Entanglement in the steady state of a collective-angular-momentum (Dicke) model

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We model the behavior of an ion trap with all ions driven simultaneously and coupled collectively to a heat bath. The equations for this system are similar to the irreversible dynamics of a collective angular momentum system known as the Dicke model. We show how the steady state of the ion trap as a dissipative many-body system driven far from equilibrium can exhibit quantum entanglement. We calculate the entanglement of this steady state for two ions in the trap and in the case of more than two ions we calculate the entanglement between two ions by tracing over all the other ions. The entanglement in the steady state is a maximum for the parameter values corresponding roughly to a bifurcation of a fixed point in the corresponding semiclassical dynamics. We conjecture that this is a general mechanism for entanglement creation in driven dissipative quantum systems.

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I. INTRODUCTION

Generically, many-body quantum systems are known to be difficult to simulate efficiently on a classical computer. This is because the quantum system may explore regions of state space with nonzero entanglement, giving these systems access to a vastly larger state space than is possible classically. In an open quantum system we may, in some circumstances, be able to resort to stochastic methods, such as Monte Carlo simulations. However, this will not be possible for open systems in which the steady state itself is entangled, as in the example we describe here. Terhal and DiVincenzo [1] have considered the possibility of using a quantum computer to simulate open quantum systems in thermal equilibrium. Plenio *et al.* [2] have considered how decay can lead to entanglement rather than destroying it. Cabrillo *et al.* [3] discuss creating entanglement in two or more atoms, by driving the atoms with a weak laser pulse and detecting the spontaneous emission. In a recent paper by Arnesen *et al.* [4], the authors look at a situation where the spins in a Heisenberg chain with an external magnetic field show entanglement in the thermal state with nonzero temperature. In this paper we formulate a Dicke-type model of an ion-trap quantum computer, and in terms of this model analyze the irreversible dynamics of N two-level systems.

In the following we model the behavior of an ion trap with all ions driven simultaneously and coupled collectively to a heat bath. The equations are similar to the so-called Dicke model [5]. This model includes resonance fluorescence of a set of two-level atoms driven by a resonant coherent laser field as well as a collective decay mechanism. We will first describe how a collective-decay mechanism may be realized for N trapped ions interacting with a collective vibrational mode when the vibrational mode is subject to controlled heating. Simulating irreversible dynamics for a trapped ion has been previously suggested by a number of authors [6]. By coherently driving to force the system into a nontrivial steady state, we show that, for the case of two ions, this steady state can be partially entangled by explicitly calculating the Wooters entanglement measure (concurrence) [7]. Extending this result to many ions is not possible at present due to the lack of a general measure of the entanglement of mixed states in higher dimensions. However, we can calculate the entanglement between two ions or atoms by tracing over all the other ions or atoms. This will show us at least whether entanglement is present. Interestingly, the maximum entanglement occurs for parameter values for which the corresponding semiclassical system undergoes a bifurcation and loss of stability of the fixed point. We conjecture that the loss of stability of a semiclassical fixed point will generically be associated with entanglement in the steady state of the full quantum system.

II. THE MODEL

In the 1970s the Dicke model and cooperative effects were subjects of research in various groups (see, e.g., [8–10] and references therein). The model consists of a group of two-level atoms, which is placed in a volume with dimensions small compared to the wavelength associated with the atom's two-level dipole and evolves on time scales shorter than any $\hat{\mathbf{J}}^2$-breaking relaxation mechanism (see [8]), such as an angular momentum system, which has collective atomic raising and lowering operators, $\hat{J}_+$ and $\hat{J}_-$, with a fixed spin quantum number $j=N/2$, where N is the number of atoms.

In the rotating frame with Markov, electric-dipole, and rotating-wave approximations and ignoring a small atomic-frequency shift, the master equation for the density matrix of this group of atoms under the cooperative influence of an electromagnetic field is [9,11,12]

$$\frac{\partial\hat{\rho}}{\partial t}=-i\frac{\Omega}{2}[\hat{J}_++\hat{J}_-,\hat{\rho}]+\frac{\gamma_A}{2}(2\hat{J}_-\hat{\rho}\hat{J}_+-\hat{J}_+\hat{J}_-\hat{\rho}-\hat{\rho}\hat{J}_+\hat{J}_-), \tag{2.1}$$

where Ω is the Rabi frequency and γ_A is the Einstein A coefficient of each atom. This model can be solved exactly [11] and it exhibits a critical-point nonequilibrium phase transition for $\Omega/j=\gamma_A$ in the limit $\Omega,j\to\infty$ [9].

