Calculation of the positronium hyperfine interval using the Bethe-Salpeter formalism

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We use a variation of the Bethe-Salpeter equation to complete the calculation of the one-photon annihilation contribution to the hyperfine interval of positronium at order $m\alpha^6$. Our results are in accord with a quite different calculation independently done using an effective field theory approach. This completes the evaluation of all the $m\alpha^6$ terms. We give the total theoretical value for this interval and compare with experiment.

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I. INTRODUCTION

Positronium is the bound state of an electron and positron. These are pointlike, electrically charged particles, which interact primarily via the quantum electrodynamic (QED) force. The effects of the weak and strong forces are negligible. Consequently, positronium is subject to much experimental and theoretical study, as it provides a means of testing our understanding of bound-state QED in a system undisturbed by other forces.

Of particular interest is the hyperfine interval, the difference of the energies of the ground state spin$=1$ and spin $=0$ states. If we label positronium states as $n^{2S+1}L_J$, where n is the principal quantum number, S the intrinsic spin, L the orbital angular momentum, and J the total angular momentum, the hyperfine interval is $\Delta E = E(1^3S_1) - E(1^1S_0)$. The theoretical expression for this can be written as

$$\Delta E_{\rm th} = m\alpha^4\left(A + B\alpha + C\alpha^2 \ln\left(\frac{1}{\alpha}\right) + D\alpha^2 + E\alpha^3 \ln^2\left(\frac{1}{\alpha}\right) + F\alpha^3 \ln\left(\frac{1}{\alpha}\right) + G\alpha^3 + \cdots \right). \tag{1.1}$$

The theoretical calculation of the hyperfine interval began with the advent of modern quantum field theory in the late 1940s and early 1950s with the calculation of the coefficient A [1–3], followed shortly thereafter by the calculation of B, a one-loop calculation, by Karplus and Klein [4].

The computations of the order α^6 terms, C and D, are much more complex because these are two-loop calculations. Consequently, there are many more graphs, most considerably more complicated than those that contribute to lower orders. As a result, the calculation of the α^6 coefficients has been done in parts by many groups and has spanned nearly 40 years. By the mid 1990s, all of C and part of D had been computed. There remained to be calculated the contributions to D coming from the two-loop corrections to the one-photon annihilation graph. Additionally, E was computed by Karshenboim [5] in 1993.

At this point, the theoretical expression for the hyperfine splitting was

$$\begin{aligned}\Delta E_{\rm th} &= m\alpha^4\left[\frac{7}{12} - \frac{\alpha}{\pi}\left(\frac{8}{9} + \frac{1}{2}\ln(2)\right) + \frac{5}{24}\alpha^2 \ln\left(\frac{1}{\alpha}\right) - \frac{7}{8\pi}\alpha^3 \ln^2\left(\frac{1}{\alpha}\right)\right] \\ &= 203\,399.34\,\text{MHz},\end{aligned} \tag{1.2}$$

which compared unfavorably to the measured values

$$\begin{aligned}\Delta E_{\rm expt} &= 203\,387.5 \pm 1.6\,\text{MHz} \quad (7.9\ \text{ppm})\ [6,7] \\ &= 203\,389.10 \pm 0.74\,\text{MHz} \quad (3.6\ \text{ppm})\ [8],\end{aligned} \tag{1.3}$$

a difference between theory and experiment on the order of 10 MHz.

To get an estimate of whether the D term could significantly reduce the size of this discrepancy, we assume that D is of the same order of magnitude as B ($|B| = 0.39$), and take it to be 1. Then, the D term would contribute 18.65 MHz to the hyperfine interval. Thus, it was clear that before any meaningful comparison of theory and experiment could be made, the calculation of D had to be completed.

In this paper we describe our calculation of the previously unknown two-loop corrections to the one-photon annihilation graph using a variant of the Bethe-Salpeter formalism. Subsequent to our calculation, the relative order $\alpha^3 \ln(1/\alpha)$ coefficient F was calculated by Kniehl and Penin [9] and Melnikov and Yelkhovsky [10]. Their results are included in the final theoretical expression at the end of the paper.

A. Bound-state formalism

Our formalism is a quasipotential variant of the Bethe-Salpeter formalism [11] and is closely related to the methods

of Barbieri, Remiddi, and Buchmüller [12–14] and Caswell and Lepage [15]. Details of the formalism are given in [16] and are summarized below.

The bound-state equation for the e^-e^+ to e^-e^+ Green's function G is $G=S_0+S_0KG$, where S_0 is a modified e^-e^+ propagator and K is the "quasipotential." For S_0 we use [17]

$$S_0(p)=2\pi\delta(p_0)\frac{-i}{2(\omega_p-E/2-i\epsilon)} \times[\Lambda_+(\vec{p})\gamma^0]^{(1)}[\Lambda_-(\vec{p})(-\gamma^0)]^{(2)T}, \quad (1.4)$$

where $\omega_p=(p^2+m^2)^{1/2}$, $p=|\vec{p}|$, and the $\Lambda_\pm(\vec{p})$ are projection operators. The reference bound-state equation has the form $G_0=S_0+S_0K_0G_0$, where K_0 is an approximation to K containing the dominant nonrelativistic physics and is chosen so that the reference equation can be solved exactly. The reference energy levels and wave functions can be found by studying the pole structure of the reference Green's-function G_0. With the particular reference kernel K_0 that we are using, the reference energy levels are $E_n^0=2m[1-\alpha^2/(4n^2)]^{1/2}$, where n is the principal quantum number. The $n=1$ reference wave functions have the form

$$\Psi^0(p)=2\pi\delta(p_0)\left(\frac{2\omega_p}{\omega_p+m}\right)\left(\frac{\omega_p+W}{2W}\right)^{1/2} \times\phi(\vec{p})[\Lambda_+(\vec{p})\Gamma\Lambda_-(\vec{p})(-\gamma^0)], \quad (1.5)$$

where $W=E_1^0/2$, Γ is a 4×4 spin matrix, and $\phi(\vec{p})$ is the nonrelativistic momentum space wave function.

B. Choice of gauge and ultraviolet regulator

The proper choice of gauge is critical to the success of precision QED bound-state calculations. For instance, calculating in the covariant and algebraically simple Feynman gauge is not well suited for such problems since spurious lower order terms are generated. These extra terms eventually cancel, but only when an infinite number of diagrams is summed [18].

The Coulomb gauge is not plagued with this problem. Its infrared behavior is sufficiently tame that no false lower order terms are produced, at least through $m\alpha^6$. So, we choose to use this gauge. However, the cost of this is much greater algebraic complexity because of the noncovariance of the photon propagator, to wit,

$$D^C_{\mu\nu}(\not\ell)=-\frac{1}{\ell^2}\left[-g_{\mu\nu}+\frac{\ell_\mu\ell_\nu}{\ell^2-(n\ell)^2}-\frac{n\not\ell}{\ell^2-(n\ell)^2} \times(\not\ell_\mu n_\nu+\not\ell_\nu n_\mu)\right], \quad (1.6)$$

where $n=(1,\vec{0})$. This problem can be ameliorated to some degree with the assistance of symbolic algebra computer programs. For our calculation, we wrote programs using MACSYMA [19] and MATHEMATICA [20] to compute traces, do the noncovariant tensor contractions, and to express the integrals in terms of Feynman parameters. This made the calculation tractable.

To regulate the ultraviolet divergences, we use dimensional regularization in $n=4-2\epsilon$ dimensions. The noncovariant formalism does not present a problem as the formulas for n dimensional noncovariant Feynman integrals are not much more complicated than their covariant counterparts [21].

C. The energy shift formula

Corrections to the energy levels can be calculated from the systematic perturbation series [13,15,22]

$$\begin{aligned}E_n=&E_n^0+(\delta K)+(\delta K\hat{G}_0\delta K)+(\delta K)(\delta K)'\\&+(\delta K\hat{G}_0\delta K\hat{G}_0\delta K)+(\delta K)(\delta K\hat{G}_0\delta K)'\\&+(\delta K)'(\delta K\hat{G}_0\delta K)+(\delta K)[(\delta K)']^2\\&+\tfrac{1}{2}(\delta K)^2(\delta K)''+O(\delta K)^4,\end{aligned} \quad (1.7)$$

where δK is the difference between the full interaction kernel and the reference kernel and $\hat{G}_0$ is the reduced reference Green's function. The parentheses indicate expectation values between the reference wave functions, while the primes denote differentiation with respect to the reference energy. (See [16] for details.)

This series can be manipulated by using the reference wave equation and by expanding $\hat{G}_0$ to give

$$\begin{aligned}E_n=&E_n^0+(K_{\rm BS}SK_{\rm BS}SK_{\rm BS})-(K_{\rm BS}SK_{\rm BS})\\&+(K_{\rm BS}-K_0)(K_{\rm BS}-K_0)'+([K_{\rm BS}-K_0]\hat{R}[K_{\rm BS}-K_0])\\&+\cdots.\end{aligned} \quad (1.8)$$

In this equation, S is the product of two full fermion propagators, $K_{\rm BS}$ is the two particle irreducible Bethe-Salpeter kernel, K_0 is the reference potential, and $\hat{R}$ is that part of $\hat{G}_0$ that comes from the exchange of two or more reference photons.

We are interested in those terms from $(K_{\rm BS}SK_{\rm BS})$ and $(K_{\rm BS}SK_{\rm BS}SK_{\rm BS})$ that have a virtual annihilation to a single photon. We call this $\Delta E_{1\gamma}$. Its graphical representation is shown in Fig. 1.

