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# Coding with finite quantum systems

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Coding using quantum states in an angular-momentum $(2j+1)$-dimensional Hilbert space $H$ is considered in this paper. A concatenated code is studied in two steps. In the first step the space $H^N$ is considered and the code is its subspace $H_A$ spanned by the direct products of $N$ angular-momentum states with the same $m$. In the second step the space $H_A^M$ is considered and the code is its subspace $H_B$ spanned by the direct products of $M$ angle states with the same $m$. It is shown that the code introduces redundancy with respect to any transformation.

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## I. INTRODUCTION

A lot of the work on quantum-information processing has been based on qubits associated with quantum systems in a two-dimensional angular-momentum $(j=1/2)$ Hilbert space (reviewed in [1]). Most recently the use of $d$-dimensional Hilbert spaces (qudits) has been considered [2]. The use of infinite-dimensional Hilbert spaces associated with continuous variables has been studied in [3]. From a practical point of view in many qubit realizations the "natural" Hilbert space of the system is multidimensional (or infinite dimensional) and only a two-dimensional subspace is utilized for information processing while the rest is "wasted." The use of a bigger $d$-dimensional subspace could be much more efficient.

The purpose of this paper is to generalize Shor's coding method [4] for the case of qudits.

Coding adds redundant information to a message so that in spite of partial corruption of the encoded message by noise, it is possible to recover the original message. In quantum coding the Hilbert space $H$ is embedded into a larger Hilbert space. The scheme considered here uses $(2j+1)$-dimensional angular-momentum Hilbert spaces (Sec. II). It is the quantum version of a concatenated code. The idea is to construct a code in two steps by starting with a code which is a subspace $H_A$ in $H^N$, spanned by angular-momentum states with the same $m$ (Sec. III). Its words are used as symbols of a new alphabet in the second step (Sec. IV). Here the code is a subspace $H_B$ of $H_A^M$, spanned by angle states with the same $m$. The code of Sec. III is associated with "anisotropic redundancy" in the $J$ direction. The code of Sec. IV introduces extra redundancy in the $\theta$ direction and the combined effect is a general redundancy in any direction. The proposed scheme will protect quantum information against "small" errors that occur on some but not all the components.

## II. DISPLACEMENTS AND SQUEEZING IN FINITE SYSTEMS

Finite quantum systems have been studied originally by Weyl and Schwinger [5]. More recently this work has been applied in various contexts by various authors [6]. In Ref. [7] we have applied these ideas in the context of the angle-angular-momentum quantum phase-space. In this section we introduce the notation and review briefly some of these ideas in the context of qudits, which are needed for quantum coding in later sections.

Performing a finite Fourier transform on the usual angular momentum states (which we denote as $|J;j,m\rangle$) we introduce the angle states (which we denote as $|\theta;j,m\rangle$). Here $m$ takes values in $\mathcal{Z}(2j+1)$ [the integers mod$(2j+1)$]. Performing the same Fourier transform on both sides of the angular-momentum operators $(J_+,J_-,J_z)$ we get the angle operators $(\theta_+,\theta_-,\theta_z)$ that form an su(2) algebra. Like the $|J;j,m\rangle$ are eigenstates of $J^2,J_z$, the $|\theta;j,m\rangle$ are eigenstates of $\theta^2$, $\theta_z$. The quantum phase space for qudits is $\mathcal{Z}(2j+1)\times\mathcal{Z}(2j+1)$. We consider displacement operators and study the corresponding Heisenberg-Weyl group.

$\mathrm{Sl}(2,\mathcal{Z}(2j+1))$ transformations in the angle-angular-momentum quantum phase space, are the analog of the $\mathrm{Sl}(2,R)$ squeezing transformations in the harmonic oscillator context. When $2j+1$ is the power of a prime, stronger results can be derived. The reason is that the formalism uses integers in $\mathcal{Z}(2j+1)$. This is in general a commutative ring with a unity; and in the case that the $2j+1$ is a power of a prime, a Galois field. The existence of inverses in the Galois case, ensures that the squeezing concepts of dilation and contraction in the $J_z$ and $\theta_z$ directions correspondingly, are well defined.

We have explained in [7] that the formulas in "Bose-Hilbert spaces" with integer $j$ are slightly different from their counterparts in "Fermi-Hilbert spaces" with half-integer $j$. For simplicity we limit our discussion to the former case (integer $j$).

