Zero-energy solutions and vortices in Schrödinger equations

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Two-dimensional Schrödinger equations with rotationally symmetric potentials $[V_a(\rho)=-a^2g_a\rho^{2(a-1)}$ with $\rho=\sqrt{x^2+y^2}$ and $a\neq 0]$ are shown to have zero-energy states. For the zero energy eigenvalue the equations for all a are reduced to the same equation representing two-dimensional free motions in the constant potential $V_a=-g_a$ in terms of the conformal mappings of $\zeta_a=z^a$ with $z=x+iy$. Namely, the zero-energy eigenstates are described by plane waves with the fixed wave numbers $k_a=\sqrt{2mg_a}/\hbar$ in the mapped spaces. All the zero-energy states are infinitely degenerate similar to the case of the parabolic potential barrier (PPB) shown by Shimbori and Kobayashi [J. Phys. A **33**, 7637 (2000)]. Following hydrodynamical arguments, we see that such states describe stationary flows around the origin, which are represented by the complex velocity potentials $W_a=\sqrt{2g_a/m}z^a$, and their linear combinations create almost arbitrary vortex patterns. Examples of the vortex patterns in constant potentials and PPB are presented. In the extension to three-dimensional problems with potentials being separable into $2+1$ dimensions we show that the states in three dimensions have the same structure as the two-dimensional states with the zero energy but they can generally have nonzero total energies.

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I. INTRODUCTION

It is known that scattering states and unstable states like resonances are generally described by states in conjugate spaces of Gel'fand triplets (GTs) [1]. In order to analyze wider phenomena in quantum mechanics, investigations in terms of solutions of GTs (GT solutions) will be indispensable. Phenomenological analyses in terms of the GT solutions, however, have not yet been performed so much. One of the reasons is due to the difficulty of obtaining the GT solutions exactly. We, therefore, do not know their common properties covering a wide range for potentials. The other is the fact that we do not yet understand what are good observables in real physical phenomena described by the GT solutions. To understand how they can be observed in real phenomena is a fundamental problem for the development of the phenomenology in terms of the GT solutions. For that purpose we would like to start from the analysis of the GT solutions for an exactly solvable potential model and then to study common properties for more general types of the potentials. A simple example of exactly solvable models is a parabolic potential barrier (PPB). The eigenstates of the PPB $V(x)=-m\gamma^2x^2/2$ in one dimension have been studied by many authors [2–7]. It has been shown that the one-dimensional PPB has pure imaginary energy eigenvalues $\mp i(n+1/2)\hbar\gamma$ with $n=0,1,2,\ldots$, and the eigenfunctions are generalized functions in the conjugate space $\mathcal{S}(\mathbb{R})^\times$ of GTs described by $\mathcal{S}(\mathbb{R})\subset L^2(\mathbb{R})\subset\mathcal{S}(\mathbb{R})^\times$, where $\mathcal{S}(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively, stand for a Schwartz space and a Lebesgue space [5,6]. In general, the energy eigenvalues $\mathcal{E}$ of the conjugate spaces in GTs are expressed by pairs of complex conjugates such that $\mathcal{E}=\varepsilon\mp i\gamma$ with $\varepsilon,\gamma\in\mathbb{R}$, and the states with the $\mp$ sign, respectively, represent resonance-decay and resonance-formation processes. This pairing property of the energy eigenvalues indicates that states in higher-dimensional PPB possibly have zero-energy eigenstates. In fact the PPB in two dimensions (generally in even dimensions) has zero energy eigenvalue, which is included in the eigenvalues expressed by $\mp i(n_x-n_y)\hbar\gamma$. We see that the zero-energy states are obtained for zero and positive integers satisfying $n_x=n_y$ and then they are infinitely degenerate. The zero-energy states are interpreted as stationary flows around the center of the PPB [8]. Furthermore, following hydrodynamics, it has been also shown that some of such flows can be expressed by complex velocity potentials and various vortex structures appear in the linear combinations of the infinitely degenerate states. Considering that states in the conjugate spaces of GTs are generally not normalizable but currents of those states are observable in quantum mechanics, the quantities observed in physical processes should be based on probability currents such as currents in hydrodynamics. Hydrodynamical considerations will play a very important role in the investigations of quantum physics in GT. The hydrodynamical approach of quantum mechanics was vigorously investigated in the earlier stage of the development of quantum mechanics [9–16]. Vortices were extensively examined by Hirschfelder and co-workers [17–20] and a review article was written by Ghosh and Deb [21]. It should be noted that such a hydrodynamical idea is still useful in present-day quantum physics [8,22,23]. Actually problems of vortices appear in many aspects of present-day physics such as vortex matters (vortex lattices) [24,25], vortices in non-neutral plasma [26–29], Bose-Einstein gases [30–34] and so on. The vortex problems will hold a very important position in the hydrodynamical approach of quantum mechanics. As noted in Ref. [8], the stationary flows in the two-dimensional PPB can create almost arbitrary patterns of vortices because of the infinite degeneracy. PPB potentials can be a good approximation to the repulsive forces that are very weak at the center of the forces such that harmonic oscillators are a good approximation to the attractive forces, being very weak at the center. In fact PPB has been applied in some chemical problems [35–37]. The PPB, however, is a very special potential and then it seems to be difficult that the results of PPB extend to more general potentials.

In this paper we shall investigate stationary flows in more general types of potentials from the hydrodynamical point of view. Especially, we study stationary flows in two dimensions discussed in the PPB [8] because interesting quantities in hydrodynamics such as complex velocity potentials and vortices are definable in the dimensions. Furthermore, it is well known that in two-dimensional hydrodynamical problems, conformal mappings can be a very strong tool to investigate velocities, complex velocity potentials, and vortices [38–41]. Some solutions solved in a special aspect are possibly extendable to others in terms of conformal mappings. Particularly, we will pay attention to the stationary flows that are described by the zero-energy eigenstates in the PPB [8]. Such zero-energy states can also play an interesting role in statistical mechanics in GTs, where a new type of entropy arises from the freedom with respect to the imaginary parts of energy eigenvalues [42–44]. That is to say, even if a many-particle system is in the ground state with a fixed energy and then it has no freedom arising from the real energy eigenvalue, it can still have freedom with respect to the imaginary parts. Remembering the fact that all the energy eigenvalues in GTs are expressed in pairs of complex conjugates, we can understand the situation very easily, because the imaginary part of every complex energy can be canceled out by adding the conjugate energy in many-particle systems. An example was presented in Ref. [43] for the PPB, where the burst of entropy from the new entropy was studied in thermal nonequilibrium. If we can find such zero-energy states in more general types of potentials, we can investigate many-particle systems from a very new aspect where every state with a fixed energy can still have huge variety arising from the degeneracy of zero-energy states. This paper has two themes. One of the themes is to show the fact that rotational symmetric potentials of the type $V_a(\rho) = -a^2 g_a \rho^{2(a-1)}$ with $\rho=\sqrt{x^2+y^2}$ and $a\neq 0$ have the same zero-energy solutions as those obtained in the PPB in two dimensions [8]. We shall show that, as far as the zero-energy solutions are concerned, Schrödinger equations with the rotational symmetric potentials $V_a(\rho)$ for all values of a except $a=0$ are reduced to the same equation by using conformal mappings and, therefore, the infinite degeneracy of the solutions obtained in the PPB case appears in all the potentials. The other is to investigate vortex formation expected from the zero-energy solutions in the hydrodynamical approach. We shall see that the infinite degeneracy of the zero-energy solutions can be observed as various patterns of vortices in real physical phenomena, and some simple vortex patterns are presented. In these discussions the conformal mappings, which are known to be very powerful tools in two-dimensional hydrodynamics, become powerful tools also in the hydrodynamical approach of quantum mechanics, and vortex patterns for all the potentials $V_a(\rho)$ can be investigated by a very simple method. In this paper we would like to show that the solutions of GTs are not only objects of mathematical interest but are also very interesting objects for describing real physical processes, especially, those in vortex phenomena.