All the diagrams with vacuum polarization insertions in the annihilation photons have been previously calculated [23,24], as have the many-potential diagrams and the derivative diagram [14,15], although not in our formalism. The remaining diagrams [see Figs. 1(a), 1(c), and 1(d)] are constructed from the amplitudes shown in Fig. 2. These combine to give a contribution to the energy of (see Appendix B)

$$\Delta E_{1\alpha+2(1c-1d)}=A_{(1R)m}\frac{1}{4W^2}A^m_{(1R)} +2A_{0m}\frac{1}{4W^2}(A^m_{(2R)}-A^m_{(1R)}), \quad (1.9)$$

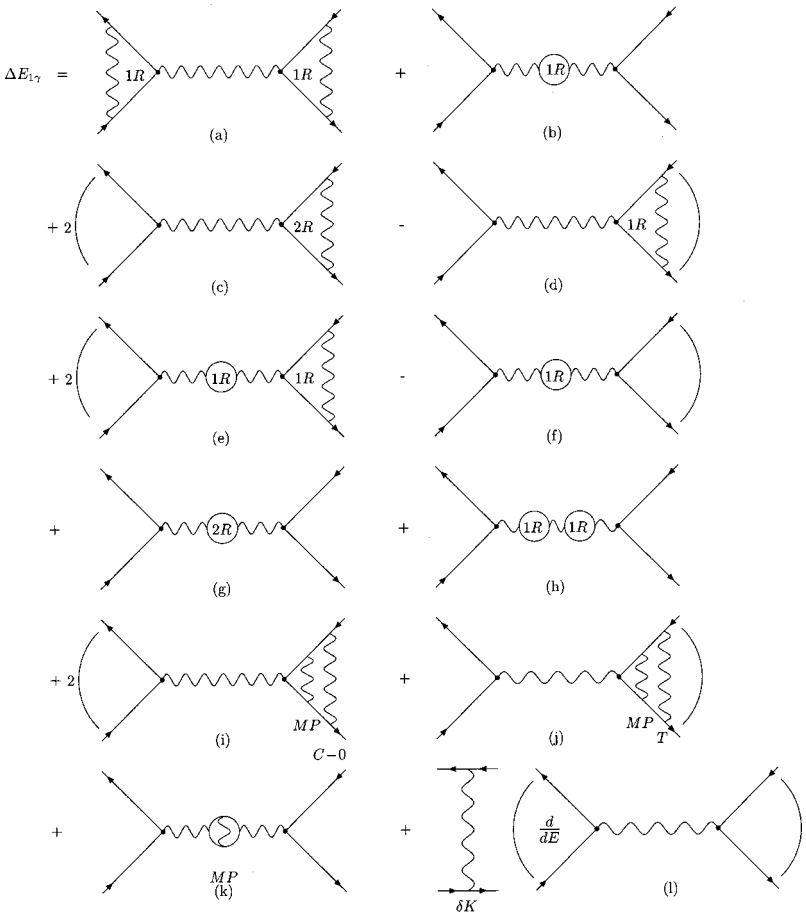

FIG. 1. The one-photon-annihilation kernel contributions to the hyperfine interval to order α^6. Wave functions are implicit on the left and right. nR refers to one-loop renormalized and two-loop renormalized kernels for $n=1$ and $n=2$. MP stands for the many potential part of the reference Green's function. $C-O$ is the difference between the Coulomb photon and the reference photon and T is the transverse photon.

where A_0^m is the lowest-order decay amplitude. $A_{(1R)}^m$ and $A_{(2R)}^m$ are the renormalized one- and two-loop vertex corrections to the annihilation amplitude. The calculation of $\Delta E_{1a+2(1c-1d)}$ is the subject of this paper. The total one-photon annihilation energy shift contribution is given by the result of (1.9) plus the previously calculated vacuum polarization contributions, plus the results for the many-potential and derivative diagrams done in our formalism.

The renormalized amplitudes have the form $A_{(1R)}^m=A_1^m-L_1A_0^m$ and $A_{(2R)}^m=A_2^m-L_1A_1^m-L_2A_0^m+L_1^2A_0^m$. A_1^m and A_2^m are the unrenormalized one- and two-loop amplitudes, L_1 and L_2 are the Coulomb gauge one- and two-loop renormalization constants [21], A_0^m is the lowest order amplitude and m is a space index.[1]

The lowest-order amplitude can be written as $A_0^m=B^mI_{\rm LO}$, where $B^m=\sqrt{2}ie\phi_0\epsilon^m$. In this formula, $\phi_0=[(m^3\alpha^3)/(8\pi)]^{1/2}$, the $n=1$, s state wave function at contact, while $\epsilon=(0,\hat{\epsilon})$ is the positronium spin vector. We also have $\hat{\epsilon}^*\cdot\hat{\epsilon}=1$, $B_m=B^{m*}$, and $L_{\rm LO}=1+\alpha/6+O(\alpha^2)$ [13]. Consequently, the lowest order contribution to the energy is $A_{0m}A_0^m/(4W^2)=m\alpha^4/4$.

A_2^m is the sum of the two-loop amplitudes shown in Fig. 2. Each of these gives an unrenormalized contribution to the energy of $2A_{0m}A_i^m/(4W^2)$, where i can take on the values SE, SV, CL, LL, or VP, which stand for self-energy, side vertex, crossed ladder, double ladder, and vacuum polarization, respectively.

As we did for A_0^m, we write each of the two-loop amplitudes displaying an explicit B^m, an α^2, and some terms related to the dimensional regularization. The result is

$$A_i^m=(\Omega e^{-\gamma_E})^{2\epsilon}B^m(\alpha/\pi)^2I_i,$$

where $\Omega=4\pi\mu^2/m^2$ and μ is the arbitrary mass parameter particular to the process of dimensional regularization. The extraction of $e^{-\gamma_E}$ is for convenience sake only. γ_E is the Euler-Mascheroni constant.

All the work is in calculating the I_i's and since an α^2 has been factored from each amplitude, we need to know each I_i to order α^0.

The self-energy, side vertex, vacuum polarization, and crossed ladder graphs contribute at leading order α^6 to the energy. Thus, we need to calculate their associated I_i's to order α^0. However, this is not true for the double-ladder graph. Consider the case when the outer photon is Coulombic. Making use of the nonrelativistic Schrödinger equation

[1]These forms for the renormalized amplitudes are a consequence of the energy perturbation series, taking into account the multiplication of the irreducible one-photon annihilation kernel by Z_2^2 and writing the bare charges as $Z_3^{-1}e$ [16]. A heuristic derivation is given by expanding Z_1A^m to order α^2 where $A^m=A_0^m+A_1^m+A_2^m+\cdots$ is the total unrenormalized annihilation amplitude and $Z_1=1/(1+L_1+L_2+\cdots)$. This is the standard way the vertex operator is renormalized.

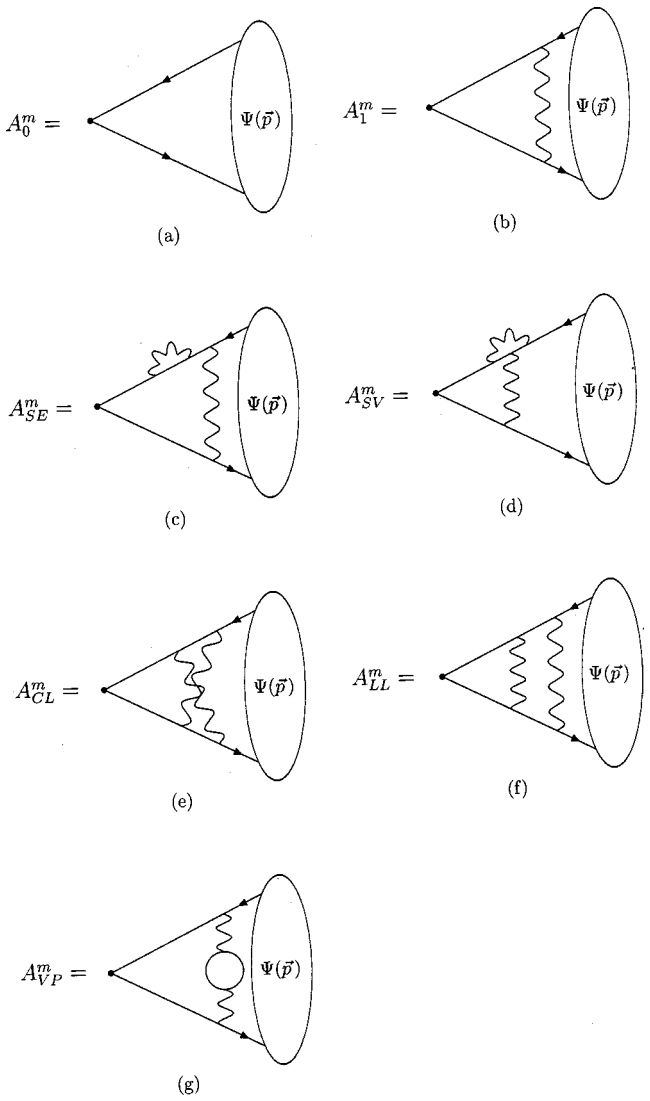

FIG. 2. Contributing amplitudes: (a) lowest order, (b) single ladder, (c) self-energy, (d) side vertex, (e) crossed ladder, (f) double ladder, and (g) vacuum polarization.

with the Coulomb potential, it can be seen that the lowest-order decay amplitude and the vertex correction to the lowest-order decay amplitude, the one-rung amplitude, are included in the double-ladder amplitude. These subamplitudes contribute at order α^4 and order α^5 to the energy. Consequently, if we write $A_{\mathrm{LL}}^m = B^m(\Omega e^{-\gamma_E})^{2\epsilon}(\alpha/\pi)^2 I_{\mathrm{LL}} = B^m(\Omega e^{-\gamma_E})^{2\epsilon}(\alpha/\pi)^2(I_{CC}+I_{TC}+I_{CT}+I_{TT})$, I_{CC} (both photons Coulomb) goes as α^{-2} and I_{TC} (inner photon transverse, outer photon Coulomb) as α^{-1}. The lower-order terms contained in I_{CC} and I_{TC} must be removed before calculating the α^0 contribution of I_{LL}.