### A. Displacements

We denote as $|J;j,m\rangle$ the usual angular-momentum states. $m$ belongs to $\mathcal{Z}(2j+1)$. The states $|J;j,m\rangle$ span the Hilbert space $H$. The finite Fourier transform is defined as

$$F=(2j+1)^{-1/2}\sum_{m,n}\omega(mn)|J;j,m\rangle\langle J;j,n|, \qquad (1)$$

$$\omega(\alpha)=\exp\left[i\frac{2\pi\alpha}{2j+1}\right]; \quad FF^{\dagger}=F^{\dagger}F=1, \quad F^4=1. \qquad (2)$$

 

Using these Fourier transforms we have introduced [7] the $\theta$ basis of angle states $|\theta;j,m\rangle$ as follows:

$$|\theta;j,m\rangle = F|J;j,m\rangle = (2j+1)^{-1/2}\sum_n \omega(mn)|J;j,n\rangle. \tag{3}$$

We have also introduced the angle operators $\theta_+, \theta_-, \theta_z$,

$$\theta_z = FJ_zF^\dagger, \quad \theta_+ = FJ_+F^\dagger, \quad \theta_- = FJ_-F^\dagger, \tag{4}$$

which obey the su(2) algebra. The $\theta$ operators act on the $\theta$ states in an analogous way to the $J$ operators acting on the $J$ states. The displacement operators are defined as

$$X = \exp\left[-i\frac{2\pi}{2j+1}\theta_z\right], \quad Z = \exp\left[i\frac{2\pi}{2j+1}J_z\right], \tag{5}$$

$$X^{2j+1} = Z^{2j+1} = 1, \quad X^\beta Z^\alpha = Z^\alpha X^\beta \omega(-\alpha\beta), \tag{6}$$

where $\alpha,\beta$ are integers in $\mathcal{Z}(2j+1)$, and perform displacements along the $J_z$ and $\theta_z$ axes, as follows:

$$X^\beta|J;j,m\rangle = |J;j,m+\beta\rangle, \quad X^\beta|\theta;j,m\rangle = \omega(-\beta m)|\theta;j,m\rangle, \tag{7}$$

$$Z^\alpha|J;j,m\rangle = \omega(m\alpha)|J;j,m\rangle, \quad Z^\alpha|\theta;j,m\rangle = |\theta;j,m+\alpha\rangle. \tag{8}$$

It has been explained in Ref. [8] that the displacement operators

$$D(\alpha,\beta) = Z^\alpha X^\beta \omega(-2^{-1}\alpha\beta), \tag{9}$$

where $2^{-1}$ is the inverse of 2 within $\mathcal{Z}(2j+1)$, are generators for the $U(1)\times SU(2j+1)$ transformations in the Hilbert space $H$. They are an alternative to the usual Cartan-Weyl generators. Their commutator is

$$\begin{aligned}[D(\alpha_1,\beta_1),D(\alpha_2,\beta_2)] &\equiv D(\alpha_1,\beta_1)D(\alpha_2,\beta_2) \\ &\quad - D(\alpha_2,\beta_2)D(\alpha_1,\beta_1) \\ &= 2i\sin\left[\frac{2\pi}{2j+1}2^{-1}(\alpha_1\beta_2-\alpha_2\beta_1)\right] \\ &\quad \times D(\alpha_1+\alpha_2,\beta_1+\beta_2).\end{aligned} \tag{10}$$

Therefore infinitesimal $U(1)\times SU(2j+1)$ transformations can be written as

$$g = 1 + \sum_{\alpha,\beta} \lambda_{\alpha\beta} X^\alpha Z^\beta, \tag{11}$$

where $\lambda_{\alpha\beta}$ are infinitesimal coefficients. For later purposes we stress here that $X$ and $Z$ should be seen not only as operators that perform shifts in the $J$ and $\theta$ directions (which is a very special case of transformations), but through $X^\alpha Z^\beta$, also as generators of general $U(1)\times SU(2j+1)$ transformations in the Hilbert space $H$.