A. Collective driving

How do we get a similar master equation to Eq. (2.1) in an ion trap? The coherent evolution is easy: We just shine the same laser at the carrier frequency on all the ions at the same time, thus forcing each ion to undergo Rabi oscillations at the same frequency. If we start initially with all the ions in their electronic ground state $|g\rangle$, the ions will not leave the $j=N/2$ space. From there we can then define collective angular momentum operators in the following way:

$$\hat{J}_- = \sum_{i=1}^{N} \hat{\sigma}_-^{(i)}, \tag{2.2}$$

$$\hat{J}_+ = \sum_{i=1}^{N} \hat{\sigma}_+^{(i)}, \tag{2.3}$$

where the raising and lowering operators for each ion are defined by $\hat{\sigma}_- = |g\rangle\langle e|$ and $\hat{\sigma}_+ = |e\rangle\langle g|$. With this the Hamiltonian for simultaneous resonant driving of all the ions can be written as

$$\hat{H} = \hbar \frac{\Omega}{2}(\hat{J}_+ + \hat{J}_-), \tag{2.4}$$

where Ω is the Rabi frequency for the electronic transition.

B. Cooperative damping

For the collective-decay mechanism we need to couple the ions equally to the same heat reservoir. In this paper we will argue that the reservoir may be taken to be the center-of-mass vibrational mode. It is subject to heating and we assume that it is in a thermal state. To couple the ions to the vibrational mode we need another laser, which, again, illuminates all the ions at the same time, but which is detuned from the carrier frequency to the red region by the trap frequency so that the electronic state of each atom gets coupled simultaneously to the center-of-mass vibrational mode. This is described by a Hamiltonian for the ith ion of the form

$$\hat{H}_{\mathrm{red}}^{(i)} = \hbar \Omega_2(\hat{\sigma}_+^{(i)}\hat{a} + \hat{\sigma}_-^{(i)}\hat{a}^\dagger), \tag{2.5}$$

where we have introduced the bosonic annihilation operator $\hat{a}$ for the vibrational mode and the coupling constant is $\Omega_2 = \eta\Omega_0$. The parameter $\eta^2 = E_r/(\hbar M\omega_0)$ is the Lamb-Dicke parameter, where E_r is the recoil kinetic energy of the atom, ω_0 is the trap vibrational frequency, and M is the effective mass for the center-of-mass mode. The Lamb-Dicke limit assumes $\eta \ll 1$, which is easily achieved in practice. The frequency Ω_0 is the effective Rabi frequency for the electronic transition involved. This sideband transition is used to efficiently remove thermal energy from the vibrational degree of freedom. If the rate of this cooling process can overcome heating due to external fluctuations in the trap potential, the ion may eventually be prepared in the vibrational ground state. However, in general, the vibrational state will reach a thermal mixture, $\hat{\rho}_v = \mathcal{Z}^{-1}\exp(-\hbar\omega_0\hat{a}^\dagger\hat{a}/k_BT)$, where $\mathcal{Z} = \mathrm{Tr}[\exp(-\hbar\omega_0\hat{a}^\dagger\hat{a}/k_BT)]$, at some effective temperature T. If the heating and cooling rates are such that the system relaxes at a rate α, large compared with any other time scale for ion motion, the ion can effectively be regarded as interacting with a thermal reservoir at temperature T. We can also arrange that the associated rate of energy dissipation is small, $\gamma_A \ll \omega_0$, which simply requires that the coupling to the vibrational degree of freedom is weak. Finally we assume that the temperature of the vibrational degree of freedom is such that $\gamma_A \ll k_BT/\hbar$. Under these assumptions we may eliminate the description of the vibrational motion from the dynamics and obtain a master equation for the electronic state $\hat{\rho}^{(i)}i$ of the ion,

$$\frac{d\hat{\rho}^{(i)}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}^{(i)}] + \gamma_A\bar{n}\mathcal{D}[\hat{\sigma}_+^{(i)}]\hat{\rho}^{(i)} + \gamma_A(\bar{n}+1)\mathcal{D}[\hat{\sigma}_-^{(i)}]\hat{\rho}^{(i)}, \tag{2.6}$$