D. Determining orders of α

The Feynman diagrams generate a multitude of integrals that must be calculated to an accuracy of order α^0. How this is done is best explained by looking at a few examples [25]. First, consider (from hereon, the electron mass has been set to 1)

$$Ex_1 = \int dp \frac{1}{(p^2+\gamma^2)(p^2+1)} = \frac{\pi(\gamma-1)}{2\gamma(\gamma^2-1)} = \frac{\pi}{\alpha} - \frac{\pi}{2} + \cdots . \tag{1.10}$$

Here, $\gamma = \alpha/2$ and the limits of integration are from 0 to ∞, as in all momentum integrations unless otherwise stated. The value of the integral and the first two terms in its Laurent series in α are given. We now are going to make a series of approximations that will reproduce the first two terms of its series expansion.

Assuming we do not know the value of Ex_1, we can get an estimate of its leading order by the following scaling argument. If the momentum is restricted to the nonrelativistic region, $p \sim \gamma$, then letting $p \to \gamma p$, an explicit γ^{-1} is in front of the scaled integral. However, if $p \sim 1$, then the integral has no factors of α in front. From this we deduce that:

(1) The leading order is α^{-1}.

(2) The nonrelativistic region of momentum space is the most important in the sense that the α^{-1} term is generated from this sector of momentum space.

(3) We expect the next order corrections, the order α^0 terms, to come from the relativistic region.

This is confirmed by calculating the leading and next to leading order with a series of approximations. First, note that Ex_1 is a singular function of α, for if $\alpha \to 0$ in the integrand, the integral diverges linearly at the lower end of the momentum integration. The γ protects against this divergence and cannot be ignored, confirming the conjecture that the nonrelativistic region of momentum is the dominant one. For these momentum values, $1/(p^2+1) \sim 1$. So, to extract the leading order we write $1/(p^2+1) = 1 - p^2/(p^2+1)$, giving us two integrals. We expect the integral with the 1 will give the leading order and the correction term to be of higher order. Thus,

$$\begin{aligned} Ex_1 &= \int dp \frac{1}{p^2+\gamma^2} - \int dp \frac{p^2}{(p^2+\gamma^2)(p^2+1)} \\ &= \frac{\pi}{\alpha} - \int dp \frac{p^2}{(p^2+\gamma^2)(p^2+1)} \end{aligned} \tag{1.11}$$

and our suspicion, at least about the leading order, is verified.

For the correction term, the integral converges if γ is set to 0. This means this term is a regular function of α, going as $c + f(\alpha)$, where c is a constant and f vanishes as $\alpha \to 0$. Thus, we can set γ to 0 in the remainder piece and evaluate the integral, getting $-\pi/2$, the correct next to leading order term.

For the correction term, we could also argue that the p^2 in the numerator makes the region $p \sim 1$ the dominant one, so values of $p \sim 1$ set the scale of the correction integral. Hence, we could approximate $1/(p^2+\gamma^2)$ by $1/p^2$, giving us the same next to leading order integral as before.

This type of scaling argument is the simplest way of estimating the leading order of an integral. It does not necessarily tell us how to calculate the integral but whether it is of the order of interest.

As a slightly more complicated example, consider the following integral over two momentum variables which is simple enough to do analytically:

$$Ex_2 = \int d^3p\, d^3q\, \frac{1}{(p^2+1)(\vec{p}-\vec{q})^2(q^2+\gamma^2)^2} = \frac{2\pi^4}{\gamma(\gamma+1)}. \tag{1.12}$$

This has leading order α^{-1} which we would like to calculate by finding a suitable approximation to the integrand. We do this by proceeding as above. First restrict p and $q \sim \gamma$. Then, scaling the integration variables by γ, we get a α^0 in front. For $p \sim \gamma$ and $q \sim 1$, the integral scales as α^3. However, for $p \sim 1$ and $q \sim \gamma$, the integral goes as α^{-1}, while for $p \sim 1$ and $q \sim 1$, the integral again is of order α^0. Thus, we can approximate the integral by

$$Ex_2 \sim \int d^3p\, d^3q\, \frac{1}{(p^2+1)p^2(q^2+\gamma^2)^2}, \tag{1.13}$$

which evaluates to $2\pi^4/\gamma$, the leading term.

A final example is an integral that occurs in the analysis of the double-ladder graph

$$Ex_3 = \int dp\, dx\, \frac{1}{\sqrt{x h_c}}\, \frac{p^2}{\omega_p D_p} \tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right), \tag{1.14}$$

with the definitions $h_c = p^2(1-x)+\gamma^2$, $\omega_p = \sqrt{p^2+1}$, and $D_p = p^2+\gamma^2$. The x integration goes from 0 to 1. If $\gamma \to 0$, the p integral diverges logarithmically at the low end of the momentum integration. Hence, Ex_3 probably goes as $\ln(\alpha)$ plus regular terms in α.

To isolate the singular and regular parts, we use $1/w_p = 1 + (1-\omega_p)/\omega_p$, which separates the integrand as a part divergent in the $\gamma \to 0$ limit and a part convergent in that limit. Two integrals emerge:

$$Ex_{3a} = \int dp\, dx\, \frac{1}{\sqrt{x h_c}}\, \frac{p^2}{D_p} \tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right) \tag{1.15}$$

and

$$Ex_{3b} = \int dp\, dx\, \frac{1}{\sqrt{x h_c}}\, \frac{p^2}{D_p} \tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right) \frac{1-\omega_p}{\omega_p}. \tag{1.16}$$

For Ex_{3b}, the integral does not diverge as $\gamma \to 0$ since the $1-\omega_p$ provides protection for small values of p. Hence, Ex_{3b} is a regular function of α and goes as α^0 plus terms which vanish as α goes to 0. (This is why we wrote $1/\omega_p$ as we did.)

To our order of accuracy, we only need the constant term, so we can approximate Ex_{3b} by setting γ to 0:

$$Ex_{3b} \sim \int dp\, dx\, \frac{1}{p\sqrt{x(1-x)}} \tan^{-1}\left(\frac{1}{p\sqrt{1-x}}\right) \frac{1-\omega_p}{\omega_p}. \tag{1.17}$$

This can be done numerically, possibly analytically.

For Ex_{3a}, we write $p^2 = D_p - \gamma^2$, resulting again in two integrals. The first is

$$\int dp\, dx\, \frac{1}{\sqrt{x h_c}} \tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right). \tag{1.18}$$

A way to do this integral is to introduce a third parameter via $\int_0^\infty da/(h_c+a^2) = 1/\sqrt{h_c}\, \tan^{-1}(1/\sqrt{h_c})$ and then do the p integral. The remaining x and a integrals are simple, giving us

$$\int dp\, dx\, \frac{1}{\sqrt{x h_c}} \tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right) = \pi^2 \sinh^{-1}(1/\gamma)/2 = \pi^2/2 \ln(4/\alpha) + \cdots$$

with the anticipated logarithm.

The remaining integral is

$$-\gamma^2 \int dp\, dx\, \frac{1}{\sqrt{x h_c}} \tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right) \frac{1}{D_p}. \tag{1.19}$$

In this, if we make the replacement $p \to \gamma p$, the γ^2 in front cancels, leaving an integral that is regular in the $\gamma \to 0$ limit. So, using the same arguments as above, to our order of accuracy, we have

$$\begin{aligned} -\gamma^2 \int dp\, dx\, \frac{1}{\sqrt{x h_c} D_p} \tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right) &\overset{p\to p\gamma}{=} -\int dp\, dx\, \frac{1}{\sqrt{x(1-x)p^2+1}}\, \frac{1}{(p^2+1)} \tan^{-1}\left(\frac{1}{\gamma\sqrt{(1-x)p^2+1}}\right) \\ &\overset{\gamma\to 0}{=} -\frac{\pi}{2} \int dp\, dx\, \frac{1}{\sqrt{x(1-x)p^2+1}}\, \frac{1}{(p^2+1)} \\ &= -\frac{\pi^2}{2} \ln(2). \end{aligned} \tag{1.20}$$

E. Orders of the amplitudes

Using similar scaling arguments as above, we find the self-energy, side vertex, vacuum polarization, and crossed ladder amplitudes contribute to the energy at leading order α^6. This means that $I_{\rm SE}$, $I_{\rm SV}$, $I_{\rm VP}$, and $I_{\rm CL}$ have leading order α^0. Their calculation to this degree of accuracy is done by setting all occurrences of the relative momentum in the fermion propagators to 0 (these are higher-order terms), and replacing factors of γ by 0 in the propagator denominators. The wave function can be replaced by its nonrelativistic value, leaving an easy integration over the relative momentum: $\int d^3p/(2\pi)^3(8\pi\gamma)/(p^2+\gamma^2)^2=1$. We are left with integrals that are independent of α and are the correct α^0 approximations to the full integrals. These integrals are expressed as integrations over the loop momentum variables. The momentum integrations are done by Feynman parameters. The parameter integrals are then numerically integrated with VEGAS [26]. This procedure works well, except for those occasions when numerically unstable integrands are generated. These cases are discussed later in the text.

The double-ladder graph presents more of a challenge since, as commented on earlier, it contains α^{-2}, α^{-1}, and $\ln(\alpha)$ terms from the lowest-order one-annihilation photon diagram and the one-rung ladder diagram.