### B. Galois qudits

In order to introduce squeezing transformations in the qudit quantum phase space $\mathcal{Z}(2j+1)\times\mathcal{Z}(2j+1)$ we need to introduce the concept of dilation by a factor $\alpha$ [in $\mathcal{Z}(2j+1)$] in the $J_z$ direction; and dilation by a factor $\alpha^{-1}$ (i.e., contraction by a factor $\alpha$) in the $\theta_z$ direction. The question of the existence of the ''inverse'' of an element of $\mathcal{Z}(2j+1)$ arises here.

When $(2j+1)$ is not a power of a prime $p$, the $\mathcal{Z}(2j+1)$ is a commutative ring with a unity, and inverses do not necessarily exist. Only when the $(2j+1)$ is a power of a prime $p$ $(2j+1=p^n)$, the $\mathcal{Z}(p^n)$ is a field and all nonzero elements have an inverse. This a famous result by Galois and the corresponding fields are called Galois fields. In Ref. [7] we called the systems with a Hilbert space whose dimension is the power of a prime Galois quantum systems. Here we call the qudits associated with Hilbert spaces with dimension $p^n$ Galois qudits. The phase space $\mathcal{Z}(p^n)\times\mathcal{Z}(p^n)$ of Galois qudits is a finite geometry [9] and dilations, contractions, and discrete rotations are well defined. In contrast, the phase space of non-Galois qudits is a set of points with no geometrical structure.

In [7] we have studied a factorization of a finite system into subsystems. This is based on the Chinese remainder theorem and is similar to the factorization used in ''fast Fourier transforms'' in order to reduce the computation time. In order to present the result in the present context, we factorize an arbitrary $2j+1$ in terms of powers of prime numbers,

$$2j+1 = (p_1)^{n_1}\cdots(p_N)^{n_N}. \tag{12}$$

We also introduce the integers

$$r_i = \frac{2j+1}{p_i^{n_i}}, \quad t_i r_i = 1; \ (\mathrm{mod}\, p_i^{n_i}). \tag{13}$$

If $m$ belongs to $\mathcal{Z}(2j+1)$, we define its ''remainders'' $m_i = m(\mathrm{mod}\, p_i^{n_i})$ and $\overline{m}_i = mt_i(\mathrm{mod}\, p_i^{n_i})$ where $m_i$ belongs to $\mathcal{Z}(p_i^{n_i})$. We then have the one-to-one mappings

$$m \leftrightarrow (m_1,\ldots,m_N), \quad m \leftrightarrow (\overline{m}_1,\ldots,\overline{m}_N), \tag{14}$$

which we use to define an isomorphism between the Hilbert space $H_{2j+1}$ and the direct product of all the Hilbert spaces $H_{p_i^{m_i}}$ as

$$|J;j,m\rangle \leftrightarrow \otimes_{i=1}^N |J;\tfrac{1}{2}(p_2^{n_i}-1)p_i^{n_i},\overline{m}_i\rangle \tag{15}$$

or equivalently as

$$|\theta;j,m\rangle \leftrightarrow \otimes_{i=1}^N |\theta;\tfrac{1}{2}(p_2^{n_i}-1)p_2^{n_i},m_i\rangle. \tag{16}$$

The proof of the above has been given in [7]. Using these results we can factorize any qudit as a product of Galois qudits.

We finally point out that Galois fields and finite geometries play an important role in classical-information processing. Although in this paper we use the Galois qudits only in the context of squeezing, we anticipate that they might play a wider role in quantum-information processing.

### C. Squeezing: $\mathrm{Sl}(2,\mathcal{Z}(2j+1))$ transformations

We consider the transformations

$$X'=X^{\alpha}Z^{\beta},\quad Z'=X^{\gamma}Z^{\delta},\quad \alpha\delta-\beta\gamma=1[\mathrm{mod}(2j+1)], \tag{17}$$

where $\alpha,\beta,\gamma,\delta$ are integers in $\mathcal{Z}(2j+1)$. These transformations preserve Eq. (6) and form the $\mathrm{Sl}(2,\mathcal{Z}(2j+1))$ group. They are the analog of the $\mathrm{Sl}(2,R)$ transformations in the harmonic oscillator phase space, which lead to the Bogoliubov transformations. We note that for Galois qudits, for a given triplet $\alpha,\beta,\gamma$ (with $\alpha\neq 0$) there exist $\delta=\alpha^{-1}(\beta\gamma+1)$, which satisfies Eq. (17).