We shall perform our considerations as follows. In Sec. II, a general property of Schrödinger equations with rotational symmetric potentials is investigated in terms of conformal mappings. In Sec. III it is shown that for zero-energy solutions, all the equations in the mapped spaces can be reduced to one equation describing free motion in a constant potential. This means that, as far as the zero-energy eigenstates are concerned, all the symmetric potentials have the same solutions with infinite degeneracy as those obtained in the PPB [8]. Following the considerations of the PPB, hydrodynamical arguments are performed and velocities, complex velocity potentials, and vortices are investigated in Sec. IV. In Sec. IV C an extension of the argument to three-dimensional problems is also discussed. Non-plane-wave solutions for zero energy are briefly discussed in Sec. V. Remarks on nonzero energy solutions are presented in Sec. VI. Some remarks and comments on the present work are given in Sec. VII.

II. CONFORMAL MAPPINGS OF SCHRÖDINGER EQUATIONS WITH SYMMETRIC POTENTIALS

We shall investigate the general structure of Schrödinger equations,

$$i\hbar\frac{\partial}{\partial t}\Psi(t,x,y)=H\Psi(t,x,y),$$

where the Hamiltonian H is described by rotational symmetric potentials in two-dimensional space (x,y). The eigenvalue problems with the energy eigenvalue $\mathcal{E}$ are explicitly written as

$$\left[-\frac{\hbar^2}{2m}\triangle+V_a(\rho)\right]\psi(x,y)=\mathcal{E}\psi(x,y), \tag{1}$$

where $a\in\mathbb{R}$ $(a\neq 0)$

$$\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

$$V_a(\rho)=-a^2 g_a \rho^{2(a-1)},$$

with $\rho=\sqrt{x^2+y^2}$, m and g_a are, respectively, the mass of the particle and the coupling constant. Note here that V_a represents repulsive potentials for $(g_a>0, a>1)$ and $(g_a<0, a<1)$ and attractive potentials for $(g_a>0, a<1)$ and $(g_a<0, a>1)$. Since we investigate the equations in the conjugate spaces of GT, the energy eigenvalues $\mathcal{E}$ of Eq. (1) are generally complex numbers.

Following the hydrodynamical argument [38–41], let us consider the following conformal mappings:

$$\zeta_a=z^a \quad \text{with} \quad z=x+iy. \tag{2}$$

Note that the conformal mappings are singular at the origin except in the cases of a being positive integers, and the conformal mapping for $a=1$ is trivial because nothing is changed by the mapping. We further notice that a complex factor A can be multiplied in the mappings such as $\zeta_a=Az^a$, which will be discussed in the case for $A=e^{-i\alpha}$ with $\alpha\in\mathbb{R}$. When we use the notation

$$\zeta_a = u_a + i v_a,$$

we see that

$$u_a = \rho^a \cos a\varphi, \quad v_a = \rho^a \sin a\varphi, \tag{3}$$

where $u_a, v_a \in \mathbb{R}$ and $\rho = \sqrt{x^2+y^2}, \varphi = \arctan(y/x)$. Using the notations

$$\rho_a^2 = u_a^2 + v_a^2 (= \rho^{2a}) \quad \text{and} \quad \varphi_a = a\varphi,$$

we have

$$u_a = \rho_a \cos\varphi_a \quad \text{and} \quad v_a = \rho_a \sin\varphi_a. \tag{4}$$

In the (u_a, v_a) plane Eqs. (1) are written down as

$$a^2 \rho_a^{2(a-1)/a} \left[-\frac{\hbar^2}{2m} \triangle_a - g_a \right] \psi(u_a, v_a) = \mathcal{E} \psi(u_a, v_a), \tag{5}$$

where

$$\triangle_a = \frac{\partial^2}{\partial u_a^2} + \frac{\partial^2}{\partial v_a^2}.$$

We can rewrite the equations as

$$\left[-\frac{\hbar^2}{2m} \triangle_a - g_a \right] \psi(u_a, v_a) = a^{-2} \mathcal{E} \rho_a^{2(1-a)/a} \psi(u_a, v_a). \tag{6}$$

Exchanging the second term on the left-hand side and the term on the right-hand side, we obtain

$$\left[-\frac{\hbar^2}{2m} \triangle_a - a^{-2} \mathcal{E} \rho_a^{2(1-a)/a} \right] \psi(u_a, v_a) = g_a \psi(u_a, v_a). \tag{7}$$

It is quite interesting that we can read this equations as follows: the eigenvalue problem for the potential $V_a(\rho)$ in the (x,y) plane given by Eq. (1) is replaced by the eigenvalue problem for the potential $V_{1/a}(\rho_a)$ in the (u_a, v_a) plane, where the roles of the eigenvalue $\mathcal{E}$ and the coupling constant g_a are exchanged. We may consider that this relation represents a kind of duality between the energy and the coupling constant. From the relation we see that by solving the eigenvalues for fixed $\mathcal{E}$ in the (u_a, v_a) plane we can determine the strength of the coupling constant g_a to reproduce the eigenvalue $\mathcal{E}$ for the potential $V_a(\rho)$ in the (x,y) plane. We shall return to the relations between the problems for V_a in the (x,y) plane and $V_{1/a}(\rho_a)$ in the (u_a, v_a) plane in Sec. VI, because this theme is not the main subject of this section.