To calculate the α^0 part of the integrals associated with the double-ladder graph, the lower-order parts must first be determined analytically. These singular parts are then subtracted from the full integrals, leaving subtracted integrals that have leading order α^0, and can be treated in much the same way as the integrals from the other amplitudes.

We now describe in more detail the calculation of the individual diagrams, starting first with a brief examination of the one-loop amplitude and then following with an in depth discussion of the double-ladder graph, the most difficult to calculate. Our comments on the remaining diagrams will generally be limited to those parts of their calculation where we had to amend our standard numerical procedure to get numerically stable integrals.

II. ONE-LOOP AMPLITUDE

In this section we outline the evaluation of the one-loop annihilation amplitude A_1^m. We illustrate some of our calculational methods here in this relatively simple setting, and also define several of the quantities that will appear in the two-loop calculation.

The one-loop amplitude [see Fig. 2(b)] can be written as

$$A_1^m=-ie\int (dp)_4'\,\mathrm{Tr}[\Lambda_1^m(-Wn+p,Wn+p)\Psi^0(p)]$$
$$=-\frac{B^m}{4}\int (dp)_3'\,T_{rr}[\Lambda_1^m((-W,\vec{p}),(W,\vec{p}))\Phi(\vec{p})]\times\left(\frac{8\pi\gamma}{D_p^2}\right), \tag{2.1}$$

where $(dp)_n'=(d^np)/(2\pi)^n$ and Tr stands for the trace. The wave function $\Psi^0(p)$ and the spin part $\Phi(\vec{p})$ are given in Appendix A. The symbol T_{rr} stands for the trace with the spin vector ϵ^m factored out and removed. This can be done because after the momentum integral is completed, the trace is proportional to ϵ^m. Finally, Λ_1^m is the unrenormalized one-loop vertex operator

$$\Lambda_1^m((-W,\vec{p}),(W,\vec{p}))$$
$$=\int (d\not{\ell})_n'(-ie(n)\gamma_\mu)\frac{i}{\gamma(\not{\ell}-Wn)-1}\times\gamma^m\frac{i}{\gamma(\not{\ell}+Wn)-1}(-ie(n)\gamma_\nu)iD_C^{\mu\nu}(\not{\ell}-(0,\vec{p}))$$
$$=-\left(\frac{\alpha}{4\pi}\right)\Omega^\epsilon\int (d\not{\ell})_n''\times\frac{\gamma_\mu(\gamma(\not{\ell}-Wn)+1)\gamma^m(\gamma(\not{\ell}+Wn)+1)\gamma_\nu}{D(W,\not{\ell})D(-W,\not{\ell})}\times D_C^{\mu\nu}(\not{\ell}-(0,\vec{p})), \tag{2.2}$$

where $e(n)=e\mu^\epsilon$ with μ an arbitrary mass parameter, $e^2=4\pi\alpha$, $(d\not{\ell})_n''=(d^n\not{\ell})/(i\pi^{n/2})$, $n=(1,\vec{0})$, $\not{\ell}=(\not{\ell}_0,\vec{\not{\ell}})$, $D(\pm W,\not{\ell})=[-(\not{\ell}\pm Wn)^2+1]$ and $\Omega=4\pi\mu^2/m^2$. For calculational purposes we take $m=1$. We find that

$$A_1^m=\frac{\alpha}{16\pi}B^m\int (dp)_3'(d\not{\ell})_n''\times\frac{\mathrm{Tr}[\gamma_\mu(\gamma(\not{\ell}-Wn)+1)\gamma^m(\gamma(\not{\ell}+Wn)+1)\gamma_\nu\Phi(\vec{p})]}{D(W,\not{\ell})D(-W,\not{\ell})}\times D_C^{\mu\nu}(\not{\ell}-(0,\vec{p}))\left(\frac{8\pi\gamma}{D_p^2}\right). \tag{2.3}$$

The Coulomb part of A_1^m has μ, $\nu\to 0$, 0 and $D_C^{\mu\nu}(\not{\ell}-(0,\vec{p}))\to 1/(\vec{\not{\ell}}-\vec{p})^2$. The trace for the Coulomb part, as a function of $\not{\ell}$, has the form

$$T(\not{\ell})=A+B_\mu\not{\ell}^\mu+C_{\mu\nu}\not{\ell}^\mu\not{\ell}^\nu. \tag{2.4}$$

The leading binding singularity is in A, while only the C term contains an ultraviolet divergence. We evaluate the "low-energy" and "ultraviolet" contributions $T_{\rm LO}(\not{\ell})=A+B_\mu\not{\ell}^\mu$ and $T_{\rm UV}(\not{\ell})=C_{\mu\nu}\not{\ell}^\mu\not{\ell}^\nu$ separately. For the low-energy contribution we do the $\not{\ell}^0$ integral via the integration formula

$$\int\frac{d\not{\ell}^0}{2\pi}\frac{(\gamma(\not{\ell}-Wn)+1)\gamma^m(\gamma(\not{\ell}+Wn)+1)-\gamma\not{\ell}\gamma^m\gamma\not{\ell}}{D(W,\not{\ell})D(-W,\not{\ell})}$$
$$=\frac{i}{4\omega_l}\frac{1}{D_l}[(\gamma(\not{\ell}-Wn)+1)\gamma^m(\gamma(\not{\ell}+Wn)+1)-\gamma\not{\ell}\gamma^m\gamma\not{\ell}]_{\not{\ell}^0\to 0} \tag{2.5}$$

and analyze carefully the various regimes contained in the $\vec{p}$ and $\vec{\not{r}}$ integrals as described in Sec. I D. The low-energy contribution is

$$A^m_{\rm CLO}=B^m\left\{1-\frac{2\alpha}{\pi}+\frac{\alpha^2}{8}\ln\left(\frac{1}{\alpha}\right)+\alpha^2[0.312\,499\,4(8)]+O(\alpha^3)\right\}. \tag{2.6}$$

For the ultraviolet contribution we worked out a parametric form for the corresponding part of the vertex function, separated off the UV divergence, and then did the expansion in α. We found

$$A^m_{\rm CUV}=B^m\left\{\frac{2\alpha}{3\pi}I_{\rm LO}+L^C_{(1)}I_{\rm LO}+\alpha^2[-0.288\,194\,452(7)]+O(\alpha^3)\right\}, \tag{2.7}$$

where

$$I_{\rm LO}=\int (dp)'_3 g(p)\left(1+\frac{p^2}{3(\omega_p+1)^2}\right)\left(\frac{8\pi\gamma}{D_p^2}\right)=1+\frac{\alpha}{6}+\frac{\alpha^2}{16}+O(\alpha^3) \tag{2.8}$$

with

$$g(p)=\left(\frac{\omega_p+1}{2\omega_p}\right)\left(\frac{\omega_p+W}{2W}\right)^{1/2} \tag{2.9}$$

being the relativistically correct zero-loop annihilation graph ($A^m_0=B^m I_{\rm LO}$), and

$$L^C_{(1)}=\left(\frac{\alpha}{4\pi}\right)\Omega^\epsilon\Gamma(\epsilon)\frac{2(n-2)}{(n-1)}=\left(\frac{\alpha}{4\pi}\right)\left\{\frac{4}{3}\Omega^\epsilon\Gamma(\epsilon)-\frac{4}{9}+O(\epsilon)\right\} \tag{2.10}$$

being the Coulomb part of the one-loop renormalization constant. The complete Coulomb contribution is the sum of the LO and UV parts:

$$A^m_C=B^m\left\{1-\frac{2\alpha}{\pi}+\frac{2\alpha}{3\pi}I_{\rm LO}+\frac{\alpha^2}{8}\ln\left(\frac{1}{\alpha}\right)+L^C_{(1)}I_{\rm LO}+\alpha^2[0.024\,304\,9(8)]+O(\alpha^3)\right\}. \tag{2.11}$$

The transverse part of A^m_1 has $\mu,\ \nu\to i,\ j$ with

$$D^{ij}_C(k)=\frac{-1}{k^2}\left(\delta_{ij}-\frac{k^i k^j}{\vec{k}^2}\right). \tag{2.12}$$

We found an appropriate parametric form for the transverse part of the vertex function, separated off the UV divergence, and expanded in α. Our result is

$$A^m_T=B^m\left\{-\frac{2\alpha}{3\pi}I_{\rm LO}+\frac{\alpha^2}{2}\ln\left(\frac{1}{\alpha}\right)+L^T_{(1)}I_{\rm LO}+\alpha^2[-0.271\,370\,60(12)]+O(\alpha^3)\right\}, \tag{2.13}$$

where

$$L^T_{(1)}=\left(\frac{\alpha}{4\pi}\right)\Omega^\epsilon\Gamma(\epsilon)\frac{-(n-3)}{(n-1)} \tag{2.14}$$

is the tranverse part of the one-loop renormalization constant. The total one-loop amplitude is the sum of the Coulomb and transverse parts. It is

$$A^m_1=B^m\left\{1-\frac{2\alpha}{\pi}+\frac{5\alpha^2}{8}\ln\left(\frac{1}{\alpha}\right)+L_{(1)}I_{\rm LO}+\alpha^2[-0.247\,065\,7(9)]+O(\alpha^3)\right\}, \tag{2.15}$$

where $L_{(1)}$ is the full one-loop renormalization constant

$$L_{(1)}=L^C_{(1)}+L^T_{(1)}=\left(\frac{\alpha}{4\pi}\right)\Omega^\epsilon\Gamma(\epsilon). \tag{2.16}$$

III. DOUBLE-LADDER AMPLITUDE

This amplitude is best calculated by dividing it into four diagrams according to whether the inner and outer photons are Coulomb or transverse. Then, the α^{-2} and some of the α^{-1} terms are isolated in the Coulomb-Coulomb amplitude, while the transverse-Coulomb amplitude has the remaining α^{-1} parts and a $\ln(\alpha)$. We will also discover that the Coulomb-transverse diagram has a $\ln(\alpha)$, while the transverse-transverse part is leading order α^0.