The general operators that lead to the transformations (17) have been given in [7]. For later purposes we give here the "dilation-contraction" operators that perform the transformations (17) with $\beta=\gamma=0$ and $\delta=\alpha^{-1}$,

$$\begin{aligned} S(\alpha)&=\sum_{n=-j}^{j}|J;j,\alpha n\rangle\langle J;j,n|=\sum_{n=-j}^{j}|J;j,n\rangle\langle J;j,\alpha^{-1}n| \\ &=\sum_{n=-j}^{j}|\theta;j,\alpha^{-1}n\rangle\langle\theta;j,n|=\sum_{n=-j}^{j}|\theta;j,n\rangle\langle\theta;j,\alpha n|. \end{aligned} \tag{18}$$

We can prove that

$$S(\alpha)XS^{\dagger}(\alpha)=X^{\alpha},\quad S(\alpha)ZS^{\dagger}(\alpha)=Z^{\alpha^{-1}}. \tag{19}$$

They are the analogs of the $x'=\alpha x$ and $p'=\lambda^{-1}p$ in the $x-p$ plane (the phase space of the harmonic oscillator). Indeed the "dilation" $n'=\alpha n$ provides a one-to-one map $\mathcal{Z}(p^m)\rightarrow\mathcal{Z}(p^m)$ in the "$J_z$ direction," and the "contraction" $n''=\alpha^{-1}n$ provides a one-to-one map $\mathcal{Z}(p^m)\rightarrow\mathcal{Z}(p^m)$ in the "$\theta_z$ direction."

## III. *J* REDUNDANCY

In classical coding we start with an alphabet $h$ of several distinct symbols and introduce redundancy by constructing words of length $N$. The code is a subset of $h^N$. Noise can change a letter into another letter. In quantum coding we start with a Hilbert space $H$ (in our case the one considered in the preceding section). The code is a subspace of $H^K$. Noise performs transformations in $H^K$.

In this paper we generalize Shor's coding method for qudits. This is the quantum version of a concatenated code (e.g., [10]). The idea is to construct a code in two steps. In the first step considered in this section the code is the subspace $H_A$ of $H^N$ spanned by the direct products of $N$ angular-momentum states with the same $m$. We consider various transformations and show that this introduces "anisotropic redundancy" in the $J$ direction.

### A. Hilbert space and operators

We consider states in the direct product $H^N\equiv H\otimes\cdots\otimes H$ ($N$ times). In this $(2j+1)^N$-dimensional space we consider the $(2j+1)$-dimensional subspace spanned by the vectors

$$H_A=\{|J_A;j,m\rangle\equiv|J;j,m\rangle\otimes\cdots\otimes|J;j,m\rangle,\ \ m=-j,\ldots,j\}. \tag{20}$$

The Hilbert space $H_A$ is isomorphic to the Hilbert space $H$ through the mapping $|J_A;j,m\rangle\leftrightarrow|J;j,m\rangle$. Given some states and operators in $H$, we use the same notation for their counterparts in $H_A$ with an additional index $A$. For example, the operator $J_{zA}$ and its eigenstates $|J_A;j,m\rangle$ in $H_A$ correspond to the operator $J_z$ and its eigenstates $|J;j,m\rangle$ in $H$.

We call $\Pi_A$ the projection operators in $H_A$,

$$\Pi_A=\sum_m |J;j,m\rangle\langle J;j,m|\otimes\cdots\otimes|J;j,m\rangle\langle J;j,m|. \tag{21}$$

We use the notation $W_i\equiv 1\otimes\cdots\otimes W\otimes\cdots\otimes 1$ ($i=1,\ldots,N$) for operators acting on $H^N$, with the operator $W$ acting on the $i$ Hilbert space $H$. It can be easily seen that

$$[\Pi_A,J_{zi}]=[\Pi_A,Z_i]=0,\quad i=1,\ldots,N. \tag{22}$$

Using this we can show that

$$\begin{gathered} J_{zA}=J_{zi}\Pi_A, \\ (J_{zA})^2=J_{zi}J_{zk}\Pi_A,\ldots,(J_{zA})^N=J_{z1},\ldots,J_{zN}\Pi_A, \end{gathered} \tag{23}$$

and also that

$$\begin{gathered} Z_A=Z_i\Pi_A, \\ (Z_A)^2=Z_iZ_k\Pi_A,\ldots,(Z_A)^N=Z_1,\ldots,Z_N\Pi_A. \end{gathered} \tag{24}$$

We note, however, that analogous relations for $J_{+i}$ and $J_{-i}$ are not true. For example, $\Pi_AJ_{+i}\Pi_A=\Pi_AJ_{-i}\Pi_A=0$.