Here let us briefly comment on the conformal mappings $\zeta_a = z^a$. We see that the transformation maps the part of the (x,y) plane described by $0 \leq \rho < \infty, 0 < \varphi < \pi/|a|$ in the upper half plane of the (u_a, v_a) plane for $a > 0$ and in the lower half plane for $a < 0$. Note here that the maps in the part of the (u_a, v_a) plane with the angle $\varphi_a = \varphi - \alpha$ can be carried out by using the conformal mappings

$$\zeta_a(\alpha) = z^a e^{-i\alpha}. \tag{8}$$

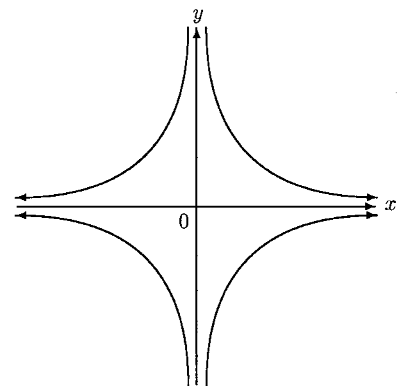

FIG. 1. Corner flows for $\psi_0^+(u_a)$ in the two-dimensional PPB.

In the maps, the variables

$$u_a(\alpha) = \rho^a \cos(a\varphi - \alpha) \quad \text{and} \quad v_a(\alpha) = \rho^a \sin(a\varphi - \alpha) \tag{9}$$

should be used. We also have the relations

$$u_a(\alpha) = u_a \cos\alpha + v_a \sin\alpha \quad \text{and} \quad v_a(\alpha) = v_a \cos\alpha - u_a \sin\alpha. \tag{10}$$

Of course, the relations $u_a(0) = u_a$ and $v_a(0) = v_a$ are obvious.

In the following, we comment on the meaning of the choice of the variables u_a and v_a given in Eq. (3). It is obvious that u_a and v_a are not suitable variables for representing the states having definite properties with respect to rotations, such as the states with definite angular momentum, in comparison with the polar coordinates ρ and φ. In the following discussions, however, we will be interested only in the states describing stationary flows, which are basic elements in hydrodynamics. In general, stationary flows, such as those in scattering problems, cannot be described by the states with definite angular momentum, because every stationary flow has specific directions representing the incoming and outgoing flows. (Examples of the stationary flows in PPB will be presented in Sec. IV. See Figs. 1 and 2.) Such stationary flows, of course, have no definite rotational sym-

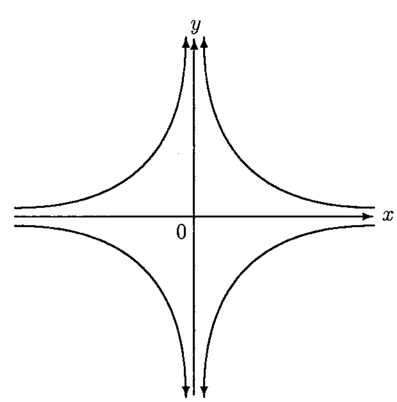

FIG. 2. Corner flows for $\psi_0^-(u_a)$ in the two-dimensional PPB.

metry except rotations with respect to some specific angles. We can understand such situations by considering the fact that the directions of the incoming flows are chosen by hand in scattering experiments. Actually it will be shown that the freedom of the phase α in the conformal mapping (8) is related to such choices (see Sec. III). The choice of the variables u_a and v_a is, therefore, important in the following hydrodynamical approach, where the relations between the potentials $V_a(\rho)$ with different values of a are studied. An explicit example of the difference between the choice of the polar coordinates and that of u_2 and v_2 has been shown in the case of the two-dimensional PPB in Sec. 3 of Ref. [8].

III. ZERO-ENERGY SOLUTIONS OF THE SCHRÖDINGER EQUATIONS

We shall here study the special solutions having zero energy eigenvalue, $\mathcal{E}=0$. As noted in Sec. I, energy eigenvalues in GTs are generally complex and all energy eigenvalues appear as pairs of complex-conjugate values such as $\varepsilon \mp i\gamma$ ($\varepsilon, \gamma \in \mathbb{R}$) [1]. This indicates that, provided a potential in one-dimensional space has pure imaginary eigenvalues, the potentials extended in two dimensions possibly have zero-energy states. This situation really occurs in PPB, that is, one-dimensional PPB has pure imaginary eigenvalues [2–7] and hence two-dimensional (generally in even dimensions) PPB has zero-energy states that are described by the stationary flows round the origin and are infinitely degenerate [8]. Let us investigate zero-energy solutions for the potentials $V_a(\rho)$.

A. Zero-energy solutions

We see that for the zero-energy $\mathcal{E}=0$ the Schrödinger equations (6) obtained by the conformal mappings become very simple such that

$$\left[-\frac{\hbar^2}{2m}\triangle_a - g_a\right]\psi(u_a,v_a)=0. \tag{11}$$

Note that the zero-energy solutions have no time dependence. It is remarkable that the equation becomes same for all a, that is, the potential is expressed by the constant g_a for all a. As far as the zero-energy solutions are concerned, the equations transformed by the conformal mappings can be written in the same form for all the potentials $V_a(\rho)$ with $a\neq 0$. It should be noticed here that only in the case of the constant potential $V_1=-g_1$ for $a=1$, the energy eigenvalues can take arbitrary values satisfying the condition $\mathcal{E}+g_1>0$, because the right-hand side of Eq. (6) does not have any ρ dependence.

It is trivial that the solutions of Eq. (11) are given by the two-dimensional plane waves with energy g_a. The solutions are, therefore, represented by

$$\psi_0^{\pm}(\boldsymbol{\rho}_a)=N_a e^{\pm i\boldsymbol{k}_a(\theta)\cdot\boldsymbol{\rho}_a} \quad \text{for} \quad g_a>0 \tag{12}$$

and

$$\phi_0^{\pm}(\boldsymbol{\rho}_a)=M_a e^{\pm \boldsymbol{k}_a(\theta)\cdot\boldsymbol{\rho}_a} \quad \text{for} \quad g_a<0, \tag{13}$$

where the angle θ denotes the moving direction of the plane wave in the (u_a,v_a) plane, $\boldsymbol{k}_a(\theta)=(\sqrt{2m|g_a|}/\hbar)\times(\cos\theta,\sin\theta)$ and $\boldsymbol{\rho}_a=(u_a,v_a)$ are two-dimensional vectors, and N_a and M_a are, in general, complex numbers. Comparing the equation $\boldsymbol{k}_a(\theta)\cdot\boldsymbol{\rho}_a=k_a(u_a\cos\theta+v_a\sin\theta)$ where $k_a=\sqrt{2m|g_a|}/\hbar$ with $u_a(\alpha)$ of Eq. (10), we see that the angle θ can be adjusted to the phase α introduced in the conformal mappings (8). By using the phase α and the variable $u_a(\alpha)$, the solutions (12) and (13) are written by

$$\psi_0^{\pm}(u_a(\alpha))=N_a e^{\pm ik_a u_a(\alpha)} \quad \text{for} \quad g_a>0 \tag{14}$$

and

$$\phi_0^{\pm}(u_a(\alpha))=M_a e^{\pm k_a u_a(\alpha)} \quad \text{for} \quad g_a<0. \tag{15}$$