where the superoperator is defined by

$$\mathcal{D}[\hat{A}]\hat{\rho} = \hat{A}\rho\hat{A}^\dagger - \frac{1}{2}(\hat{A}^\dagger\hat{A}\hat{\rho} + \hat{\rho}\hat{A}^\dagger\hat{A}), \tag{2.7}$$

and where $\hat{H}$ is the Hamiltonian for any other reversible electronic dynamics and $\bar{n}$ is the mean thermal occupation number of the vibrational degree of freedom. In what follows we assume that the cooling is very efficient and set $\bar{n}=0$. At any time we may turn off the cooling lasers, thus reducing γ_A suddenly to zero. We note that the irreversible dynamics of the electronic state is due entirely to the interaction with the phonons associated with the vibrational degree of freedom.

If the external laser field on each ion is identical (in amplitude and phase) the interaction Hamiltonian is

$$\hat{H}_I = \hbar\Omega_2(\hat{a}\hat{J}_+ + \hat{a}^\dagger\hat{J}_-), \tag{2.8}$$

where $\hat{J}_+$ and $\hat{J}_-$ are defined in Eqs. (2.3) and (2.2). For the case of a linear ion trap, with separately addressable ions, identical laser fields could easily be obtained by splitting the cooling laser into multiple beams. In this way we can simulate an angular momentum system with quantum number $j=N/2$. This imposes a permutation symmetry on the system, which reduces the effective Hilbert-space dimension from 2^N to $2N+1$. Thus an exponentially large portion of the available Hilbert space, i.e., all the states with $j<N/2$, is not used in this simulation. However, it is easy to generate the relevant unitary transformations to simulate the $j=N/2$ angular momentum quantum system.

It is not trivial to keep the vibrational mode in a thermal state of fixed temperature. One way of doing this was recently suggested by Kielpinski *et al.* [13]: They propose to put one ion, which is of a different species than all the other ions, in the center of a string of ions, so that they have an odd number of ions in the trap. Through this center ion, which can be cooled at will without disturbing the coherence of the other ions, all the other ions get sympathetically cooled and this allows for keeping the string of ions at a

well-defined temperature. The authors conclude that such a scheme of sympathetic cooling is ''well within the reach of current experimental technique'' [13]. We assume for further calculations that the center-of-mass mode is kept in such a thermal state by the outlined technique. In the following derivation of the master equation we do not explicitly put laser cooling into the equation. We just assume that the vibrational state instantly, i.e., on a time scale fast compared to all the other processes involved, relaxes back into the thermal state.

With all these assumptions we get the master equation describing the collective motion of the density matrix of all the ions,

$$\frac{\partial\hat{\rho}}{\partial t}=-i\frac{\Omega}{2}[\hat{J}_{+}+\hat{J}_{-},\hat{\rho}]+\gamma_A\frac{\overline{n}}{2}(2\hat{J}_{+}\hat{\rho}\hat{J}_{-}-\hat{J}_{-}\hat{J}_{+}\hat{\rho}-\hat{\rho}\hat{J}_{-}\hat{J}_{+})$$
$$+\gamma_A\frac{\overline{n}+1}{2}(2\hat{J}_{-}\hat{\rho}\hat{J}_{+}-\hat{J}_{+}\hat{J}_{-}\hat{\rho}-\hat{\rho}\hat{J}_{+}\hat{J}_{-}), \tag{2.9}$$

where $\overline{n}$ is the mean phonon number of the vibrational center-of-mass mode and $\gamma_A=2\Omega_2^2\eta^2$. Note that for $\overline{n}=0$ this equation is identical to Eq. (2.1) for the Dicke model.

III. STEADY STATE AND ENTANGLEMENT FOR $j=1$

With two ions we have $j=1$ and from the master equation, Eq. (2.9), we can write down the equation of motion for the components of the 3×3 density matrix of the state of the system, taking into account that $\mathrm{Tr}(\hat{\rho})=1$ and that $\hat{\rho}$ is Hermitian. Getting the steady state is then a matter of simple algebra.

Once we have determined the steady state of the $j=1$ system, we can rewrite this state in the underlying two-qubit basis. What we are interested in is the change of entanglement in the system as the parameters γ and $\overline{n}$ change. The entanglement of two qubits is well defined [7,14,15] and we choose the concurrence [7] as a measure for it.