It is easily seen that

$$Z_iZ_k^{-1}|J_A;j,m\rangle=|J_A;j,m\rangle,\quad [Z_iZ_k^{-1}]^{2j+1}=1. \tag{25}$$

Therefore the $Z_iZ_k^{-1}$ are stabilizers of all states in $H_A$. They form an Abelian finite group with $N-1$ generators, $Z_1Z_2^{-1},Z_2Z_3^{-1},\ldots,Z_{N-1}Z_N^{-1}$. We call $G_i$ ($i=1,\ldots,N-1$) the cyclic group of order $2j+1$, generated by $Z_iZ_{i+1}^{-1}$. The total Abelian finite group of the stabilizers is the direct product

$$G=G_1\times\cdots\times G_{N-1};\quad G_i=\{1,Z_iZ_{i+1}^{-1},\ldots,[Z_iZ_{i+1}^{-1}]^{2j}\}, \tag{26}$$

and is of order $(2j+1)^{N-1}$.

The Fourier operator $F$, acting on the states of the Hilbert space $H$, corresponds to the Fourier operator $F_A$ acting on the states of the Hilbert space $H_A$, given by

$$F_A=(2j+1)^{-1/2}\sum_{m,n}\omega(mn)|J;j,m\rangle\langle J;j,n|\otimes\cdots\otimes|J;j,m\rangle \times\langle J;j,n|. \quad (27)$$

We call $\overline{N}$ the remainder of $N$ divided over $2j+1$ [$\overline{N}=N(\mathrm{mod}(2j+1))$]. Using Eq. (18) we can show that

$$F_A=(2j+1)^{(N-1)/2}S(\overline{N})\Pi_A F_1,\ldots,F_N\Pi_A. \quad (28)$$

The states $|\theta;j,m\rangle$ in $H$ correspond to the states $|\theta_A;j,m\rangle\equiv F_A|J_A;j,m\rangle$, which form an orthonormal basis in $H_A$. It is easily seen that

$$\Pi_A|\theta;j,m\rangle\otimes\cdots\otimes|\theta;j,m\rangle=(2j+1)^{(1-N)/2}|\theta_A;j,Nm\rangle, \quad (29)$$

where $Nm$ is defined $\mathrm{mod}(2j+1)$. We can also prove

$$\Pi_A\theta_{zi}\Pi_A=\Pi_A X_i\Pi_A=0. \quad (30)$$

More generally the above equation is true if $\theta_{zi}$ is replaced by a product of several but *not all* $\theta_{zi}$; and also if $X_i$ is replaced by a product of several but *not all* $X_i$. For the product $X_1,\ldots,X_N$ we can show that

$$[\Pi_A,X_1,\ldots,X_N]=0;\quad X_A=X_1,\ldots,X_N\Pi_A. \quad (31)$$

We can interpret the equations of this section as a "dilution" in the $J$ direction of the Hilbert space $H_A$ within the Hilbert space $H^N$. The states $|J_A;j,m\rangle$ are far from each other in the sense that the operator $X_A$, which performs shifts among them, is equal to the product of all $X_i$ [Eq. (31)]. Therefore, it is unlikely that noise will perform such transformations causing errors. In contrast there is no dilution in the $\theta$ direction. The operator $Z_A$ that performs shifts between the various $|\theta_A,j,m\rangle$ states requires the action of one $Z_i$ [Eq. (24)]. This dilution in the $J$ direction is also seen in the appearance of the "dilation-contraction" operator $S(\overline{N})$ in the Fourier operator of Eq. (28).

The shifts in the $J$ and $\theta$ directions are, of course, a very special case of transformations, and in Sec. III B we discuss more general transformations.

