of distillation and activation. For instance in Sec. IV, we review in a simple manner the fact that every bipartite NPPT state is either one-distillable or one-activatable [14].

#### 2. Proofs

The reader who is not interested in the proofs can continue reading in the next section. For simplicity we prove the statements for  $N=1$ , since the argument holds for arbitrary  $N$ . The technical details and the definitions, which are needed to follow the proofs can be found in the appendices. Let us start out by determine the properties of the operator  $W_X$ .

(a)  $W_X$  is not positive semidefinite iff  $X^{T_{A_2}}$  is not positive semidefinite, i.e., iff  $X$  is NPPT.

(b) If  $W_X$  fulfills condition (i) then both,  $W_X^{T_{A_1}}$  and  $W_X^{T_{B_1}}$  are optimal EW's.

(c)  $W_X$  is decomposable iff  $W_X = R \geq 0$ .

*Proof.* Property (a) It is clear since  $P_{A_1, B_1}$  is positive. Property (b) If  $W_X$  fulfills condition (i) then so do  $W_X^{T_{A_1}}$  and  $W_X^{T_{B_1}}$ . On the other hand  $P$  is NPPT, implying that both  $W_X^{T_{A_1}}$  and  $W_X^{T_{B_1}}$  are not positive semidefinite and therefore they are both EW's. It remains to show that they are optimal. Using

that  $S_{\rho_{T_A}} = \{|e\rangle|e^\perp\rangle\forall|e\rangle \in \mathbb{C}^2\}$  (Appendix A, Proof of Lemma 4) we find that  $W_X^{T_A}$  vanishes on the set  $\{|e\rangle_{A_1}|e^\perp\rangle_{B_1}|\psi\rangle, \forall|e\rangle \in \mathbb{C}^2, |\psi\rangle \in \mathcal{H}_2\}$  (note that these are not only product states). This contains the set  $S_{W_X^{T_A}} = \{|e, g\rangle_A|e^\perp, f\rangle_B \forall|e\rangle \in \mathbb{C}^2, \forall|g\rangle \in \mathcal{H}_{A_2} \forall|f\rangle \in \mathcal{H}_{B_2}\}$ . Using the fact that  $\{|e\rangle|e^\perp\rangle\forall|e\rangle \in \mathbb{C}^2\}$  spans  $\mathbb{C}^2 \otimes \mathbb{C}^2$ , we have that  $S_{W_X^{T_A}}$  spans the whole Hilbert space,  $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{H}_2$ . Thus,  $W_X^{T_A}$  is optimal. Now, using the fact that  $W$  is an optimal EW iff  $W^T$  is an optimal EW, we also have that  $W_X^{T_B}$  is optimal. Property (c): We only have to prove the only if part. Let us assume that  $W_X$  is decomposable. Then we can write it as  $W_X = R + Q^{T_A}$  [25] and therefore  $W_X^{T_A} = R^{T_A} + Q$ . Using property (b) and the discussion concerning optimality in Appendix A we find that  $Q = 0$ , which proves the statement. ■

Note, that property (a) tells us that since we only have to consider NPPT states,  $W_\rho$  is not an EW iff it does not fulfill (i). With that we are now in the position to prove the results of the previous section.

*Proof of Theorem 1.* Using Corollary 2 (Appendix C) we have that  $\rho$  is one-distillable iff there exists a separable CPM (Appendix B),  $\mathcal{E}: \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  such that  $0 > \text{tr}_1[P_1^{T_B} \mathcal{E}(\rho)]$ . Using Eq. (B4) we can write this inequality as  $0 > \text{tr}_1[P_1^{T_B} \text{tr}_2(\rho_2^T E_{1,2})] = \text{tr}_{1,2}[P_1^{T_B} \otimes \rho_2^T E_{1,2}]$ . Taking now the partial transpose with respect to  $B_1, B_2$  within the trace we have  $0 > \text{tr}_{1,2}[P_1 \otimes \rho_2^{T_A} E_{1,2}^{T_B}] = \text{tr}(W_\rho E_{1,2}^{T_B})$ . Thus, we have that  $\rho$  is one-distillable iff  $W_\rho$  “detects” the state  $E^{T_B}$ . Now, since  $\mathcal{E}$  is separable iff  $E$  is separable [Appendix B (p1)], we have that the inequality above is true iff  $W_\rho$  is not an EW (since it detects  $E^{T_B}$  which is separable). ■

Note that this proof implies the following fact. Given a distillable state  $\rho$ , we determine the separable state  $E^{T_B}$ , which is “detected” by  $W_\rho$ . Then the state  $E$  corresponds to the CPM,  $\mathcal{E}$  (see Appendix B) which fulfills the property that  $\mathcal{E}(\rho)$  is a two-qubit entangled state. Thus, we found the LOCC, which distills the state  $\rho$ .