A numeric evaluation of the concurrence leads to the plot in Fig. 1. What we see is that we can get a certain amount of entanglement in the steady state of a coherently driven system that is coupled to a thermal reservoir. This is remarkable as the steady state is independent of the initial state, which can be unentangled. The coherent evolution alone does not lead to any entanglement for an initially unentangled state either, as it only consists of (simultaneously acting) single-qubit rotations and no coupling between the qubits is present. Thus the entanglement is due to the cooperative decoherence in the system acting together with the coherent evolution.

IV. ZERO-TEMPERATURE CASE

The analysis here is restricted to the case $j=1$. For $j>1$ and $\overline{n}\neq0$, numerical methods will need to be employed to derive the steady state; however, in this case another problem will arise due to the fact that there is currently no measure of entanglement for N coupled qubits. Nevertheless, other phase transitions analogous to that in the Dicke model

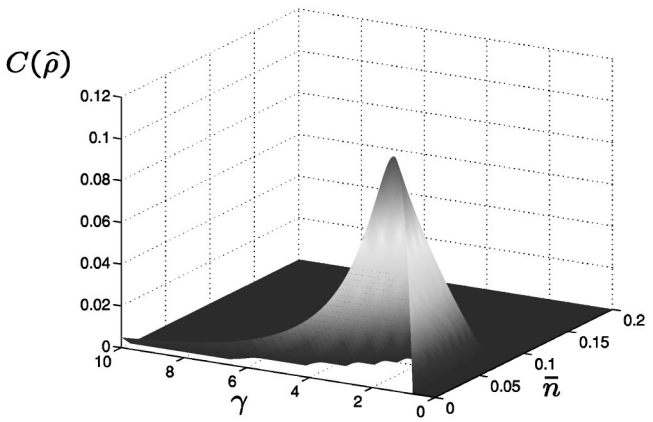

FIG. 1. Plot of the concurrence as a measure of entanglement depending on the parameters $\gamma=\gamma_A/\Omega$ and $\overline{n}$, the mean phonon number of the thermal vibrational state.

(see, e.g., [8,9] and references therein) will appear.

For $\overline{n}=0$ we can compare our results for the steady state to those calculated by Puri and Lawande [16] (see also Lawande *et al.* [17]). They calculate the steady state to be

$$\hat{\rho}_S=\frac{1}{D}\sum_{m,n=0}^{2}(g^*)^{-m}(g)^{-n}\hat{J}_{-}^{m}\hat{J}_{+}^{n}, \tag{4.1}$$

where

$$D=\sum_{k=0}^{2}H_{2,k}|g|^{-2k} \tag{4.2}$$

is a normalization constant, $g=i/\gamma$, where $\gamma=\gamma_A/\Omega$ as defined above, and

$$H_{2,m}=\frac{(2+m+1)!(m!)^2}{(2-m)!(2m+1)!}. \tag{4.3}$$

With this we can write the density matrix of the steady state in matrix form as

$$\hat{\rho}_S=\frac{1}{D}\begin{pmatrix}1 & -i\sqrt{2}\gamma & -2\gamma^2\\ i\sqrt{2}\gamma & 1+2\gamma^2 & -i\sqrt{2}\gamma-i2\sqrt{2}\gamma^3\\ -2\gamma^2 & i\sqrt{2}\gamma+i2\sqrt{2}\gamma^3 & 1+2\gamma^2+4\gamma^4\end{pmatrix}, \tag{4.4}$$

where we have calculated D as

$$D=3+4\gamma^2+4\gamma^4. \tag{4.5}$$

This gives the same result as above if we set $\overline{n}=0$. The concurrence for this special case is plotted in Fig. 2, but this time we plot it against $|g|=1/\gamma$ as well. Thus the Dicke model, as a special case of our model, shows entanglement in the steady state.