### B. Transformations

We consider general unitary $[U(1)\times SU(2j+1)]^N$ transformations on the states in the Hilbert space $H^N$. Infinitesimal action of these transformations can be written as [Eq. (11)]

$$g=1+\sum_{i,\alpha,\beta}\lambda_{i\alpha\beta}X_i^{\alpha}Z_i^{\beta}, \quad (32)$$

where $\lambda_{i\alpha\beta}$ are infinitesimal coefficients.

A subgroup of these transformations is the $U(1)\times SU(2j+1)$ transformation on the states in the Hilbert space $H_A$. In fact since the group $G$ of Eq. (26) are stabilizers for the states in $H_A$ we can consider the quotient group $[U(1)\times SU(2j+1)]/G$. Infinitesimal action of these transformations can be written as

$$g_A=1+\sum_{\alpha,\beta}\lambda_{\alpha\beta}X_A^{\alpha}Z_A^{\beta}, \quad (33)$$

where 1 is here the unit element in $U(1)\times SU(2j+1)$. Using Eqs. (19) and (26) we rewrite them as

$$g_A=\Pi_A\left[1+\sum_{\alpha,\beta}\lambda_{\alpha\beta}(X_1^{\alpha},\ldots,X_N^{\alpha})Z_i^{\beta}\right]. \quad (34)$$

It is seen here that we have an "anisotropic dilution" of the Hilbert space $H_A$ within the Hilbert space $H^N$, and transformations that contain $X_A^{\alpha}$ are performed with the $X_1^{\alpha},\ldots,X_N^{\alpha}$ while transformations that contain $Z_A^{\beta}$ are performed with the $Z_i^{\beta}$. So we have redundancy in the $J$ direction only.

The above are unitary transfomations. A rather general class of noise transformations describing the interaction of a quantum system with density matrix $\rho$ with the environment can be written as

$$\rho'=\sum_i E_{\ell}\rho E_{\ell}^{\dagger},$$

$$E_{\ell}=\sum\lambda_{\ell;\alpha_1\beta_1,\ldots,\alpha_N\beta_N}(X_1^{\alpha_1}Z_1^{\beta_1})\cdots(X_N^{\alpha_N}Z_N^{\beta_N}). \quad (35)$$

Here we have written the general expression where noise acts on all qudits, but in the case of "small noise" we can assume that noise acts only on some of the qudits. Our coding so far provides protection against noise in the $J$ direction only. In the following section we introduce redundancy in the $\theta$ direction and show that the combined effect is redundancy in any direction.

## IV. GENERAL REDUNDANCY

In this section we consider the second step of the quantum concatenated code. Here the code is the subspace $H_B$ of $H_A^M$ spanned by the direct products of $M$ angle states with the same $m$. We consider various transformations and show that the combined effect is general redundancy in any direction.

### A. Hilbert space and operators

We consider the space $(H_A)^M\equiv H_A\otimes\cdots\otimes H_A$ ($M$ times), which is clearly a $(2j+1)^M$-dimensional subspace of the space $H^{NM}$. The operator $P_A=\Pi_A\otimes\cdots\otimes\Pi_A$ projects the space $H^{NM}$ to the space $(H_A)^M$. We use the notation $W_{A\mu}\equiv\Pi_A\otimes\cdots\otimes W_A\otimes\cdots\otimes\Pi_A$ $(\mu=1,\ldots,M)$ for operators acting on $(H_A)^M$, with the operator $W_A$ acting on the $\mu$ Hilbert space $H_A$. Clearly $\Pi_{A\mu}=P_A$ for any $\mu$. For a product of two operators $W_AV_A$ it is easily seen that $(W_AV_A)_\mu=W_{A\mu}V_{A\mu}$. Most of the equations of Sec. III A can be generalized in the space $(H_A)^M$. For example, Eq. (24) will become $Z_{A\mu}=(Z_i\Pi_A)_\mu=Z_{i\mu}P_A$. In $Z_{i\mu}$ the indices $i$ and $\mu$

refer to the positions in the words considered at the first and second step of the concatenated code, respectively.