A. Coulomb-Coulomb ladder

The Coulomb-Coulomb amplitude is

$$A^m_{cc}=B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}I_{cc}, \tag{3.1}$$

where

$$I_{cc}=-\frac{e^{2\epsilon\gamma_E}}{64}\int (dq)''_n(d\not{r})''_n(dp)'_3\times\frac{T_{cc}}{(\vec{q}-\vec{\not{r}})(\vec{\not{r}}-\vec{p})^2D(W,q)D(-W,q)D(W,\not{r})D(-W,\not{r})}\times\left(\frac{8\pi\gamma}{D_p^2}\right) \tag{3.2}$$

and

$$T_{cc}=T_{rr}[\,\gamma_0(\gamma(\not\ell-Wn)+1)\gamma_0(\gamma(q-Wn)+1)\times\gamma^m(\gamma(q+Wn)+1)\gamma^0(\gamma(\not\ell+Wn)+1)\gamma_0\Phi(\vec{p})]. \tag{3.3}$$

To calculate A_{cc}, we separate the trace into two parts. The first is the term with 0 or 1 factors of q. This part of the trace has the leading $O(\alpha^{-2})$ term so we call it $T_{\rm LO}$. The remaining terms in the trace are quadratic in q and have ultraviolet divergences. This part of the trace is called $T_{\rm UV}$.

After partitioning the trace in this fashion, I_{cc} is separated into two pieces, $I_{\rm LO}$ and $I_{\rm UV}$. We consider $I_{\rm LO}$ first.

1. I_{LO}

For $I_{\rm LO}$ we have

$$I_{\rm LO}=-\frac{1}{64}\int (dq)''_4(d\not\ell)''_4(dp)'_3\frac{T_{\rm LO}}{(\vec{q}-\vec{\not\ell})^2(\vec{\not\ell}-\vec{p})^2D(W,q)D(-W,q)D(W,\not\ell)D(-W,\not\ell)}\left(\frac{8\pi\gamma}{D_p^2}\right) \tag{3.4}$$

with $T_{\rm LO}$ the full trace minus the terms quadratic in q:

$$T_{\rm LO}=T_{rr}[\,\gamma_0(\gamma(\not\ell-Wn)+1)\gamma_0[(\gamma(q-Wn)+1)\times\gamma^m(\gamma(q+Wn)+1)-\gamma q\gamma^m\gamma q]\times\gamma_0[\,\gamma((\not\ell+Wn)+1)\gamma_0\Phi(\vec{p})]. \tag{3.5}$$

The first step is to do the q integration. This can be done via poles or parameters. We choose the latter and use

$$\int (dq)''_4\frac{(1;q_0;q_i)}{(\vec{q}-\vec{\not\ell})^2D(W,q)D(-W,q)}=\frac{1}{W}\int dx\,\frac{1}{\sqrt{xh_c}}\tan^{-1}\left(\frac{W}{\sqrt{h_c}}\right)[1;0;(1-x)l_i], \tag{3.6}$$

where $|\vec{\not\ell}|=l$ and $h_c=(1-x)l^2+\gamma^2$. Then, $T_{\rm LO}$ is computed.

$T_{\rm LO}$ has many terms. Of these, there is one which has no explicit factors of $\vec{\not\ell}$, $\vec{p}$, or $\not\ell_0$. This term, as will be shown, is responsible for the α^{-2} contribution. The other terms have at least 2 or 4 factors of momentum. In these, factors of ω_p and W can be set to one. This reduces the numbers of terms significantly. What remains is a sum of terms that are quadratic and quartic in the momentum variables. Of these, only the quadratic terms must be kept. Quartic terms, which are l^2p^2 or $(\vec{\not\ell}\cdot\vec{p})^2$, can safely be dropped as contributing beyond the order of interest, as can be established using the arguments presented in Sec. I D. Then, correct to $O(\alpha^0)$,

$$T_{\rm LO}=T_A+T_B+T_C, \tag{3.7}$$

where

$$T_A=-2\,\frac{(W+1)^4}{\sqrt{x}}\left(\frac{1+\omega_p}{\omega_p}\right)\sqrt{\frac{\omega_p+W}{2W}},$$

$$T_B=\frac{128(l^2(x-1)}{\sqrt{x}}+\frac{64\not\ell_0^2}{\sqrt{x}}\quad\text{and}\quad T_C=-\frac{128\vec{\not\ell}\cdot\vec{p}}{\sqrt{x}}. \tag{3.8}$$

This separates $I_{\rm LO}$ into three subintegrals, I_A, I_B, and I_C.

2. I_A

The first integral is

$$I_A=\frac{(1+W)^4}{32W}\int (d\not\ell)''_4(dp)'_3\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{W}{\sqrt{h_c}}\right)\times\frac{1}{(\vec{\not\ell}-\vec{p})^2D(W,\not\ell)(-W,\not\ell)}\left(\frac{1+\omega_p}{\omega_p}\right)\sqrt{\frac{\omega_p+W}{2W}}\times\left(\frac{8\pi\gamma}{D_p^2}\right). \tag{3.9}$$

To calculate I_A, we must first determine its leading order. This is done first by doing the $\not\ell_0$ integration via the residue theorem, closing the contour in the upper half $\not\ell_0$ plane. This picks up the poles at $\not\ell_0=-W-\omega_l+i\epsilon$ and $\not\ell_0=W-\omega_l+i\epsilon$. The result of the $\not\ell_0$ integration is

$$I_A=\frac{(1+W)^4}{64\pi W}\int d^3l(dp)'_3\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{W}{\sqrt{h_c}}\right)\times\frac{1}{\omega_lD_l(\vec{\not\ell}-\vec{p})^2}\left(\frac{1+\omega_p}{\omega_p}\right)\sqrt{\frac{\omega_p+W}{2W}}\left(\frac{8\pi\gamma}{D_p^2}\right). \tag{3.10}$$

Now, consider the regions $p\sim\gamma$ and $l\sim\gamma$. We replace ω_p, ω_l and W by one and $\tan^{-1}(1/\sqrt{h_c})$ by $\pi/2$. Then, letting $l\to\gamma l$ and $p\to\gamma p$, we get an explicit α^{-2} in front of the approximate integral, indicating that this region of momentum space contributes to $O(\alpha^{-2})$. Corrections to these initial approximations, in the same integration region, will be $O(\alpha^0)$, because the corrections contribute two more powers of momentum in the numerator (from $\omega_p\sim1+p^2/2$, for instance).

Using similar arguments, the region $p\sim1$, $l\sim\gamma$ and the region $p\sim1$, $l\sim1$ contribute to $O(\alpha)$. However, if $p\sim\gamma$, $l\sim1$, the integral is $O(\alpha^0)$. This means, to the $O(\alpha^0)$, only nonrelativistic values of p contribute, but all values of l contribute.

With this in mind, we make the replacement

$$\left(\frac{1+\omega_p}{\omega_p}\right)\sqrt{\frac{\omega_p+W}{2W}}\sim 2+\frac{\gamma^2}{4}-\frac{p^2}{4}. \tag{3.11}$$

Substituting in I_A, we get

$$I_{A1}=\frac{(1+W)^4}{8W}\int dl\,\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{W}{\sqrt{h_c}}\right)\frac{l^2}{\omega_l D_l^2}, \tag{3.12}$$

$$I_{A2}=\frac{\gamma^2}{4}\int dl\,\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{l^2}{\omega_l D_l^2},$$

$$I_{A3}=-\frac{1}{16\pi}\int d^3l(dp)_3'\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\times\frac{1}{\omega_l D_l}\frac{p^2}{(\vec{\ell}-\vec{p})^2}\left(\frac{8\pi\gamma}{D_p^2}\right). \tag{3.13}$$

For I_{A2}, if $l\sim 1$, the integral is $O(\alpha^2)$, but for $l\sim\gamma$, it is $O(\alpha^0)$. Hence, we replace ω_l by 1, scale l by γ and approximate $\tan^{-1}(1/\sqrt{h_c})$ by $\pi/2$, and get

$$I_{A2}=\frac{\pi^2}{32}. \tag{3.14}$$

For I_{A3}, the p integral can be done, leaving us with

$$I_{A3}=-\frac{\gamma}{4}\int dl\,\frac{dx}{\sqrt{xh_c}}\frac{l^2}{\omega_l D_l}\tan^{-1}\left(\frac{1}{\sqrt{hc}}\right)\left[\frac{2}{l}\tan^{-1}\left(\frac{l}{\gamma}\right)-\frac{\gamma}{D_l}\right]. \tag{3.15}$$

We can again approximate $\omega_l\sim 1$, scale l by γ and let $\tan^{-1}(1/\sqrt{h_c})$ go to $\pi/2$ without any loss of required accuracy. The integral can be done and we obtain

$$I_{A3}=-\frac{\pi^2}{16}\left(\frac{\pi^2}{3}-\frac{1}{2}\right). \tag{3.16}$$

The evaluation of I_{A1} is more interesting. For $l\sim\gamma$, the integral has a power count of α^{-2}. This region of momentum space gives the lowest order term. If we expand the $1/\omega_l$ to order l^2, we get a correction term that, for $l\sim\gamma$, gives an integral that is $\ln(\alpha)+O(\alpha^0)$. Hence, we must calculate corrections to the replacement $\omega_l\to 1$. Also, recall that if $l\sim 1$, the integral is $O(\alpha^0)$, so the integral for the relativistic region of l must be calculated as well.