We shall use the representations given in Eqs. (14) and (15) in the following discussions. Note here that, taking account of the relations

$$u_a(\pm\pi/2)=\pm v_a, \quad u_a(\pm\pi)=-u_a, \tag{16}$$

we can represent all the solutions of Eqs. (14) and (15) by

$$\psi_0^{+}(u_a(\alpha)) \quad \text{and} \quad \phi_0^{+}(u_a(\alpha)) \quad \text{with} \quad -\pi<\alpha\leq\pi. \tag{17}$$

We also notice that the solutions $\psi_0^{\pm}(u_a(\alpha))$ for $g_a>0$ are expressed by plane waves with fixed momentum $p_a=\sqrt{2mg_a}$, whereas $\phi_0^{\pm}(u_a(\alpha))$ for $g_a<0$ are expressed by exponential growing or dumping functions. This difference is essential, because the plane-wave solutions can always be the states contained in the conjugate spaces of GTs, the nuclear space of which is given by Schwartz space [1], whereas the exponential growing functions such as $\exp[\rho^a\cos(a\varphi-\alpha)]$ with $0<\cos(a\varphi-\alpha)$ cannot find a simple nuclear space for arbitrary values of a. From now on we shall mainly discuss the plane-wave solutions for $g_a>0$.

Let us summarize the main results of the case $g_a>0$.

(i) All the potentials written as $V_a(\rho)$ have zero-energy eigenstates in GTs.

(ii) All the solutions with zero energy can be expressed by a plane wave with fixed momentum $p_a=\sqrt{2mg_a}$ in the (u_a,v_a) plane.

(iii) The zero-energy solutions have an infinite freedom arising from the arbitrary angle $-\pi<\alpha\leq\pi$, which corresponds to the freedom of the angle between the incoming particle and the x axis, which is given by $(\pi-\alpha)/a$.

(iv) In the case of the constant potential corresponding to $a=1$, though we have the same solutions obtained in the above arguments, their energy eigenvalues need not equal zero but the energies can take arbitrary values fulfilling the relation $\mathcal{E}+g_1>0$.

B. Infinite degeneracy of the zero-energy states

A kind of degeneracy arising from the angle of the incoming particle with respect to the x axis has been pointed out in Sec. II. We, however, see that the zero-energy states have another type of infinite degeneracy that has been already solved in the two-dimensional PPB [8]. In the PPB the degeneracy arises from the pairing property of the energy eigenvalues given by $\mp i(n+1/2)\hbar\gamma$, that is, the energy eigenvalues of the type $\mp i(n_x-n_y)\hbar\gamma$ appear in the two-dimensional PPB and hence the infinitely degenerate zero-energy states are derived for all the cases satisfying $n_x=n_y$. We see that the origin of the infinite degeneracy is due to the existence of an infinite number of resonances having decay widths $(n+1/2)\hbar\gamma$ in the one-dimensional PPB and the coexistence of the resonance-formation and resonance-decay processes with equal probability in the two-dimensional PPB. The zero-energy states are interpreted as stationary flows expressed by the incoming flows corresponding to the formation process and the outgoing flows corresponding to the decay process, which will be shown in Figs. 1 and 2 of Sec. IV. Let us see the degeneracy in Eq. (11) where the two-dimensional PPB is included. As an example we study the freedom for the wave function $\psi_0^{\pm}(u_a)$ given by Eq. (14). By putting the wave function $f^{\pm}(u_a;v_a)\psi_0^{\pm}(u_a)$ into Eq. (11) where $f^{\pm}(u_a;v_a)$ is a polynomial function of u_a and v_a, we obtain the equation

$$\left[\triangle_a \pm 2ik_a\frac{\partial}{\partial u_a}\right]f^{\pm}(u_a;v_a)=0. \tag{18}$$

As solved in Ref. [8], a few examples of the functions $f^{\pm}$ are given by

$$f_0^{\pm}(u_a;v_a)=1,$$

$$f_1^{\pm}(u_a;v_a)=4k_av_a,$$

$$f_2^{\pm}(u_a;v_a)=4(4k_a^2v_a^2+1\pm 4ik_au_a). \tag{19}$$

In the two-dimensional PPB, the functions are generally written as multiples of polynomials of degree n, $H_n^{\pm}(\sqrt{2k_2}x)$, such that

$$f_n^{\pm}(u_2;v_2)=H_n^{\pm}(\sqrt{2k_2}x)H_n^{\mp}(\sqrt{2k_2}y), \tag{20}$$

where x and y on the right-hand side should be considered to be functions of u_2 and v_2 [8]. Since the form of Eqs. (18) is the same for all a, the solutions can be written using the same polynomial functions that are given in Eq. (20) for the PPB. That is to say, we can obtain the polynomials for arbitrary a by replacing u_2 and v_2 with u_a and v_a in Eq. (20). Note that the polynomials $H_n^{\pm}(\xi)$ with $\xi=\sqrt{m\gamma/\hbar}x$ are defined by solutions for the eigenstates with $\mathcal{E}_n^{\pm}=\mp i(n+1/2)\hbar\gamma$ in a one-dimensional PPB of the type $V(x)=-m\gamma^2x^2/2$ and they are written in terms of Hermite polynomials $H_n(\xi)$ as

$$H_n^{\pm}(\xi)=e^{\pm in\pi/4}H_n(e^{\mp i\pi/4}\xi). \tag{21}$$

(For details, see Refs. [5,6].) It is remarkable that all the wave functions for arbitrary a can be represented by the same functions of the PPB in the (u_a,v_a) plane. For $\psi_0^{\pm}(v_a)$ we should take the polynomials $f_n^{\pm}(v_a;u_a)$ in which the variables u_a and v_a are exchanged.

Note that the eigenfunctions

$$\psi_{0n}^{\pm}(u_a(\alpha))=f_n^{\pm}(u_a(\alpha);v_a(\alpha))\psi_0^{\pm}(u_a(\alpha))$$

for $n\geqslant 1$ do not describe plane waves and hence they cannot be normalized in terms of δ functions. We essentially have to treat them as the eigenfunctions of the conjugate space $\mathcal{S}(\mathbb{R}^2)^{\times}$ in GT, which is expressed by

$$\mathcal{S}(\mathbb{R}^2)\subset L^2(\mathbb{R}^2)\subset \mathcal{S}(\mathbb{R}^2)^{\times},$$

where $\mathcal{S}(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ are, respectively, Schwartz space and Lebesgue space in two dimensions. (For details, see Refs. [1,8].) We will see in the following section that this degeneracy plays an important role in investigating vortices.