*Proof of Theorem 2.* The proof is basically the same as the one of Theorem 1, but now with  $\mathcal{E}$  a PPT-preserving CPM, which implies that  $E$  is PPT. Then  $W$ , which must be an EW (Theorem 1), detects a PPTES and is therefore (Sec. II B) a NDEW. ■

Using the same arguments as before, if we determine the PPTES,  $E^{T_B}$ , which is detected by the NDEW,  $W_\rho$ , then we know which PPTES activates the entanglement of  $\rho$ , namely,  $E$ .

*Proof of Corollary*  $W_\rho$  must either not be an EW, or be an NDEW or a DEW. It is not an EW iff  $\rho$  is one-distillable. It is an NDEW iff  $\rho$  is one-activatable. Therefore,  $\rho$  is neither one-distillable nor one-activatable iff  $W_\rho$  is a DEW.

### B. Three parties

Let us now generalize the results obtained in the previous section for the bipartite case to the case where we consider more parties. Here we will show how to do it for three, but one can generalize the methods introduced in the previous section to any number of particles.

Let us now denote by  $X$  an arbitrary positive semidefinite operator acting on  $\mathcal{H}_2 = \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2} \otimes \mathcal{H}_{C_2}$ . We define

$$W_X^a = P_{B_1, C_1} \otimes X_{A_2, B_2, C_2}^{T_{C_2}}, \quad (10)$$

$$W_X^b = P_{A_1, C_1} \otimes X_{A_2, B_2, C_2}^{T_{A_2}}, \quad (11)$$

$$W_X^c = P_{A_1, B_1} \otimes X_{A_2, B_2, C_2}^{T_{B_2}},$$

where  $P_{Y,Z} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  is defined in Eqs. (A1) for  $\{Y, Z\} \in \{A_1, B_1, C_1\}$ .

### 1. Main results

We will now, analogously to Sec. III C, characterize the distillation and activation properties of a state, by the operators  $W_\rho^a$ ,  $W_\rho^b$ , and  $W_\rho^c$ . For the seek of clarity we state our results only for the  $BC$ -distillation and  $BC$ -activation. That is we assume that  $\rho$  is  $B$ -NPPT and  $C$ -NPPT (recall that otherwise it is neither possible to distill, nor to activate the entanglement shared between Bob and Charly). All the results presented here can be formulated for the other partitions,  $AB$  and  $AC$  too, using then the operators  $W_X^c$  and  $W_\rho^b$ , respectively.

*Theorem 1'.* A state  $\rho$ , is  $N$ - $BC$ -distillable iff  $W_{\rho^{\otimes N}}^a$  is not an EW [does not fulfill (i)].

*Theorem 2'.* A state  $\rho$ , which is  $N$ - $BC$ -undistillable, is  $N$ - $BC$ -activatable iff  $W_{\rho^{\otimes N}}^a$  is a NDEW.

Those theorems state that a state is, for instance, one fully distillable iff at least two of the operators  $W_\rho^a$ ,  $W_\rho^b$ , or  $W_\rho^c$  are not EW's. If it is not fully distillable then at least two of the operators  $W_\rho^a$ ,  $W_\rho^b$ , or  $W_\rho^c$  must be EW's (Theorem 1'). Then, its entanglement can be fully activated via a PPTES iff at least two of the operators  $W_\rho^a$ ,  $W_\rho^b$ , or  $W_\rho^c$  (the ones that are EW's) are NDEW's. Note that, the states which are “detected” by  $W_\rho^a$  are the ones which allow us to derive a distillation protocol. That is, given a one- $BC$ -distillable state  $\rho$ , the separable state which is “detected” by  $W_\rho^a$  corresponds to a LOCC (see Appendix B), that distills  $\rho$ . If the state is one- $BC$ -undistillable, but it is one- $BC$ -activatable, then the PPTES, which activates its entanglement is easily determined by the one which is detected by the NDEW  $W_\rho^a$ . On the other hand, we have that a state  $\rho$  is neither one-fully distillable nor one-fully activatable iff at least two of the operators  $W_\rho^a$ ,  $W_\rho^b$ , and  $W_\rho^c$  are decomposable. This can be, concerning the bipartite entanglement, stated as:

*Corollary 1'.* A state  $\rho$  is neither  $N$ - $BC$ -distillable nor  $N$ - $BC$ -activatable iff there exist positive semidefinite operators  $R, Q$  such that  $\rho^{\otimes N} = R^{T_C} + Q^{T_B}$ .

Note that, as mentioned before a three-partite state is fully distillable (activatable) iff it is (up to permutations)  $AB$ -distillable (activatable) and  $BC$ -distillable (activatable). Thus, Corollary 1' provides a necessary and sufficient condition for a state to be neither  $N$ -fully distillable nor  $N$ -fully activatable.

In the next section we show that using these theorems we are able to prove in a simple way that there exist three-partite

NPPT states which are neither one-distillable nor one-activatable. This is in contrast to the bipartite case, where all NPPT states are either one-distillable or one-activatable. The methods even allow us to prove that some of those states are not even two-distillable nor two-activatable. We show that by proving that all the operators  $W_\rho^a$ ,  $W_\rho^b$ , and  $W_\rho^c$  are decomposable.