We consider the $(2j+1)$-dimensional subspace spanned by the vectors

$$H_B=\{|\theta_B;j,m\rangle\equiv|\theta_A;j,m\rangle\otimes\cdots\otimes|\theta_A;j,m\rangle,\quad m=-j,\ldots,j\}. \tag{36}$$

For the states and operators in $H_B$, we use the same notation as for their counterparts in $H$ with an additional index $B$. The Hilbert space $H_B$, is isomorphic to the Hilbert space $H_A$ and also to the Hilbert space $H$, through the mapping $|\theta_B;j,m\rangle\leftrightarrow|\theta_A;j,m\rangle\leftrightarrow|\theta;j,m\rangle$. We call $P_B$ the projection operator in $H_B$,

$$P_B=\sum_m |\theta_A;j,m\rangle\langle\theta_A;j,m|\otimes\cdots\otimes|\theta_A;j,m\rangle\langle\theta_A;j,m|. \tag{37}$$

It is clear that $P_AP_B=P_BP_A=P_B$.

It can be easily seen [using Eq. (31)] that

$$[P_B,X_{A\mu}]=0,\quad X_{A\mu}=X_{1\mu},\ldots,X_{N\mu}P_A. \tag{38}$$

Using this we can show that

$$X_B=X_{A\mu}P_B=X_{1\mu},\ldots,X_{N\mu}P_B. \tag{39}$$

We have explained in the preceding section that the $Z_iZ_k^{-1}$ are stabilizers of all states in $H_A$. Therefore, the $Z_{i\mu}Z_{k\mu}^{-1}$ are stabilizers of the states $|\theta_B;j,m\rangle$. In addition to that it is easily seen that

$$X_{A\mu}X_{A\nu}^{-1}|\theta_B;j,m\rangle=|\theta_B;j,m\rangle. \tag{40}$$

Using Eq. (38) we conclude that the

$$(X_{1\mu}X_{1\nu}^{-1})\cdots(X_{N\mu}X_{N\nu}^{-1}) \tag{41}$$

are also stabilizers of the states $|\theta_B;j,m\rangle$. All these stabilizers commute with each other and they form an Abelian finite group with the $NM-1$ generators

$$Z_{1\mu}Z_{2\mu}^{-1},\quad Z_{2\mu}Z_{3\mu}^{-1},\ldots,Z_{(N-1)\mu}Z_{N\mu}^{-1},\quad [(X_{11}X_{12}^{-1})\cdots(X_{N1}X_{N2}^{-1})],\ldots,\quad [(X_{1(M-1)}X_{1M}^{-1})\cdots(X_{N(M-1)}X_{NM}^{-1})], \tag{42}$$

where $\mu=1,\ldots,M$. Each of these generators generates a cyclic group of order $2j+1$, and the full Abelian finite group $G'$ of the stabilizers is the direct product of all of them and is of order $(2j+1)^{NM-1}$.

The Fourier operator $F$ acting on the states of the Hilbert space $H$ corresponds to the Fourier operator $F_B$ acting on the states of the Hilbert space $H_B$ and is given by

$$F_B=(2j+1)^{-1/2}\sum_{m,n}\omega(mn)|\theta_A;j,m\rangle\times\langle\theta_A;j,n|\otimes\cdots\otimes|\theta_A;j,m\rangle\langle\theta_A;j,n|. \tag{43}$$

Let $M^{-1}$ be an integer in $\mathcal{Z}(2j+1)$ such that $M^{-1}M=1[\mathrm{mod}(2j+1)]$. As we explained above in the Galois case, provided that $M$ is not 0 [integer multiple of $(2j+1)$], the $M^{-1}$ exists. Using Eq. (18), it can be shown that

$$F_B=(2j+1)^{(M-1)/2}S(M^{-1})P_BF_{A1},\ldots,F_{AM}P_B. \tag{44}$$

The states $|J;j,m\rangle$ in $H$ correspond to the states $|J_B;j,m\rangle\equiv F_B|\theta_B;j,m\rangle$, which form an orthonormal basis in $H_B$. It is easily seen that

$$P_B|J_A;j,m\rangle\otimes\cdots\otimes|J_A;j,m\rangle=(2j+1)^{(1-M)/2}|J_B;j,Mm\rangle. \tag{45}$$

For the product $Z_{A1},\ldots,Z_{AM}$ we show that

$$[P_B,Z_{A1},\ldots,Z_{AM}]=0,\quad Z_B=Z_{i1},\ldots,Z_{iM}P_B. \tag{46}$$

Here $i$ can take any value from 1 to $N$. Equation (19) has been used in the proof of the second of these equations.