Note that we can obtain the polynomials for the wave functions $\phi_0^{\pm}(u_a)$ by replacing k_a with $-ik_a$ in the polynomials derived from Eq. (20). We also easily see that in one dimension the equation corresponding to Eq. (18) does not bring any new freedom to the plane-wave solutions.

IV. HYDRODYNAMICAL CONSIDERATIONS OF THE ZERO-ENERGY STATES

In hydrodynamics conformal mappings are very powerful tools for understanding structures of currents. Actually the important hydrodynamical ideas such as the property of complex velocity potentials, circulations of currents, strengths of vortices, strengths of sources and sinks, and so forth do not change in the conformal mappings [38–41]. This fact means that we can simultaneously carry out the investigation of the hydrodynamical properties of the zero-energy solutions for all the potentials $V_a(\rho)$ in the mapped spaces, i.e., in the (u_a,v_a) plane. Results for all the potentials with $a\neq 0$ can be obtained by the inverse transformations of the conformal mappings. In this section we shall study the zero-energy states from a hydrodynamical viewpoint for the $g_a>0$ cases, because the eigenstates for $g_a<0$, represented by exponential growing or damping functions, do not describe any oscillating waves, which will be briefly discussed in Sec. V. It should be remarked on the solutions with nonzero energy such as those with the energy eigenvalues $\mp i(n_x+n_y+1)\hbar\gamma$ in two-dimensional PPB, which are well expressed in terms of the eigenfunctions of the angular momentum $L=-i\hbar\partial/\partial\varphi$. (See Ref. [8].) The discussion of vortices of those states can be simply performed by using the variables ρ and φ and it has already been done in the PPB case [8]. We shall devote this section to the investigation of vortices formed from zero-energy solutions, which are the stationary flows, which are not the eigenstates of L. We shall see that zero-energy solutions with infinite degeneracy can produce a wide variety of vortex patterns.

A. Currents and velocities

Though states in GTs are generally not normalizable, the probability currents are observable in physical processes such as in scattering processes. We shall, therefore, study the currents and other quantities based on hydrodynamics. The probability density $\rho(t,x,y)$ and the probability current $\boldsymbol{j}(t,x,y)$ of a state $\Psi(t,x,y)$ in nonrelativistic quantum mechanics are defined by

$$\rho(t,x,y)=|\Psi(t,x,y)|^2, \tag{22}$$

$$\boldsymbol{j}(t,x,y)=\mathrm{Re}[\Psi(t,x,y)^*(-i\hbar\boldsymbol{\nabla})\Psi(t,x,y)]/m. \tag{23}$$

They satisfy the equation of continuity

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0. \tag{24}$$

Following the analog of the hydrodynamical approach [38–41], the fluid can be represented by the density ρ and fluid velocity $\boldsymbol{v}$. They satisfy Euler's equation of continuity,

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})=0. \tag{25}$$

Comparing this equation with Eq. (24), we are thus led to the following definition for the quantum velocity of a state $\Psi(t,x,y)$:

$$\boldsymbol{v}=\frac{\boldsymbol{j}(t,x,y)}{|\Psi(t,x,y)|^2}, \tag{26}$$

in which $\boldsymbol{j}(t,x,y)$ is given by Eq. (23). Notice that ρ and $\boldsymbol{j}$ for the zero-energy states do not depend on time t.

Let us discuss them in the (u_a,v_a) plane. All quantities O defined in the (u_a,v_a) plane will be marked as $\hat{O}$ and can easily be transformed into the quantities in the (x,y) plane. It is apparent that in the (u_a,v_a) plane the currents of the plane waves $\psi_0^+(u_a(\alpha))$ are represented by the same form for all a,

$$\hat{j}_{u_a}=|N_a|^2\hbar k_a(\cos\alpha)/m,$$

$$\hat{j}_{v_a}=|N_a|^2\hbar k_a(\sin\alpha)/m. \tag{27}$$

Note here the following relations:

$$u_a(\alpha)=u_a\cos\alpha+v_a\sin\alpha,\quad u_a(0)=u_a,\quad v_a(0)=v_a.$$

When we represent the momentum in terms of a vector of the (u_a,v_a) plane as

$$\hat{\boldsymbol{p}}_a=(\hbar k_a\cos\alpha,\hbar k_a\sin\alpha)$$

for $\psi_0^+(u_a(\alpha))$, the currents are generally written as

$$\hat{\boldsymbol{j}}_a=|N_a|^2\hat{\boldsymbol{p}}_a/m. \tag{28}$$

Hence the velocities are given by

$$\hat{\boldsymbol{v}}_a=\hat{\boldsymbol{p}}_a/m. \tag{29}$$

Following the argument of hydrodynamics (for details, see Appendix A of Ref. [8]), we can introduce the complex velocity potential W_a as

$$W_a=(\hat{p}_{u_a}-i\hat{p}_{v_a})\zeta_a/m. \tag{30}$$

The velocity potential Φ_a and the stream function Ψ_a can be introduced similarly to those in hydrodynamics as

$$W_a=\Phi_a+i\Psi_a,$$

where they satisfy the following relations in the (u_a,v_a) plane:

$$\hat{v}_{u_a}=\frac{\partial\Phi_a}{\partial u_a}=\frac{\partial\Psi_a}{\partial v_a},\qquad \hat{v}_{v_a}=\frac{\partial\Phi_a}{\partial v_a}=-\frac{\partial\Psi_a}{\partial u_a}. \tag{31}$$

It is known that Cauchy-Riemann equations are satisfied by the velocity potential and the stream function.

The velocities in the u_a and v_a directions in the (x,y) plane are given by

$$v_{u_a}=h_a\hat{v}_{u_a}\quad v_{v_a}=h_a\hat{v}_{v_a}, \tag{32}$$

where the scale factors $h_a=a(u_a^2+v_a^2)^{(a-1)/2a}$. Hydrodynamics tells us that W_a describes corner flows with the angle π/a around the origin. For example, in the case of the PPB with $a=2$ [8], the plane waves in the (u_2,v_2) plane, $\psi_0^{\pm}(u_2)$, are expressed in Figs. 1 and 2. Note that the states multiplied by the polynomials $f_0^{\pm}$ and $f_1^{\pm}$ of Eq. (19) also represent the corner flows with the angle π/a.