The calculation of I_{A1} proceeds by using the identity $1/\omega_l=1-l^2/2+[(1-\omega_l)/\omega_l+l^2/2]$. This is done instead of using $1/\omega_l=1+(\omega_l-1)$ because it is a simple way to isolate the $\ln(\alpha)$, which is in the $-l^2/2$ term.

Upon making this substitution, we have three integrals. The ones associated with the 1 and l^2 can be done analytically. For the integral with the $(1-\omega_l)/\omega_l+l^2/2$ factor, which we call I_{A1N}, γ can be set to 0 (it is order α^0), and the resulting integral done numerically. (From I_B and $I_{\rm UV}$ there will be other integrals that have to be computed numerically, so we postpone evaluating I_{A1N} for now, and choose to do all the numerical integrations at once.) So,

$$I_{A1}=-\frac{\pi^2}{\alpha^2}-\frac{2\pi}{\alpha}-\frac{\pi^2}{2}\ln\left(\frac{1}{\alpha}\right)-\frac{\pi^2}{2}\ln(2)+\frac{\pi^2}{4}+I_{A1N} \tag{3.17}$$

and for I_A, adding everything together,

$$I_A=\frac{\pi^2}{\alpha^2}-\frac{2\pi}{\alpha}-\frac{\pi^2}{2}\ln\left(\frac{1}{\alpha}\right)-\frac{\pi^2}{2}\ln(2)-\frac{\pi^4}{48}+\frac{5\pi^2}{16}+I_{A1N}. \tag{3.18}$$

3. I_B and I_C

The calculation of I_B and I_C is relatively straightforward. The ℓ_0 integration is done as before. For I_B,

$$I_B=\frac{1}{2}\int dl\,\frac{dx}{\sqrt{xh_c}}\frac{l^2}{\omega_l D_l}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)+\int dl\,\frac{dx}{\sqrt{xh_c}}\frac{l^4}{\omega_l D_l^2}(1-x)\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right). \tag{3.19}$$

Using $1/\omega_l=1+(1-\omega_l)/\omega_l$,

$$\begin{aligned}I_B&=\int dl\,\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{l^2}{D_l}\left[\frac{1}{2}+(1-x)\frac{l^2}{\omega_l D_l}\right]\times\left(1+\frac{1-\omega_l}{\omega_l}\right)\\&\to\int dl\,\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{l^2}{D_l}\left[\frac{1}{2}+(1-x)\frac{l^2}{D_l}\right]\\&\quad+\int dl\,\frac{dx}{x\sqrt{1-x}}\frac{1}{l}\tan^{-1}\left(\frac{1}{l\sqrt{1-x}}\right)\frac{1-\omega_l}{\omega_l}\left[\frac{3}{2}-x\right]\\&=\frac{\pi^2}{2}\ln\left(\frac{1}{\alpha}\right)+\frac{\pi^2}{2}\ln(2)-\frac{\pi^2}{8}+I_{BN},\end{aligned} \tag{3.20}$$

where I_{BN} is the integral with the $(1-\omega_l)/\omega_l$ in Eq. (3.20) and will be done numerically. The remaining integral is

$$I_C=\frac{1}{4\pi}\int d_3l(dp)_3'\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{1}{\omega_l D_l}\frac{\vec{\ell}\cdot\vec{p}}{(\vec{\ell}-\vec{p})^2}\times\left(\frac{8\pi\gamma}{D_p^2}\right). \tag{3.21}$$

We use $(\vec{\ell}-\vec{p})^2=l^2+\vec{p}^2-2\vec{\ell}\cdot\vec{p}$ to decouple the $\vec{\ell}$ and $\vec{p}$ integrations. Then,

$$I_C=\frac{1}{2}\int dl\,\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{l^2}{\omega_l D_l}\left[\frac{l^2}{D_l}-1\right]$$
$$+\frac{1}{8\pi}\int d_3l(dp)'_3\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{1}{D_l\omega_l}\,\frac{p^2}{(\not{l}-\vec{p})^2}$$
$$\times\left(\frac{8\pi\gamma}{D_p^2}\right). \tag{3.22}$$

The last integral is $-2I_{A3}$. In the first integral, for $l\sim 1$, the integral vanishes. For $l\sim\gamma$, if the integral is done as a sum of two terms, each term separately goes as $\ln(\alpha)+\alpha^0$. However, the log terms, which develop when l is between γ and 1, cancel, since both terms individually have the same asymptotic limit in this region. Hence, to order α^0 we can set ω_l to 1 and $\tan^{-1}(1/\sqrt{h_c})$ to $\pi/2$. We get

$$I_C=\frac{\pi^4}{24}-\frac{\pi^2}{8} \tag{3.23}$$

and, finally, for $I_{\rm LO}$

$$I_{\rm LO}=I_A+I_B+I_C=\frac{\pi^2}{\alpha^2}-\frac{2\pi}{\alpha}+\frac{\pi^4}{48}+\frac{\pi^2}{16}+I_{A1N}+I_{BN}. \tag{3.24}$$

4. The ultraviolet term

The ultraviolet contribution to I_{cc} is

$$I_{\rm UV}=\frac{e^{2\epsilon\gamma_E}}{64}\int (dq)''_n(d\not{l})''_n(dp)'_3\frac{T_{\rm UV}}{(\vec{q}-\not{l})^2(\not{l}-\vec{p})^2D(W,q)D(-W,q)D(W,\not{l})D(-W,\not{l})}\left(\frac{8\pi\gamma}{D_p^2}\right) \tag{3.25}$$

with

$$T_{\rm UV}=T_{rr}[\,\gamma_0(\gamma(\not{l}-Wn)+1)\,\gamma_0\gamma q\gamma^m\gamma q\gamma_0$$
$$\times(\gamma(\not{l}+Wn)+1)\,\gamma_0\Phi(\vec{p})]. \tag{3.26}$$

The first step is to do the q integral with parameters. The required integral is (N_{C0} and N_{C1} are the 0 and 1 contraction terms)

$$\int (dq)''_n\frac{\gamma q\gamma^m\gamma q}{(\vec{q}-\not{l})^2D(W,q)D(W,-q)}$$
$$=\int dx\,du\,x^{-1/2-\epsilon}\left[\Gamma(1+\epsilon)\,\frac{N_{CO}}{H_c^{1+\epsilon}}-\frac{1}{2}\Gamma(\epsilon)x\,\frac{N_{C1}}{H_c^{\epsilon}}\right]. \tag{3.27}$$

We next do a partial integration (in the x variable) on the one contraction term. This isolates the inner vertex divergence in a term that is independent of the outer loop (l) momentum. This is important because from the renormalized energy shift equation, we would like to cancel the $L_1A_1^m$ term without doing any extra work. This, we shall see, will be a consequence of writing the inner vertex in this manner.

So, after the partial integration,

$$\int (dq)''_n\frac{\gamma q\gamma^m\gamma q}{(\vec{q}-\not{l}^2)D(W,q)D(W,-q)}$$
$$=\eta_C(\epsilon)\,\gamma^m+\Gamma(1+\epsilon)\int dx\,du\,\frac{x^{-\epsilon}}{H_c^{1+\epsilon}}NN_{CO}^m \tag{3.28}$$

with the definitions

$$a=1-2u,$$

$$NN_{C0}^m=\frac{1}{\sqrt{x}}[-a^2W^2\gamma^m-2(1-x)aW\not{l}^m\gamma_0$$
$$+(1+x)^2\vec{\gamma}\cdot\not{l}\gamma^m\vec{\gamma}\cdot\not{l}]-\sqrt{x}\left[\frac{1}{n-3}+\frac{n-3}{n-1}x\right]\gamma^m l^2,$$

$$\eta_C(\epsilon)=\Gamma(\epsilon)\left[\frac{1}{n-3}+\frac{n-3}{n-1}\right]\int du\,\frac{1}{\tilde{h}_c^{\epsilon}},$$

and

$$H_c=(1-x)l^2+a^2W^2+\gamma^2,\tilde{h}_c=a^2W^2+\gamma^2.$$

This gives

$$A_{cc\ \rm UV}^m=\frac{\alpha}{4\pi}\Omega^{\epsilon}\eta_C(\epsilon)A_C^m+B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}I'_{\rm UV} \tag{3.29}$$

A_C^m is the Coulomb part of the vertex corrected one-annihilation photon diagram, the one-rung ladder diagram. In $I'_{\rm UV}$, there are terms that have four factors of momentum, which diverge upon integration. The remaining terms have finite integrations. We treat these separately and write $I'_{\rm UV}=I_{\rm fin}+I_{\rm div}$.

For $I_{\rm fin}$ we can set all factors of $\vec{p}$ to 0 in the trace, except in the wave function itself. We must also keep $1/(\vec{p}-\not{l})^2$ intact. Then the p integral can be done using

$$\int (dp)'_3\frac{1}{(\not{l}-\vec{p})^2}\left(\frac{8\pi\gamma}{D_p^2}\right)=\frac{1}{D_l}.$$

If this were not done, that is if $1/(\vec{p}-\vec{l})^2 \to l^2$, then $I_{\rm fin}$ would diverge in the infrared since, as will be shown, it is of order α^{-1}.