## 2. Proofs

We prove the statements above for the case  $N=1$  since all the arguments remain the same for arbitrary  $N$ . Again, if a reader is not that interested in the proofs he can skip this part of the paper and continue reading in the next section.

We start by showing the following properties of the operators  $W_X^a$ :

(A)  $W_X^a$  is not positive semidefinite iff  $X^{TC}$  is not positive semidefinite, i.e.,  $X$  is  $C$ -NPPT.

(B) If  $W_X^a$  fulfills condition (i) then  $(W_X^a)^{TB}$  and  $(W_X^a)^{TC}$  are optimal EW's.

(C)  $W_X^a$  is decomposable iff there exists  $R, Q \geq 0$  such that  $X = R^{TC} + Q^{TB}$ .

*Proof.* The proofs of property (A) and property (B) are similar to the ones of property (a) and property (b) in Sec. III A 2 and will be omitted here. Property (C): We denote by  $\tilde{Y}$  the total transpose of an operator  $Y$ . (if): If  $X = R^{TC} + Q^{TB}$  then  $W_X^a = P_{B_1, C_1} \otimes X^{TC} = P_{B_1, C_1} \otimes (R + \tilde{Q}^{TA}) = O + S^{TA}$ , with  $O = P_{B_1, C_1} \otimes R \geq 0$  and  $S = P_{B_1, C_1} \otimes \tilde{Q} \geq 0$  and so it is decomposable. (only if) If  $W_X^a$  is decomposable then there exist operators (4)  $Q_0, Q_1, Q_2, Q_3 \geq 0$  such that  $W_X^a = Q_0 + Q_1^{TA} + Q_2^{TB} + Q_3^{TC}$  and so  $(W_X^a)^{TB} = Q_0^{TB} + \tilde{Q}_1^{TC} + Q_2 + \tilde{Q}_3^{TA}$  and  $(W_X^a)^{TC} = Q_0^{TC} + \tilde{Q}_1^{TB} + \tilde{Q}_2^{TA} + Q_3$ . Since  $(W_X^a)^{TB}$  and  $(W_X^a)^{TC}$  are both optimal EW's, because of property (b), we have that  $Q_2 = Q_3 = 0$  and so  $W_X^a = Q_0 + Q_1^{TA}$ . Let us now use that  $P_{BC}^{TC}$  and  $P_{BC}^{TB}$  are optimal decomposable EW's (Appendix A, Lemma 4) and that the sets  $S_{P^{TC}_{BC}} = S_{P^{TB}_{BC}} = \{|e\rangle|e^\perp\rangle\} \forall |e\rangle \in \mathbb{C}^2$  span  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . This implies that  $(W_X^a)^{TC} = Q_0^{TC} + \tilde{Q}_1^{TB}$  vanishes on  $\{|e\rangle_{B_1}|e^\perp\rangle_{C_1}|\phi\rangle_{A_2}|\psi\rangle_{B_2}|\chi\rangle_{C_2}\} \forall e, \phi, \psi, \chi$ . Thus,  $Q_0^{TC}$  and  $\tilde{Q}_1^{TB}$  must vanish on those states. This implies that  $Q_0^{TC} = P_{B_1, C_1}^{TC} \otimes R^{TC}$ ,  $\tilde{Q}_1^{TB} = P_{B_1, C_1}^{TB} \otimes Q^{TB}$  and so  $Q_0 = P_{B_1, C_1} \otimes R$ , and  $Q_1 = P_{B_1, C_1} \otimes \tilde{Q}$  where  $R, Q \geq 0$ . Thus, we have that  $W_X^a = P_{B_1, C_1} \otimes X^{TC} = P_{B_1, C_1} \otimes (R + \tilde{Q}^{TA})$  and so  $X = R^{TC} + Q^{TB}$ . ■

Property (A) implies that, since we only have to consider  $C$ -NPPT states, the operator  $W_\rho^a$  is not positive semidefinite. Thus,  $W_\rho^a$  is not an EW iff it does not fulfill (i) [29].

The proofs of the theorems are basically the same as the one in the previous section, but now with  $\mathcal{E}: \mathcal{B}(\mathcal{H}_2^{\otimes N}) \rightarrow \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ , where  $\mathcal{H}_2 = \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2} \otimes \mathcal{H}_{C_2}$  and the corresponding operator  $E_{1,2} \equiv E_{A_2, B_1, B_2, C_1, C_2} \in \mathcal{B}(\mathcal{H}_2^{\otimes N} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ .

*Proof of Theorem 1'.* Using the same arguments as in the proof of Theorem 1 we find  $\text{tr}[P_{B_1, C_1}^{TB} \mathcal{E}_a(\rho)] = \text{tr}[W_\rho^a (E_{1,2}^T)^{TC}]$ , where  $E_{1,2}$  is the operator corresponding to the CPM  $\mathcal{E}_a$ . Recall that  $\mathcal{E}_a$  is separable iff  $(E_{1,2}^T)^{TC}$  is separable. Now, the left-hand side of the last equation is negative, for  $\mathcal{E}_a$  separable, iff  $\rho$  is  $BC$ -distillable. And on the other hand, (looking at the right-hand side of this equation) this is true iff  $W_\rho^a$  is not an EW, since it “detects” the separable state  $(E_{1,2}^T)^{TC}$ . ■

*Proof of Theorem 2'.* Same proof as for Theorem 1', but now with  $E$  a PPTES and  $\mathcal{E}$  a PPT-preserving CPM. Using that the operator  $W_\rho^a$  must be an EW (Theorem 1'), we have that the PPTES,  $(E^T)^{TC}$  is detected by the EW,  $W_\rho^a$ , implying that it is an NDEW. ■

*Proof of Corollary 1'.* Same proof as for Corollary 1, but now  $W_\rho^a$  is a DEW iff there exist operators  $R, Q \geq 0$  such that  $\rho = R^{TC} + Q^{TB}$  [property (C)]. ■

## IV. APPLICATIONS

In this section we use the formalism introduced in the previous section to show the following facts. On the one hand, the entanglement of one copy of any bipartite NPPT can either be distilled or activated [14]. On the other hand, this formalism allows us to show that this is not the case for multipartite states. That is, there exist states describing a system composed of more than two subsystems which are neither one-distillable nor one-activatable. Indeed, in the following subsections we will show that, using the connection between EW's and the distillation and activation properties of a state, there are NPPT states which are not even two-distillable nor two-activatable.

### A. Two parties

*Observation* [14]. Any bipartite NPPT state,  $\rho$  is either one-distillable or one-activatable. This can be easily seen using Corollary 1, which states that  $\rho$  is neither one-distillable nor one-activatable iff  $W_\rho$  is a DEW. Using then property (c) we have that  $W_\rho$  is a DEW iff it is a positive semidefinite operator, which is true iff  $\rho$  is PPT [property (a)].

### B. Example of a one-undistillable and one-inactivatable tripartite state

In this section we present a family of density operators,  $\{\rho_\alpha\}$ , which describe the state of a system composed of three qubits. We show, using the formalism of the previous section, that for  $\alpha \leq 1$  one copy of these states can neither be distilled nor activated. Then we prove that for  $\alpha \leq \alpha_0$ , with  $\alpha_0 \approx 0.8507$  even two copies of the states can neither be distilled, nor activated. Recall that in the bipartite case there exists no such state. The states of interest are

$$\rho_\alpha = \mathbb{1}_8 + \alpha |\Psi_W\rangle\langle\Psi_W|, \quad (12)$$

where  $|\Psi_W\rangle = |001\rangle + |010\rangle + |100\rangle$ . Note first, that  $\rho_\alpha$  is NPPT iff  $\alpha > 1/\sqrt{2}$ , which implies that only in this region the state might be distillable or activatable. Note further that  $\rho_\alpha$  is symmetric under all the permutations of the three parties,  $A, B, C$ . This symmetry implies that this state is  $N$ -AB-

distillable iff it is  $N$ -AC-distillable iff it is  $N$ -BC-distillable. Let us therefore, without loss of generality, consider the situation where Alice performs a measurement and Bob and Charly distill out of the remaining density operator a maximally entangled state. That is, we are interested in the  $N$ -BC-distillation of the state and therefore in the properties of the operator  $W_{\rho_{\alpha}^{\otimes N}}^a$  Eq. (10). Note that, according to our definitions the state is (because of the symmetry)  $N$ -BC-distillable iff it is  $N$ -distillable. Then the theorems and the corollary of the previous section simplify to

*Remark 1'*.  $\rho_{\alpha}$  is  $N$ -distillable iff  $W_{\rho_{\alpha}^{\otimes N}}^a$  is not an EW.

*Remark 2'*. If  $\rho_{\alpha}$  is  $N$ -undistillable, it is  $N$ -activatable iff  $W_{\rho_{\alpha}^{\otimes N}}^a$  is a NDEW.

*Remark 3'*.  $\rho_{\alpha}$  is neither  $N$ -distillable nor  $N$ -activatable iff there exist positive semidefinite operators  $R_{\alpha}, Q_{\alpha}$  such that  $(\rho_{\alpha}^{\otimes N})^{TC} = R_{\alpha} + Q_{\alpha}^{TA}$ .