B. Vortices in the zero-energy states

In hydrodynamics vortices are very important objects. In quantum mechanics, since the velocity defined by Eq. (26) diverges at the zero points of the wave functions, the vortices generally appear at such nodal points of the wave functions [17–20]. The situation is, however, not so simple to determine the positions of vortices, because the vortices do not always appear at the points where the wave functions vanish, when the currents also vanish at the same points. Since the zero-energy states have an infinite degeneracy and also the freedom of the angle α, we will be able to create vortex patterns having an arbitrary number of vortices at arbitrary positions. A general study of quantized vortices is carried out in Ref. [20]. We shall here discuss the vortex patterns in a few simple cases of the linear combinations in terms of the infinite degeneracy. It should be noted that the search for the nodal points of the wave functions, where the currents do not vanish, is not enough to determine the positions of vortices. We have one more criterion on the circulation that characterizes the strength of the vortex. That is to say, a vortex must have a nonzero circulation Γ defined by the integral round a closed contour C encircling the vortex such that

$$\Gamma=\oint_C\boldsymbol{v}\cdot d\boldsymbol{s}. \tag{33}$$

Even at the nodal points with nonvanishing currents the circulations can be zero, for instance, at the positions of sources or sinks of currents, vortex dipoles, vortex quadrupoles, and so forth [38–41,45]. For the confirmation of vortices we have to evaluate the circulation Γ. Note also that the circulation is quantized as

$$\Gamma = 2\pi l\hbar/m, \tag{34}$$

where the circulation number l is an integer [18,20,23].

Let us study the vortex structures appearing in the linear combinations of two eigenstates constructed from Eqs. (14) and (19). The following discussions are carried out in the (u_a, v_a) plane, because the singularities of the velocity do not change in the conformal mappings except the singularity of the mappings at the origin, for a taking nonpositive integeral values. The general form of the linear combination of two states can be written as

$$\Psi = \psi_1 + \psi_2, \tag{35}$$

where, since the two states are not normalized, the complex coefficients appearing in the linear combination are included in the two wave functions ψ_1 and ψ_2. The absolute square of Ψ is evaluated as

$$|\Psi|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\mathrm{Re}(\psi_1^* \psi_2). \tag{36}$$

In general, a component of the current of Ψ is written as

$$\hat{j}_\mu = \frac{\hbar}{m}\mathrm{Re}[\Psi^*(A_\mu \psi_1 + B_\mu \psi_2)], \tag{37}$$

where $\mu = u_a$ or v_a and A_μ and B_μ are complex functions defined by

$$A_\mu = -i\frac{\partial \psi_1}{\partial \mu}\psi_1^{-1}, \quad B_\mu = -i\frac{\partial \psi_2}{\partial \mu}\psi_2^{-1}. \tag{38}$$

Let us study the nodal points of $|\Psi|^2$, where the vortices appear. We have

$$|\Psi|^2 = |\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2|\cos\theta, \tag{39}$$

where θ denotes the phase between ψ_1 and ψ_2. It is trivial that nodal points appear when the following two conditions are fulfilled:

$$|\psi_1| = |\psi_2| \quad \text{and} \quad \cos\theta = -1. \tag{40}$$

We put the first relation into Eq. (39) and thus obtain

$$|\Psi|^2 = 2|\psi_1|^2(1+\cos\theta). \tag{41}$$

Taking account of the same relation $|\psi_1| = |\psi_2|$, the current is written as

$$\hat{j}_\mu = \frac{\hbar}{m}|\psi_1|^2[(|A_\mu|\cos\phi_A + |B_\mu|\cos\phi_B)(1+\cos\theta) + (|A_\mu|\sin\phi_A - |B_\mu|\sin\phi_B)\sin\theta], \tag{42}$$

where ϕ_A and ϕ_B are, respectively, the phases of A_μ and B_μ. The velocity is evaluated as

$$\hat{v}_\mu = \frac{\hbar}{2m}\left[(|A_\mu|\cos\phi_A + |B_\mu|\cos\phi_B) + (|A_\mu|\sin\phi_A - |B_\mu|\sin\phi_B)\frac{\sin\theta}{1+\cos\theta}\right]. \tag{43}$$

We see that the second term in the brackets diverges by l'Hôpital's theorem, when the second condition for the angle, $\cos\theta = -1$, is fulfilled. Thus we can obtain the condition for the divergence of the velocity,

$$|A_\mu|\sin\phi_A - |B_\mu|\sin\phi_B \neq 0. \tag{44}$$

This equation means that the functions A_μ and B_μ must not be real and also the imaginary parts of A_μ and B_μ must not be equal at least for one of the components $\mu = u_a$ and v_a.

Now we can summarize the conditions for the determination of the vortex positions in the linear combination of two wave functions ψ_1 and ψ_2 as follows:(i) $|\psi_1| = |\psi_2|$; (ii) $\theta = (2l-1)\pi$, l is an integer (θ is the phase between ψ_1 and ψ_2); (iii) $|A_\mu|\sin\phi_A - |B_\mu|\sin\phi_B \neq 0$ [A_μ and B_μ are defined in Eq. (38)]. Let us investigate the above conditions with a few simple examples.

Example (i). It is trivial that any linear combination composed of the wave functions with the lowest polynomial (19) has no vortex, because the condition (iii) is not fulfilled whereas nodal points satisfying the conditions (i) and (ii) appear in the linear combinations.

Example (ii). The combination of the lowest polynomial and the second one such that

$$\Psi = \psi_0^+(u_a(\alpha)) - C f_1^+(u_a(0); v_a(0))\psi_0^+(u_a(0))$$

has vortices at positions fulfilling the following conditions derived from (i) and (ii):

$$v_a(0) = (-1)^n/4|C|k_a,$$

$$\hat{\theta} + \theta_C = n\pi \quad (n \text{ is an integer}), \tag{45}$$

where θ_C is the phase of C and

$$\hat{\theta} = k_a[u_a(0) - u_a(\alpha)] = k_a[u_a(0)(1-\cos\alpha) - v_a(0)\sin\alpha]. \tag{46}$$

Let us examine the relations (45) in two cases for $a=1$ and 2, where C is taken to be a real number, i.e., $\theta_C = 0$.