So, for $I_{\rm fin}$ $(a=1-2u)$,

$$I_{\rm fin}=\frac{1}{64}\int (d\not{\ell})''_4 dx\, du\, \frac{1}{\sqrt{x}}\frac{1}{D_l D(W,\not{\ell})D(-W,\not{\ell})H_c} \times\left[4a^2\not{\ell}_0^2-\frac{4}{3}(20x+a^2-4)\not{\ell}^2-16a^2\right]. \tag{3.30}$$

$I_{\rm fin}$ has a leading order α^{-1} term that comes from the $16a^2$ expression in Eq. (3.30). The other terms, if calculated individually, have $\ln(\alpha)$'s, but these eventually cancel. The remainder of the calculation of $I_{\rm fin}$ is straightforward, and we get ($I_{{\rm fin}N}$ is the part of $I_{\rm fin}$ that is done numerically)

$$I_{\rm fin}=-\frac{\pi}{2\alpha}+I_{{\rm fin}N}. \tag{3.31}$$

For $I_{\rm div}$, the calculation follows standard techniques after we realize that it is of order α^0. Thus, we can set all factors of γ to 0. Then, after its trace, $T_{\rm div}$, is calculated, we find it can be written as $T_{\rm div}=l^2 t_{\rm div}$. The l^2 cancels against a similar term in the denominator so our expression to evaluate is

$$I_{\rm div}=-\Gamma(1+\epsilon)\frac{e^{2\epsilon\gamma_E}}{256}\int (d\not{\ell})''_n\, dx\, du\, \frac{x^{-\epsilon}}{(1-x)^{1+\epsilon}} \times\frac{t_{\rm div}}{[l^2+a^2/(1-x)]^{1+\epsilon}(-\not{\ell}^2+2\not{\ell}_0)(-\not{\ell}^2-2\not{\ell}_0)}. \tag{3.32}$$

The calculation is straightforward and we get

$$I_{\rm div}=-\frac{1}{18\epsilon^2}-\frac{11}{54\epsilon}+I_{{\rm div}\,N}. \tag{3.33}$$

5. $\eta_C(\epsilon)$ term

The last part of the Coulomb-Coulomb term to calculate is the first expression of Eq. (3.29). For this we need the Coulomb part of the one-rung annihilation diagram to order α. This is

$$A_C^m=\frac{4(1-\epsilon)}{3-2\epsilon}L_1 I_{\rm LO}B^m+\Omega^\epsilon I_{CS}B^m, \tag{3.34}$$

where L_1, the one loop renormalization constant, equals $\alpha/(4\pi)\Omega^\epsilon\Gamma(\epsilon)$ and $I_{CS}=1-(4\alpha)/(3\pi)$. $I_{\rm LO}$ and B^m are as before. Then, after making a Laurent expansion of $\eta_C(\epsilon)$ in ϵ and α and collecting terms,

$$\frac{\alpha}{4\pi}\Omega^\epsilon\eta_C(\epsilon)A_C^m=\frac{4}{3}L_1A_C^m+\left[(\Omega e^{-\gamma_E})^{2\epsilon}\frac{19}{54}\left(\frac{\alpha}{\pi}\right)^2\frac{1}{\epsilon}+\frac{19}{18}\frac{\alpha}{\pi} -\frac{1}{6}\alpha^2-\frac{83}{162}\left(\frac{\alpha}{\pi}\right)^2\right]B^m. \tag{3.35}$$

6. A_{cc}^m results

Putting the various pieces together and computing the numerical integrals,

$$A_{cc}^m=\left(\frac{4}{3}L_1\right)A_C^m+B^m\left\{1-\frac{13}{9}\frac{\alpha}{\pi}+\left(\frac{\alpha}{\pi}\right)^2[2.878\,360(7)] +\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}\left[-\frac{1}{18}\frac{1}{\epsilon^2}+\frac{4}{27}\frac{1}{\epsilon}\right]\right\}. \tag{3.36}$$

B. Transverse-Coulomb ladder

For the transverse-Coulomb amplitude we have

$$A_{tc}^m=-\frac{B^m}{64}\left(\frac{\alpha}{\pi}\right)^2(\Omega)^{2\epsilon}\int (dq)''_n(d\not{\ell})''_n(dp)'_3\frac{T_{tc}}{(-(\not{\ell}-q)^2)(\vec{\ell}-\vec{p})^2D(W,q)D(-W,q)D(W,\not{\ell})D(-W,\not{\ell})}\left(\frac{8\pi\gamma}{D_p^2}\right) \tag{3.37}$$

with the trace

$$T_{tc}=T_{rr}[\gamma_0(\gamma(\not{\ell}-Wn)+1)\gamma_i((q-Wn)+1) \times\gamma^m(\gamma(q+Wn)+1)\gamma_j(\gamma(\not{\ell}+Wn)+1)\gamma_0\Phi(\vec{p})] \times\delta_{ij}^T(\vec{\ell}-\vec{q}) \tag{3.38}$$

and $\delta_{ij}^T(\vec{\ell})=\delta_{ij}-\vec{\ell}_i\vec{\ell}_j/l^2$.

Unlike the Coulomb-Coulomb calculation, here it is more efficient to treat the inner vertex as a whole. So, we write

$$\Lambda_T^m(\not{\ell}-Wn,\not{\ell}+Wn)=\frac{\alpha}{4\pi}\overline{\Lambda}_T^m(\not{\ell}-Wn,\not{\ell}+Wn), \tag{3.39}$$

$$\begin{aligned}\overline{\Lambda}_T^m&(\not{\ell}-Wn,\not{\ell}+Wn)\\ &=\int (dq)''_n\frac{\gamma_i(\gamma(q-W)+1)\gamma^m(\gamma(q+W)+1)\gamma_j}{(-(\not{\ell}-q)^2)D(W,q)D(-W,q)} \times\delta_{ij}^T(\vec{\ell}-\vec{q})\\ &=\int dz\,dx\,du\,z^{3/2}x\left[\Gamma(2+\epsilon)\frac{N_{T0}}{\Delta_T^{2+\epsilon}} -\frac{1}{2}\Gamma(1+\epsilon)\frac{N_{T1}}{\Delta_T^{1+\epsilon}}+\frac{1}{4}\Gamma(\epsilon)\frac{N_{T2}}{\Delta_T^{\epsilon}}\right].\end{aligned} \tag{3.40}$$

N_{T0}, N_{T1}, and N_{T2} are the zero, one, and two contraction terms, $\Delta_T=zxH_T$, and $H_T=-(1-x)\vec{\ell}_0^2+(1-xz)l^2+2a(1-x)\not{\ell}nW+xa^2W^2+\gamma^2$.

The divergence of $\overline{\Lambda}_T^m$ is separated by doing a partial integration on the two contraction term, which, after some further manipulations, gives

$$\overline{\Lambda}_T^m = \eta_T(\epsilon)\gamma^m + \overline{\Lambda}_{TA}^m + \overline{\Lambda}_{TB}^m \tag{3.41}$$

with

$$\eta_T(\epsilon) = (1-\epsilon)\left[\frac{1-2\epsilon}{3-2\epsilon} - (1+2\epsilon)\right]\Gamma(\epsilon)\int dx\,du\,\frac{x^{1-\epsilon}}{\tilde{h}_t^{\epsilon}}, \tag{3.42}$$

$$\overline{\Lambda}_{TA}^m = \int dz\,dx\,du\,\frac{z^{3/2}x}{(zx)^{\epsilon}}\left[\Gamma(2+\epsilon)\frac{N(N_{T0})}{H_T^{2+\epsilon}} - \frac{1}{2}\Gamma(1+\epsilon)\frac{N(N_{T1})}{H_T^{1+\epsilon}}\right], \tag{3.43}$$

$$\overline{\Lambda}_{TB}^m = (\epsilon-1)\left[\frac{1-2\epsilon}{3-2\epsilon} - (1+2\epsilon)\right]\Gamma(1+\epsilon)\gamma^m \times \int dz\,dx\,du\,\frac{x^{1-\epsilon}}{\tilde{H}_T^{1+\epsilon}}B, \tag{3.44}$$

and the following definitions:

$$\tilde{h}_e = xa^2W^2 + \gamma^2,$$

$$N(N_{T0}) = \frac{N_{T0}}{(zx)^2},$$

$$NN_{T1} = \frac{N_{T1}}{zx} + 2(1-\epsilon)l^2x\left[\frac{n-3}{n-1} - (5-n)z\right]\gamma^m,$$

$$\tilde{H}_T = -(1-x)z\not{\ell}^2 + 2z(1-x)a\not{\ell}nW + xa^2W^2 + \gamma^2,$$

$$B = (1-x)(-\not{\ell}^2 + 2a(\not{\ell}n)W).$$

The separation of the inner transverse vertex divergence in this way is done for the same reasons as for the Coulomb part of the inner vertex.