### 1. One copy

We show that one copy of the state,  $\rho_{\alpha}$  cannot be distilled for  $\alpha \leq 1$ , i.e., (Remark 1)  $W_{\rho_{\alpha}}^a$  is an EW for  $1/\sqrt{2} < \alpha \leq 1$ . Note that the remaining state, after Alice performs a measurement is a state of two qubits, which is distillable iff it is NPPT (Sec. I). And so we only have to find the measurement in  $A$ ,  $|\psi\rangle\langle\psi|$ , which maximizes the region of  $\alpha$ , for which the state  $\langle\psi|\rho_{\alpha}|\psi\rangle$  is NPPT. It can be easily shown that the best measurement Alice can do is to measure  $|0\rangle\langle 0|$ . Then the remaining state,  $\langle 0|\rho_{\alpha}|0\rangle$ , is NPPT iff  $\alpha > 1$ , which implies that the state,  $\rho_{\alpha}$  can be distilled,  $\forall \alpha > 1$ . On the other hand, using Remark 1 we have that  $W_{\rho_{\alpha}}^a$  is an EW for  $1/\sqrt{2} < \alpha \leq 1$ . Now we show that  $W_{\rho_{\alpha}}^a$  is for  $1/\sqrt{2} < \alpha \leq 1$  a DEW and therefore  $\rho$  cannot be activated for  $\alpha \leq 1$  (Remark 2). Using Remark 3 we have to find  $R_{\alpha}, Q_{\alpha} \geq 0$  such that  $\rho_{\alpha}^{TC} = R_{\alpha} + Q_{\alpha}^{TA}$ . It can be easily verified that the operators  $R_{\alpha} = \alpha|\Psi^+\rangle\langle\Psi^+|_{A,B} \otimes |0\rangle\langle 0|_C$ , where  $|\Psi^+\rangle = |01\rangle + |10\rangle$ , and  $Q_{\alpha} = \rho_{\alpha}^{TB} - (R_{\alpha})^{TA}$  are both positive semidefinite and lead to the desired decomposition. Thus, we have shown that the NPPT state  $\rho_{\alpha}$ , is neither one-distillable nor one-activatable  $\forall \alpha \in ]1/\sqrt{2}, 1]$ . Note that the given decomposition of  $\rho_{\alpha}$  proves this statement already.

### 2. Two copies

Using the same method as above one can also show that two copies of  $\rho_{\alpha}$  can neither be distilled, nor activated if  $\alpha \in ]1/\sqrt{2}, \alpha_0]$ , with  $\alpha_0 \approx 0.8507$ . In this case  $R_{\alpha} = y(\rho_{1/\sqrt{2}}^{TC} R_1 + R_1 \otimes \rho_{1/\sqrt{2}}^{TC})$ , with  $R_1 = |\Psi^+\rangle\langle\Psi^+|_{A,B} \otimes |0\rangle\langle 0|_C$ , where  $|\Psi^+\rangle = |01\rangle + |10\rangle$ , and  $Q_{\alpha} = (\rho_{\alpha}^{\otimes 2})^{TB} - R_{\alpha}^{TA}$  are both positive semidefinite for  $y \approx 0.4953$  and fulfill  $(\rho_{\alpha}^{\otimes 2})^{TC} = R_{\alpha} + Q_{\alpha}^{TA}$ .

## V. CONCLUSIONS

We have shown that one can connect the problem of entanglement witnesses to the one of distillation and activation of entanglement. We defined, depending on the state  $\rho$  and on the number of copies  $N$  of it, an operator which has the

following properties: It is an entanglement witness iff  $\rho$  is  $N$ -undistillable, i.e., those  $N$  copies cannot be distilled via LOCC; it is a decomposable entanglement witness iff the state is  $N$ -unactivatable; i.e., those  $N$  copies cannot be distilled via LOCC, even if we allow for a PPTES in addition. Using those methods we have shown that there exist tripartite NPPT states, which are neither two-distillable, nor two-activatable. We showed it by proving that the corresponding operator is a decomposable entanglement witness.

## ACKNOWLEDGMENTS

This work has been supported in part by the Deutsche Forschungsgemeinschaft (SFB 407 and Schwerpunkt ‘‘Quanteninformationsverarbeitung’’), the DAAD, the Austrian Science Foundation (SFB ‘‘control and measurement of coherent quantum systems’’), the ESF PESC Program on Quantum Information, TMR network ERB-FMRX-CT96-0087, the IST Program EQUIP, and the Institute for Quantum Information GmbH.

## APPENDIX A: ENTANGLEMENT WITNESSES

In [17] we showed how to optimize two-partite EW's; that is how to construct a new EW out of a given one, which detects the same entangled states and in addition some others. There we also showed that an EW,  $W$ , is optimal iff  $\forall R \geq 0, \epsilon > 0 W' = W - \epsilon R$  is not an EW, in the sense that it does not fulfill (i). This method of optimization can be easily generalized to the case of more parties. In order to recall a sufficient condition for an EW to be optimal, we define, for an EW,  $W$ , the set  $S_W = \{ |a, \dots, z\rangle \in \mathcal{H} \text{ such that } \langle a, \dots, z | W | a, \dots, z \rangle = 0 \}$ . Then we have

*Lemma 3.* If  $S_W$  spans  $\mathcal{H}$  then  $W$  is optimal.