Case $a=1$. In this case we have $u_1(0) = x$ and $v_1(0) = y$ and then the relations are reduced to

$$y = (-1)^n/4|C|k_1,$$

$$x(1-\cos\alpha) - y\sin\alpha = n\pi/k_1. \tag{47}$$

All vortices appear on the two lines $y = \pm 1/4|C|k_1$ parallel to the x axis and they are at the cross points of the two lines and the lines $x = [n\pi + (-1)^n \sin\alpha/4|C|]/k_1(1-\cos\alpha)$ for $\alpha \neq 0$. The positions of vortices for $n = 0, \pm 1, \pm 2, \pm 3$ are

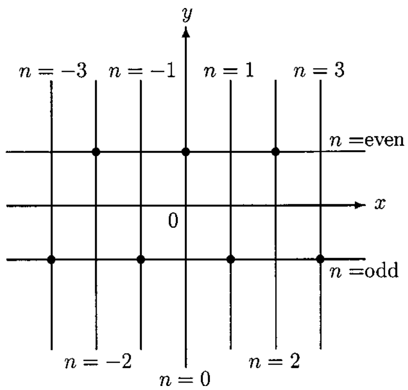

FIG. 3. Positions of vortices for $n=0,\ \pm 1,\ \pm 2,\ \pm 3$ in the constant potential $(a=1)$, denoted by ●.

presented in Fig. 3, where $\alpha=\pi$ is taken. This situation is quite similar to the vortices called parallel vortex lines obtained in hydrodynamics.

Case $a=2$ (PPB). Since the inverse transformation of the conformal mapping is described by the equations $u_2(0)=x^2-y^2$ and $v_2(0)=2xy$ in PPB [8], the relations are given by

$$2xy=(-1)^n/4|C|k_2,$$

$$(x^2-y^2)(1-\cos\alpha)-2xy\sin\alpha=n\pi/k_2. \tag{48}$$

Vortices appear at the cross points of $x^2-y^2=(n\pi+(-1)^n\sin\alpha/4|C|)/k_2(1-\cos\alpha)$ and $xy=(-1)^n/8|C|k_2$. The positions of the two vortices for $n=0$ and the other four for $n=\pm 1$ are shown in Fig. 4, where $\alpha=\pi$ is taken.

The vortices appear at symmetric positions with respect to the origin, which are described by the cross points of the two equations

$$x^2-y^2=n\pi/2k_2, \quad xy=(-1)^n/8|C|k_2. \tag{49}$$

We can make a large variety of vortex patterns by changing the parameters α and C and the zero-energy states in terms of the polynomials (19). Here we stress that, as shown in the above discussions, the higher polynomial solutions with $n\neq 0$, which are not described by the plane waves in the (u_a,v_a) plane, play essential roles in creating vortices.

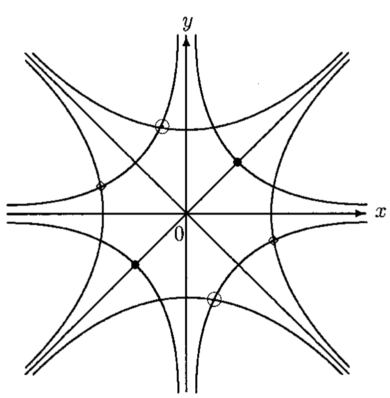

FIG. 4. Positions of vortices for $n=0,\pm 1$ in the PPB $(a=2)$, which are denoted by ● for $n=0$, ◇ for $n=1$, and ⊙ for $n=-1$.

For the confirmation of the vortices let us calculate the circulation Γ defined by Eq. (33). After some elementary calculations we obtain that $l=-1$ for the vortices with $n=$ even and $l=1$ for the vortices with $n=$ odd.

Before closing this section we point out the fact that we can realize almost all of the vortex patterns because of the infinite degeneracy of the zero-energy solutions. The study of the vortex patterns will be carried out by determining the parameter a (the type of potential) and by finding the best linear combination in terms of the infinitely degenerate zero-energy states to describe the vortex patterns.

C. Vortices in three dimensions

Let us briefly study vortices in three dimensions. It is obvious that the conformal mappings given in Eq. (2) cannot apply in three dimensions. Schrödinger equations in three dimensions are generally written as

$$\left[-\frac{\hbar^2}{2m}\left(\triangle+\frac{\partial^2}{\partial z^2}\right)+V_a(x,y,z)\right]\psi(x,y,z)=\mathcal{E}\psi(x,y,z), \tag{50}$$

where $\triangle=\partial^2/\partial x^2+\partial^2/\partial y^2$. The equations, however, can be reduced to two-dimensional ones in cases where potentials are separable into $2+1$ dimensions such that $V_a(x,y,z)=V_a(\rho)+V_z(z)$. In general, in three-dimensional models where the vortex plane (x,y) and the other axis (z) perpendicular to the vortex plane are completely separable, the wave functions are written by in multiplicative forms such as $\psi(x,y)\psi(z)$ and then the zero-energy solutions $\psi_{0n}^{\pm}(u_a,v_a)$ for $V_a(\rho)$ are applicable. Provided that the eigenstates $\psi_{E_z}(z)$ for the eigenvalues E_z are obtained in the z direction, the states written as $\psi_{0n}^{\pm}(u_a,v_a)\psi_{E_z}(z)$ are the eigenstates having the energy eigenvalues E_z. In these cases all vortices are described by the axial type and the toroidal vortices do not appear [20], because the positions of the vortices in the (x,y) plane do not depend on z.

Here we would like to note the construction of vortices in the case with $V_a(x,y,z)=0$. Let us put the plane-wave solution

$$\psi_0(x,y,z)=N_a e^{i(k_x x+k_y y)}e^{ik_z z} \tag{51}$$

into Eq. (50), where $V_a(x,y,z)=0$ is taken. Taking $\hbar^2k^2/2m=\mathcal{E}-\hbar^2k_z^2/2m$, the equation has solutions same as those for the constant potential $V_a=-g_a$ in two dimensions, where $g_a=\hbar^2k^2/2m$ and then $k_x^2+k_y^2=k^2$. This fact implies that the parallel vortices obtained in Sec. IV B are producible from $\psi_0(x,y,z)$ and the polynomial solution $f_1^+(x,y)\psi_0(x,y,z)$ with the nonzero energy $\mathcal{E}$ in three dimensions.

Real vortex phenomena [24–34] appear in three-dimensional spaces. Some of the vortex phenomena will be understood in the cases discussed above.

V. SHORT NOTES ON ZERO-ENERGY SOLUTIONS FOR $g_a<0$

As noted in Sec. III A, we have the zero-energy solutions $\phi_0^{\pm}(u_a(\alpha))=M_a e^{\pm k_a u_a(\alpha)}$ of Eq. (15). In general, they are unnormalizable in the (x,y) plane. In some special cases, however, they can be normalizable. For example, provided that the parameters a and α are taken so as to fulfill the relation

$$\cos(a\varphi-\alpha)>0 \quad \text{for} \quad 0\leqslant\varphi<2\pi, \tag{52}$$

$\phi_0^{-}(u_a(\alpha))$ can be normalizable. The relation can be fulfilled by suitable choices of the parameters such that $0<a<1/2$ and $-(1/2-2a)\pi<\alpha<\pi/2$. There are, of course, different choices, when we take the different solutions from $\phi_0^{\pm}(u_a(\alpha))$. It is very hard to answer the question whether the choice of the solutions is physically meaningful or not. Such solutions, however, possibly have some meanings in phenomena limited to very special regions, provided that the solutions are used only in the limited regions and smoothly connected to other functions defined outside the regions. In fact the solutions are used for constructing the vortices from the plane-wave solutions in three-dimensional space. (See the argument of Sec. IV C.)