In Sec. III we have introduced redundancy in the $J$ direction. In this section we have introduced further redundancy in the $\theta$ direction. The operator $X_B$ that performs shifts among the states $|J_B;j,m\rangle$ is equal to the product of all $X_{i\mu}$ [Eq. (39), all $i$, fixed $\mu$]. The operator $Z_B$ that performs shifts among the states $|\theta_B;j,m\rangle$, is equal to the product of all $Z_{i\mu}$ [Eq. (46), all $\mu$, fixed $i$]. Therefore, it is unlikely that small noise will perform such transformations causing errors. In the following section we discuss more general transformations.

### B. Transformations

We first consider general unitary $[\mathrm{U}(1)\times\mathrm{SU}(2j+1)]^{NM}$ transformations on the states in the Hilbert space $H^{NM}$. Infinitesimal action of these transformations can be written as

$$g=1+\sum_{i,\mu,\alpha,\beta}\lambda_{i\mu\alpha\beta}.X_{i\mu}^{\alpha}Z_{i\mu}^{\beta}. \tag{47}$$

A subgroup of these transformations is the $[\mathrm{U}(1)\times\mathrm{SU}(2j+1)]^M$ transformations on the states in the Hilbert space $(H_A)^M$. Infinitesimal action of these transformations can be written as

$$g_A=1+\sum_{\mu,\alpha,\beta}\lambda_{\mu\alpha\beta}X_{A\mu}^{\alpha}Z_{A\mu}^{\beta}, \tag{48}$$

where 1 is here the unit element in $[\mathrm{U}(1)\times\mathrm{SU}(2j+1)]^M$.

A subgroup of these transformations is the $\mathrm{U}(1)\times\mathrm{SU}(2j+1)$ transformations on the states in the Hilbert space $H_B$. In fact we can consider the quotient group $[\mathrm{U}(1)\times\mathrm{SU}(2j+1)]/G'$, where $G'$ is the group of stabilizers for the states in $H_B$, discussed in Sec. IV A. Infinitesimal action of these transformations can be written as

$$g_B = 1 + \sum_{\alpha,\beta} \lambda_{\alpha\beta} X_B^{\alpha} Z_B^{\beta}, \tag{49}$$

where 1 is here the unit element in $U(1)\times SU(2j+1)$. Using Eqs. (39) and (46) we rewrite them as

$$g_B = P_B\left[1 + \sum_{\alpha,\beta} \lambda_{\alpha\beta}(X_{1\mu}^{\alpha},\ldots,X_{N\mu}^{\alpha})(Z_{i1}^{\beta},\ldots,Z_{iM}^{\beta})\right]. \tag{50}$$

It is seen that we have redundancy in all directions.

Noise transformations can be written as

$$\rho' = \sum_i E_\ell \rho E_\ell^\dagger,$$

$$E_\ell = \sum \lambda_{\ell;\alpha_{11}\beta_{11},\ldots,\alpha_{NM}\beta_{NM}} (X_{11}^{\alpha_{11}} Z_{11}^{\beta_{11}}) \cdots (X_{NM}^{\alpha_{NM}} Z_{NM}^{\beta_{NM}}). \tag{51}$$

Our coding provides protection against noise in any direction provided that we have ''small noise'' that acts only on some qudits.

## V. DISCUSSION

A lot of the work on quantum computers uses qubits as building blocks. We have considered qudits associated with states in $d=2j+1$ angular-momentum Hilbert spaces. Practical implementation of qubits in many cases uses a two-dimensional subspace of a bigger Hilbert space. In such cases, the use of a bigger $d$-dimensional subspace could be much more efficient.

We have considered a concatenated code that involves two steps. In the first one the space $H^N$ is considered and the code is the subspace $H_A$ spanned by the direct products of $N$ angular-momentum states with the same $m$. We have shown that this introduces ''anisotropic redundancy'' in the $J$ direction. In the second step, words of the first step, are used as letters. The space $H_A^M$ is considered and the code is the subspace $H_B$ spanned by the direct products of $M$ angle states with the same $m$. We have shown that this introduces general redundancy in any direction. The proposed scheme will protect quantum information against ''small'' errors that occur on some the components.

The work provides the theoretical background for qudit quantum computation.


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