Let us now give an example of an optimal decomposable two-partite EW in  $\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ . This EW will allow us to draw the connection between EW's and the criterion that a state in  $\mathbb{C}^2 \otimes \mathbb{C}^2$  is entangled iff it is NPPT [21,22].

We denote by  $P_{AB} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  the projector onto the maximally entangled state, that is

$$P_{AB} = |\Phi_2\rangle\langle\Phi_2|, \quad (\text{A1})$$

with  $|\Phi_2\rangle$  given in Eq. (1). Then we have

*Lemma 4.*  $P_{AB}^{TA}$  is an optimal DEW.

*Proof.* Using Lemma 3 it is sufficient to show that  $S_{P_{AB}^{TA}}$  spans  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . It can be easily verified that  $S_{P_{AB}^{TA}} = \{ |e, e^{\perp}\rangle \mid |e\rangle \in \mathbb{C}^2 \}$ . On the other hand, one can easily check that there exists no state orthogonal to this set, which implies that  $S_{P_{AB}^{TA}}$  spans  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . ■

Note that an EW,  $W$ , is optimal iff  $W^T$  is optimal, implying that  $P_{AB}^{TB}$  is optimal too. This EW detects, up to local operations, all NPPT states, which follows from:

*Lemma 5.* [9] A state  $\rho_{AB} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  is NPPT iff there exist some local operators  $A, B$  such that  $\text{tr}(P_{AB}^{TB} A \otimes B \rho A^{\dagger} \otimes B^{\dagger}) < 0$ .

## APPENDIX B: COMPLETELY POSITIVE MAPS

Any physical action can be mathematically described by a completely positive map (CPM), which is a linear, Hermitian map,  $\mathcal{E}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ , that is positive, i.e.,  $\forall \rho \geq 0 \mathcal{E}(\rho) \geq 0$ , and fulfills that the extended map,  $\mathcal{E} \otimes \mathbb{1}_n$  is positive for any  $n$ . Note that any CPM can be written as,  $\mathcal{E}(\rho) = \sum_k O_k \rho O_k^\dagger$ , where  $O_k \in \mathcal{B}(\mathcal{H})$ . Note further that any separable CPM, that is any map that can be, up to a proportionality constant, implemented locally, can be written as

$$\mathcal{E}(\rho) = \sum_{k=1}^l (O_k^A \otimes O_k^B \cdots O_k^Z) \rho (O_k^A \otimes O_k^B \cdots O_k^Z)^\dagger, \quad (\text{B1})$$

where  $l$  is some finite number. Thus, we can reformulate condition (2) as follows: A state,  $\rho$ , is separable if there exists a separable map  $\mathcal{E}$  such that  $\rho \propto \mathcal{E}(|00 \cdots 0\rangle\langle 00 \cdots 0|)$ .

A CPM  $\mathcal{E}$  is called *Y-PPT-preserving* if for all  $\rho$  *Y-PPT*,  $\mathcal{E}(\rho)$  is *Y-PPT*. We call a CPM *PPT-preserving*, if it is *Y-PPT-preserving* for all systems  $Y$ .

Let us also recall the isomorphism between CPM's and positive semidefinite operators [28]. We consider a CPM,  $\mathcal{E}$ , acting on the  $N$  systems  $A_1, B_1 \cdots Z_1$  and define the operator

$$E_{A_1, A_2, \dots, Z_1, Z_2} = \frac{1}{d^{2N}} \mathcal{E}(P_{A_1, A_2} \otimes \cdots \otimes P_{Z_1, Z_2}), \quad (\text{B2})$$

where  $P_{X_1, X_2}$  is the projector onto the maximally entangled state (1).  $E_{A_1, A_2, \dots, Z_1, Z_2}$  is acting on  $\mathcal{H}_A \otimes \cdots \otimes \mathcal{H}_Z$ , with  $\mathcal{H}_X = \mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}$  and  $\dim(\mathcal{H}_{X_i}) = \dim(\mathcal{H}_{Y_i}) = d$ , for  $i = 1, 2$  and  $X, Y \in \{A, \dots, Z\}$ . In Eq. (B2) the map  $\mathcal{E}$  is understood to act as the identity on the operators in  $\mathcal{B}(\mathcal{H}_2)$ . The interpretation of Eq. (B2) is the following: Each of the  $N$  parties prepares his system in a maximally entangled state (locally) with an auxiliary system, e.g.,  $A_1$  and  $A_2$  are in a maximally entangled state. Then the operation  $\mathcal{E}$  acts on the systems  $A_1, B_1, \dots, Z_1$ . The state which the parties share then is proportional to  $E$ . On the other hand one can show that

$$\begin{aligned} \mathcal{E}(\rho_{A_2, \dots, Z_1}) &= \text{tr}_{A_2, A_3, \dots, Z_1, Z_3} (E_{A_1, A_2, \dots, Z_1, Z_2} P_{A_3, \dots, Z_3} \\ &\quad \times P_{A_2, A_3} \otimes \cdots \otimes P_{Z_2, Z_2}), \end{aligned} \quad (\text{B3})$$

which can be also written as

$$\mathcal{E}(\rho_{A_1, \dots, Z_1}) = \text{tr}_{A_2, \dots, Z_2} (E_{A_1, A_2, \dots, Z_2, Z_2} \rho_{A_2, \dots, Z_2}^T). \quad (\text{B4})$$