Note also here that the solutions $\phi_0^{\pm}(u_a(\alpha))=M_a e^{\pm k_a u_a(\alpha)}$ have no current because they can be taken as real. The higher polynomial solutions with $n\geqslant 2$ of Eq. (19) or Eq. (20) can, however, have currents because they are generally complex. This means that we have a possibility for producing vortices from these solutions even if they will appear only in very limited regions.

VI. REMARKS ON NONZERO-ENERGY SOLUTIONS

We shall briefly discuss the equation for nonzero-energy given by Eq. (7),

$$\left[-\frac{\hbar^2}{2m}\triangle_a - a^{-2}\mathcal{E}\,\rho_a^{2(1-a)/a}\right]\psi(u_a,v_a)=g_a\psi(u_a,v_a).$$

As noted in Sec. II, this equation can be read as the equation for determining the strength of the coupling constant g_a of the original potential $V_a(\rho)=-a^2 g_a\rho^{2(a-1)}$ for the given energy $\mathcal{E}$. We shall, however, discuss it from a slightly different standpoint. If we can solve the eigenvalue problem for the potential of $-a^{-2}\mathcal{E}\rho_a^{2(1-a)/a}$, we can obtain the eigenvalues of the original equation

$$\left[-\frac{\hbar^2}{2m}\triangle - a^2 g_a\rho^{2(a-1)}\right]\psi(x,y)=\mathcal{E}\psi(x,y).$$

Let us show one example for $a=1/2$, where the original potential is written as

$$V_{1/2}(\rho)=-\frac{1}{4}g_{1/2}\frac{1}{\rho} \quad \text{for} \quad g_{1/2}>0. \tag{53}$$

For real and negative eigenvalues ($\mathcal{E}<0$) Eq. (7) can be understood as a two-dimensional harmonic oscillator with spring constant $k=8|\mathcal{E}|$. The eigenvalues of the two-dimensional harmonic oscillator are well known as

$$E_{n_x n_y}=(n_x+n_y+1)\hbar\omega, \tag{54}$$

where n_x and n_y are zero or positive integers and $\omega=2\sqrt{2|\mathcal{E}|/m}$. Thus we have the relation

$$g_{1/2}=E_{n_x n_y}. \tag{55}$$

From this relation we obtain the eigenvalue $\mathcal{E}$ as

$$\mathcal{E}=-\frac{m g_{1/2}^2}{8(2N+1)^2\hbar^2}, \tag{56}$$

with $N=(n_x+n_y)/2$. We can directly confirm the eigenvalues by solving the original equation for the solutions $\psi(x,y)=R(\rho)e^{il\varphi}$ (l is an integer), which correspond to the symmetric solutions of the harmonic oscillator described by $n_x=n_y$. We see that, provided that one of the eigenvalue problems can be solved, we can also obtain the eigenvalues of the other equation. It is interesting that harmonic-oscillator (ρ^2) and Coulomb-type (ρ^{-1}) potentials are mapped to each other by conformal mapping and that there is a relation between the energy eigenvalues of the two potentials in two dimensions.

VII. CONCLUDING REMARKS

We have shown that all Schrödinger equations with symmetric potentials of the type $V_a(\rho)$ in two dimensions can be reduced to the same equation with a constant potential for the zero-energy eigenstates in terms of conformal mappings, and the states with the zero-energy are in the infinite degeneracy. The degeneracy becomes not only the origin of the huge variety of vortex patterns but it will possibly be an interesting tool to investigate complicated problems of surface physics including boundaries as well. And the idea can be extended to phenomena in three dimensions. Particularly this scheme will become a powerful tool for studying vortex phenomena. Actually a vortex-lattice solution has been found in this scheme [45]. We may expect that the hydrodynamical approach in quantum mechanics presented here will open many interesting aspects in physics such as the investigation of vortex patterns [24–34]. We have to note here that many vortex phenomena are discussed in nonlinear problems [46], whereas our scheme is based on the linear equation. In realistic phenomena we have to solve vortex problems in the cases with many potential sources. In such cases interactions among vortices, which are known in hydrodynamics [38–41], must be taken into account. We have also to consider effects from boundaries of systems. In order to complete vortex dynamics in quantum mechanics and to analyze real vortex phenomena, the introduction of such interactions and effects must be performed in the present scheme. At present, however, the relation between the nonlinear approach and the present one is still an open question.

We briefly note here that in order to represent the whole

(u_a, v_a) plane, the double sheets of the (x,y) plane (Riemann surface) are needed for the choice of $a = \pm 1/2$. In general, for the choice of $a = \pm 1/p$, the p sheets of the (x,y) plane, such as p spiral sheets, are required to cover the whole (u_a, v_a) plane. We may consider that the case for $a = 0[V_0(\rho) \propto \rho^{-2}]$ can be examined in the limit of $p \to \infty$, where the infinite spiral sheets are needed in the (x,y) plane. From this fact we can understand that the zero-energy solutions for the ρ^{-2} potential behave as power types such as ρ^{iq}, which are expressed by logarithmic exponents $e^{iq \ln \rho}$. Actually we obtain $q = \pm \sqrt{2mg_0/\hbar^2 - l^2}$ for the potential $V_0(\rho) = -g_0 \rho^{-2}$, where l is an integer defined by the eigenvalue $\hbar l$ of the angular momentum.

It should be noticed that some kinds of equations in hydrodynamics [38–41] are obtainable from the original eigenvalue equation (1) by changing parameters such as $\hbar$ and mass m. This means that the conformal mappings (2) are applicable to hydrodynamical problems in two dimensions and infinite degeneracy can also take place. The analysis in terms of the functions obtained in this paper will also become an interesting approach in many aspects of hydrodynamical problems.

Finally we would like to note that the infinite degeneracy of the zero-energy solutions brings infinite variety to many-body systems with a fixed energy, which possibly becomes the origin of an entropy different from the Boltzmann entropy [42–44]. This entropy has nothing to do with the determination of usual temperatures in thermal equilibrium but the freedom stored in the entropy can be released in thermal nonequilibrium [43]. These considerations will also give rise to a different aspect in statistical mechanics extended from Hilbert spaces to Gel'fand triplets [42,44].

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