To date, no definite measure of entanglement exists for $N>2$ up . But we can calculate the entanglement between

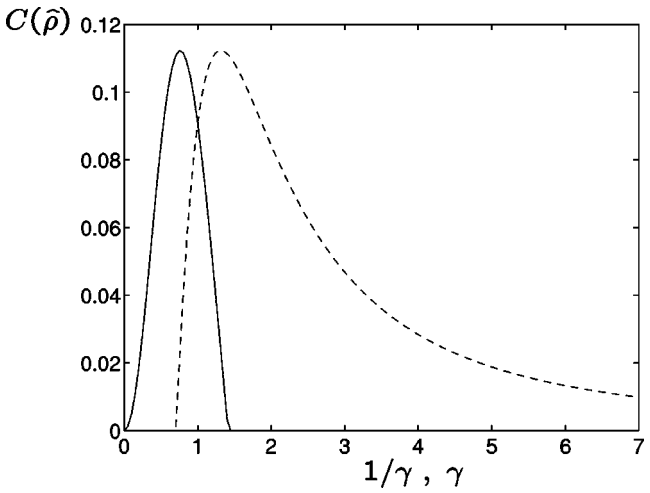

FIG. 2. Plot of the concurrence as a measure of entanglement depending on the parameters $|g|=1/\gamma=\Omega/\gamma_A$ and $1/|g|=\gamma$ (dashed line) with $\bar{n}=0$, i.e., for the Dicke model.

just two of the N ions at least for the Dicke model, where the temperature of the bath is zero. The steady is then given by [11,16,17]

$$\hat{\rho}_S=\frac{1}{D}\sum_{l=0}^{2j}\sum_{l'=0}^{2j}\left(\frac{\hat{J}_-}{g}\right)^l\left(\frac{\hat{J}_+}{g^*}\right)^{l'}. \tag{4.6}$$

By writing this as a sum of states with angular momentum $j_1=1$ and $j_2=j-1$ we can trace over the part of the Hilbert space with $j_2=j-1$ and thus get the density matrix in the steady state for just two ions (or atoms in the original Dicke model). From there we can again calculate the concurrence. This time we plot it against the relative Rabi frequency [11] $\Omega_r=\Omega/(j\gamma)$. We note that the maxima of the entanglement occur close to the critical point in the cooperative limit $j,\Omega\to\infty$ of the Dicke model, i.e., around $\Omega_r=1$. The two-ion entanglement is not the real measure of entanglement in the system. Thus we cannot take the cooperative limit as the two-ion entanglement goes to zero in this limit. However, we note from the plots in Fig. 3 that the maximum value of the two-ion entanglement indeed does move closer to the point $\Omega_r=1$ for increasing N.

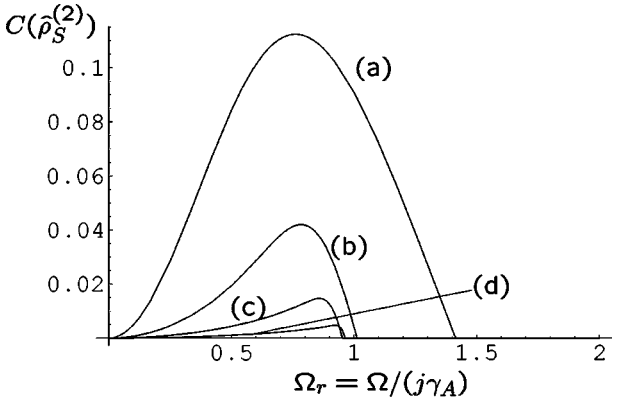

FIG. 3. Plot of the two-ion concurrence as a measure of entanglement depending on the parameters $\Omega_r=\Omega/(j\gamma_A)$ for (a) $j=1$, (b) $j=4$, (c) $j=16$, and (d) $j=64$.

V. CONCLUSION

In this paper we have demonstrated how the steady state of a dissipative many-body system, driven far from equilibrium, may exhibit nonzero quantum entanglement. This result is significant for two reasons. First, the steady state is a mixed state and the study of quantum entanglement for mixed states is a very active field of inquiry [18]. It immediately raises the question of whether the entanglement can be distilled and used as a resource for some quantum communication or computation task [19]. Second the maximum entanglement occurs at the same parameter values for which the semiclassical dynamics of the system undergoes a bifurcation of the fixed point corresponding to the quantum steady state. At the bifurcation point the time constant associated with the fixed point goes to zero as the bifurcation is approached. This is reminiscent of a phenomenon that characterizes quantum phase transitions, in which a morphological change in the ground state, as a parameter is varied, is associated with a frequency gap tending to zero [20,21]. We conjecture that the association between the bifurcation of a fixed point of the semiclassical description and the maximum of entanglement will be a general feature of dissipative many-body systems driven far from equilibrium.

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