Equation (B3) has a simple physical interpretation too. If we have the state  $E$  at our disposal we can locally implement the operation  $\mathcal{E}$  on a state of the systems  $A_3, \dots, Z_3$  (with certain probability). For that the parties perform a joint measurement locally such that the systems  $(A_2, A_3), \dots, (Z_2, Z_3)$  are projected onto a maximally entangled state (1).

The importance of this isomorphism is that [28]

(P1)  $\mathcal{E}$  is a separable CPM iff  $E$  is separable with respect to the systems  $(A_1, A_2), \dots, (Z_1, Z_2)$ .

(P2)  $\mathcal{E}$  can create entanglement iff  $E$  is entangled with respect to the systems  $(A_1, A_2), \dots, (Z_1, Z_2)$ .

(P3)  $\mathcal{E}$  represents a *Y-PPT-preserving* CPM iff  $E$  is *Y-PPT*.

We can generalize this isomorphism for CPM's of the form  $\mathcal{E}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\bar{\mathcal{H}})$ . Then the corresponding operator  $E$  is an element of  $\mathcal{B}(\mathcal{H} \otimes \bar{\mathcal{H}})$ .

Note that (P1)–(P3) imply the following facts: First, we can implement a separable (*Y-PPT-preserving*) CPM,  $\mathcal{E}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\bar{\mathcal{H}})$ , on a state  $\rho$  if we allow the parties to apply LOCC on the state  $\rho \otimes E$ , where  $E \in \mathcal{B}(\mathcal{H} \otimes \bar{\mathcal{H}})$  is separable (*Y-PPT*). Second, the scenario, where the parties are allowed to apply LOCC on the state  $\rho \otimes E$ , where  $E$  is separable (*Y-PPT*), is equivalent to the one where the parties are allowed to implement a separable (*Y-PPT-preserving*) CPM.

## APPENDIX C: DISTILLATION AND ACTIVATION OF ENTANGLEMENT IN TERMS OF CPM'S

Here we use the notion of CPM's to characterize distillable and activatable states.

### 1. Two parties

We consider a bipartite state  $\rho$  acting on  $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}$ . Using Lemma 2 and Lemma 5 we have the following:

*Corollary 2.* A state,  $\rho$  is distillable iff there exists a positive integer  $N$  and a separable CPM  $\mathcal{E}: \mathcal{B}(\mathcal{H}_1^{\otimes N}) \rightarrow \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  such that

$$\text{tr}[P^{TB} \mathcal{E}(\rho^{\otimes N})] < 0, \quad (\text{C1})$$

where  $P = P_{AB}$  is given in Eq. (A1). Note that condition (C1) is equivalent to  $\text{tr}[P^{TA} \mathcal{E}(\rho^{\otimes N})] < 0$ . Note further that the operator  $E$  corresponding to the CPM  $\mathcal{E}$  is separable and acts on the Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ , where  $\mathcal{H}_A = \mathcal{H}_{A_1}^{\otimes N} \otimes \mathbb{C}^2$  and  $\mathcal{H}_B = \mathcal{H}_{B_1}^{\otimes N} \otimes \mathbb{C}^2$ .

Using the isomorphism between density operators and CPM's (Appendix B), we know that the possibility for Alice and Bob to use a PPTES and then apply LOCC equals the possibility for them to use a PPT-preserving CPM. And so we have

*Lemma 6.* A state  $\rho$  which is  $m$ -undistillable, is activatable iff there exists a positive integer  $N \leq m$  and a PPT-preserving CPM  $\mathcal{E}: \mathcal{B}(\mathcal{H}_1^{\otimes N}) \rightarrow \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  such that  $\text{tr}[P^{TB} \mathcal{E}(\rho^{\otimes N})] < 0$ , or, equivalently  $\text{tr}[P^{TA} \mathcal{E}(\rho^{\otimes N})] < 0$ .

Note that the operator  $E$  corresponding to the CPM  $\mathcal{E}$  is PPT and acts on the Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ , where  $\mathcal{H}_A = \mathcal{H}_{A_1}^{\otimes N} \otimes \mathbb{C}^2$  and  $\mathcal{H}_B = \mathcal{H}_{B_1}^{\otimes N} \otimes \mathbb{C}^2$ . An  $N$ -undistillable state  $\rho$ , is  $N$ -activatable if for this integer  $N$ , condition (C1) is fulfilled, for a PPT-preserving CPM,  $\mathcal{E}$ .

### 2. Three parties

Let us now consider a density operator  $\rho$ , describing the state of a system composed of three subsystems. The Hilbert Space  $\rho$  is acting on is  $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{C_1}$ . Analogous to the previous section we reformulate the problem of distillation and activation in terms of CPM's. In the following we will, without loss of generality concentrate on the distillation and activation of the entanglement between Bob and Charly.



*Corollary 2'*. A state  $\rho$  is  $BC$ -distillable iff there exists a positive integer  $N$  and a separable CPM  $\mathcal{E}_a: \mathcal{B}(\mathcal{H}_1^{\otimes N}) \rightarrow \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  such that

$$\text{tr}[P_{B,C}^{T_C} \mathcal{E}_a(\rho^{\otimes N})] < 0, \quad (\text{C2})$$

where  $P_{B,C}$  is defined in Eq. (A1) and the small letter  $a$  indicates that  $\mathcal{E}_a$  maps a density operator describing the state of the particles  $A, B, C$  to a density operator describing the state of two qubits, one held by Bob and the other one by Charly.

Then  $\rho$  is  $N$ - $BC$ -distillable if for the integer  $N$  condition (C2) is fulfilled, for  $\mathcal{E}_a$  separable.

*Lemma 6'*. A state,  $\rho$ , which is not  $m$ - $BC$ -distillable, is  $BC$ -activatable iff there exists an integer  $N \leq m$  and a PPT-preserving CPM  $\mathcal{E}_a: \mathcal{B}(\mathcal{H}_1^{\otimes N}) \rightarrow \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  such that condition (C2) is fulfilled.

An  $N$ - $BC$ -undistillable state,  $\rho$ , is  $N$ - $BC$ -activatable if for the integer  $N$  condition (C2) is fulfilled, for a PPT-preserving CPM,  $\mathcal{E}_a$ .

Let us summarize this section: A state is  $N$ - $(XY)$  distillable (activatable) iff there exists a separable (PPT-preserving) CPM, which transforms  $N$  copies of the state into an entangled state acting on  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . This automatically implies that all PPTES are bound entangled, in the sense that their entanglement cannot be distilled nor activated.

- 
- [1] S. Lloyd, Sci. Am. Oct 1995, p. 140; C. H. Bennett, Phys. Today **48**(10), 24 (1995).
- [2] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
- [3] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
- [4] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
- [5] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Phys. Rev. Lett. **77**, 2818 (1996).
- [6] N. Gisin, Phys. Lett. A **210**, 151 (1996).
- [7] N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **81**, 3279 (1998).
- [8] A. Kent, Phys. Rev. Lett. **81**, 2839 (1998).
- [9] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).
- [10] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
- [11] W. Dür, J. I. Cirac, M. Lewenstein, and D. Bruss, Phys. Rev. A **61**, 062313 (2000).
- [12] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and A. V. Thapliyal, Phys. Rev. A **61**, 062312 (2000).
- [13] P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. Lett. **82**, 1056 (1999).
- [14] T. Eggeling, K. G. H. Vollbrecht, R. F. Werner, and M. M. Wolf, e-print quant-ph/0104095.
- [15] P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **86**, 2681 (2001).
- [16] In [12] the problem of distillability is connected to the problem of two-positivity of linear maps, in contrast to the results presented here.
- [17] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A **62**, 052310 (2000).
- [18] M. Lewenstein, B. Kraus, P. Horodecki, and J. I. Cirac, Phys. Rev. A **63**, 044304 (2001).
- [19] S. Karnas and M. Lewenstein, Phys. Rev. A **64**, 042313 (2001).
- [20] M. Horodecki, P. Horodecki, and R. Horodecki, e-print quant-ph/0006071.
- [21] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
- [22] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **233**, 8 (1996).
- [23] Note that partial transposition depends on the used basis, but the eigenvalues do not.
- [24] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
- [25] Consider, for instance, a bipartite DEW,  $W$ . It can be written, according to Eq. (4) as  $W = Q_0 + Q_1^{T_A} + Q_2^{T_B} = Q_0 + (Q_1 + Q_2^{T_A})^{T_A} = Q_0 + \tilde{Q}_1^{T_A}$ , with  $Q_0, Q_1, Q_2 \geq 0$ , where we used that  $Q_2^{T_B} = (Q_2^{T_A})^{T_A}$ .
- [26] Note that a Hermitian operator can either not be an EW, a DEW, or a NDEW.
- [27] D. M. Greenberger, M. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989).
- [28] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).
- [29] Note that  $W_\rho^a$  does not fulfill (i) iff  $(W_\rho^a)^{T_A} = P_{B_1 C_1} \otimes (\rho_{A_2 B_2 C_2}^T)^{T_{B_2}}$  does not fulfill (i). Thus, if  $W_\rho^a$  does not fulfill (i) then  $(W_\rho^a)^{T_A}$  must be in particular nonpositive semidefinite. But this is the case iff  $\rho^{T_B}$  is nonpositive semidefinite. And so we have, if  $W_\rho^a$  does not fulfill (i) then  $\rho$  is  $B$ -NPPT.