Putting this all together, we can write for the transverse-Coulomb vertex

$$A_{tc}^m = \frac{\alpha}{4\pi}\Omega^{\epsilon}\eta_T(\epsilon)A_C^m + \left(\frac{\alpha}{\pi}\right)^2 B^m(\Omega e^{-\gamma_E})^{2\epsilon}J_{tc} \tag{3.45}$$

with

$$J_{tc} = \frac{e^{2\epsilon\gamma_E}}{64}\int (d\not{\ell})_n''(dp)_3' \frac{T_{rr}[\gamma_0(\gamma(\not{\ell}-nW)+1)(\overline{\Lambda}_{TA}^m + \overline{\Lambda}_{TB}^m)(\gamma(\not{\ell}+nW)+1)\gamma_0\Phi(\vec{p})]}{(\vec{\ell}-\vec{p})^2 D(W,\not{\ell})D(-W,\not{\ell})}\left(\frac{8\pi\gamma}{D_p^2}\right) = J_A + J_B. \tag{3.46}$$

1. J_A

Since J_A has leading order α^{-1} and not α^{-2}, it is sufficient to use the nonrelativistic approximation of the wave function, leaving an easy integration over the relative momentum. What remains is an expression that we separate into an ultraviolet convergent and ultraviolet divergent part. These are, respectively, J_{ALO} and J_{AUV}, with

$$J_{\mathrm{ALO}} = -\frac{1}{64}\int (d\not{\ell})_4'' \frac{T_{rr}[\gamma_0\{(\gamma(\not{\ell}-nW)+1)\overline{\Lambda}_{TA}^m(\gamma(\not{\ell}+nW)+1) - (\gamma\not{\ell}\overline{\Lambda}_{TA}^m\gamma\not{\ell})\}(1+\gamma_0)\gamma\epsilon]}{D(W,\not{\ell})D(-W,\not{\ell})D_l} \tag{3.47}$$

and

$$J_{\mathrm{AUV}} = -\frac{e^{2\epsilon\gamma_E}}{64}\int (d\not{\ell})_4'' \frac{T_{rr}[\gamma_0\gamma\not{\ell}\overline{\Lambda}_{TA}\gamma\not{\ell}(1+\gamma_0)\gamma\epsilon]}{D(W,\not{\ell})D(-W,\not{\ell})D_l}. \tag{3.48}$$

For the low-energy term, after the trace is computed,

$$J_{\mathrm{ALO}} = -\frac{1}{64}\int (d\not{\ell})_4''\,dx\,du\,dz\,\frac{z^{3/2}x}{D_lD(W,\not{\ell})D(-W,\not{\ell})} \times\left[\frac{T_0}{H_T^2} - \frac{1}{2}\frac{T_1}{H_T}\right]. \tag{3.49}$$

T_0 and T_1 are the traces associated with the $N(N_{T0})$ and $N(N_{T1})$ terms. The calculation is made easier if a partial integration with respect to z is done on the T_0 term. This results in ($t_0 = T_0|_{z=1}, h_T = H_T|_{z=1}$)

$$J_{\mathrm{ALO}} = -\frac{1}{64}\int (d\not{\ell})_4''\,dx\,du\,dz\,\frac{1}{D_lD(W,\not{\ell})D(-W,\not{\ell})} \times\left[(4a^2x^2 + (4-a^2)x^2l^2)\frac{1}{h_T} + 4(a^2x^2-12)\sqrt{z}l^2\frac{1}{H_T}\right]. \tag{3.50}$$

The next step is to determine which terms are singular, and as was the case with the Coulomb-Coulomb calculation, this is simpler after the $\not{l}_0$ integration is done.

If we do the $\not{l}_0$ integration by closing the contour in the upper half plane, the nonanalytic terms come from the pole at $\not{l}_0 = W - \omega_l + i\epsilon$. The other poles contribute $O(\alpha^0)$ terms only. So, for the purposes of calculating the α^{-1} and $\ln(\alpha)$ terms only, it would suffice to set l_0 to 0 in h_T and H_T. So,

$$\frac{1}{h_T} = \frac{1}{h_{T0}} + \frac{h_{T0} - h_T}{h_{T0} h_T}, \tag{3.51}$$

$$\frac{1}{H_T} = \frac{1}{H_{T0}} + \frac{H_{T0} - H_T}{H_{T0} H_T}, \tag{3.52}$$

where $H_{T0} = H_T|_{\not{l}_0 = 0}$ and $h_{T0} = h_T|_{\not{l}_0 = 0}$. This isolates the nonanalytic terms in the integrals associated with h_{T0} and H_{T0}. The correction terms are at most $O(\alpha^0)$. The reason is the expressions $(h_{T0} - h_T)$ and $(H_{T0} - H_T)$ each have an explicit factor of $\not{l}_0$, thereby suppressing the singular behavior at the $W - \omega_l + i\epsilon$ pole. So, for the correction parts we set α to 0 and calculate the integrals numerically.

Putting this together,

$$J_{\mathrm{ALO}} = -\frac{\pi}{12}\frac{1}{\alpha} + \frac{11}{24}\pi^2 \ln\left(\frac{1}{\alpha}\right) + J_{\mathrm{ALON}}, \tag{3.53}$$

where, as usual, J_{ALON} is the part of J_{ALO} that has to be done numerically.

The analysis of J_{AUV} proceeds rather smoothly once it is realized that it is $O(\alpha^0)$. Parameters are used throughout, which facilitates the isolation of the divergent part. We have

$$\begin{aligned} J_A &= J_{\mathrm{ALO}} + J_{\mathrm{AUV}} \\ &= \left(\frac{371}{432} - \frac{\pi^2}{12}\right)\frac{1}{\epsilon} - \frac{\pi}{12}\frac{1}{\alpha} + \frac{11}{24}\pi^2 \ln\left(\frac{1}{\alpha}\right) + J_{\mathrm{ALON}} + J_{\mathrm{UVN}}. \end{aligned} \tag{3.54}$$

2. J_B

From Eq. (3.46),

$$J_B = \frac{e^{2\epsilon\gamma_E}}{64}\int (d\not{l})_n'' (dp)_3' \frac{T_{rr}\{\gamma_0[\gamma(\not{l} - nW) + 1]\bar{\Lambda}_{BT}^m(\gamma(\not{l} + nW) + 1)\gamma_0\Phi(\vec{p})\}}{(\vec{\not{l}} - \vec{p})^2 D(W,\not{l}) D(-W,\not{l})} = J_{\mathrm{BLO}} + J_{\mathrm{BUV}}, \tag{3.55}$$

where a separation into finite and divergent parts is made. The finite part has a leading $\ln(\alpha)$ and we call it J_{BLO}. We have for this integral

$$\begin{aligned} J_{\mathrm{BLO}} &= \frac{1}{6}\int (d\not{l})_4'' \, dx\, du\, dz \frac{l^2}{D_l D(W,\not{l}) D(-W,\not{l})}\frac{x(1-x)}{\tilde{H}_T} \\ &\quad - \frac{1}{3}\int (d\not{l})_4'' \, dx\, du\, dz \frac{1}{D_l D(W,\not{l}) D(-W,\not{l})} \\ &\quad \times \frac{x(1-x)}{\tilde{H}_T}\left[\frac{1}{2}\not{l}_0^2 - a\not{l}_0 + \frac{1}{2}a\not{l}^2\not{l}_0 + \frac{1}{6}a\not{l}_0^3\right]. \end{aligned} \tag{3.56}$$

Because of the factors of l_0, the second part of Eq. (3.56) is leading order α^0, so we put α to 0 and integrate numerically. Setting this aside for the moment, let us turn our attention to the first integral of Eq. (3.56).

The analysis for this is similar that done for J_{ALO}. To extract the leading log, we write $1/\bar{H}_T$ in the form

$$\frac{1}{\bar{H}_T} = \frac{1}{\bar{H}_{T0}} + \frac{\bar{H}_{T0} - \bar{H}_T}{\bar{H}_{T0}\bar{H}_T} \tag{3.57}$$

and $\bar{T}_{T0} = \bar{H}_T|_{l_0 = 0}$. This separates J_{BLO} into two subintegrals. Since $\bar{H}_{T0} - \bar{H}_T = (1-x)x\not{l}_0^2 - 2(1-x)za\not{l}_0$, the integral associated with this factor is leading order α^0 (because of the $\not{l}_0$'s). We do this part numerically. The integral associated with $1/\bar{H}_{T0}$ is similar to others already done. The $\not{l}_0$ integration is completed first, yielding

$$\frac{1}{3}\int dl\, dx\, du\, dz \frac{l^4}{D_l^4}\frac{x(1-x)}{\omega_l \bar{H}_{T0}}. \tag{3.58}$$

Then, the log term is extracted by writing $1/\omega_l$ as $1 + (\omega_l - 1)/\omega_l$. The subintegral associated with $(\omega_l - 1)/\omega_l$ is leading $O(\alpha^0)$ and can be done numerically. The integral that has the one from the expression for $1/\omega_l$ above is

$$\frac{1}{3}\int dl\, dx\, du\, dz \frac{l^4}{D_l^4}\frac{x(1-x)}{\bar{H}_{T0}}, \tag{3.59}$$

and can be done analytically. This gives for J_{BLO},

$$J_{\mathrm{BLO}} = \frac{\pi^2}{24}\ln(1/\alpha) + \frac{\pi^2}{24}\ln(2) - \frac{\pi^2}{72} + J_{\mathrm{BLON}}, \tag{3.60}$$

where J_{BLON} is the sum of the various subintegrals done numerically.

The divergent part of J_B is

$$J_{BUV}=\frac{e^{2\epsilon\gamma_E}}{64}\Gamma(1-\epsilon)(1+\epsilon)\left[\frac{1-2\epsilon}{3-2\epsilon}-(1+2\epsilon)\right]\times\int (d\not{\ell})''_n\,dx\,du\,dz\,\frac{(1-x)x^{1-\epsilon}l^2T_{rr}[\gamma\not{\ell}\gamma^m\gamma\not{\ell}\gamma^\epsilon]}{D_lD(W,\not{\ell})D(-W,\not{\ell})\overline{H}_T^{1+\epsilon}}. \tag{3.61}$$

The trace is $4/(2\epsilon-3)(2\not{\ell}_0^2-2\epsilon l^2+l^2)$. From this point, parameters are used and we get

$$J_{BUV}=\frac{1}{72}\frac{1}{\epsilon^2}+\frac{53}{432}\frac{1}{\epsilon}+J_{BUVN}, \tag{3.62}$$

and for J_B,

$$J_B=J_{BLO}+J_{BUV}=\frac{1}{72}\frac{1}{\epsilon^2}+\frac{53}{432}\frac{1}{\epsilon}+\frac{\pi^2}{24}\ln(1/\alpha)+\frac{\pi^2}{24}\ln(2)-\frac{\pi^2}{72}+J_{BLON}+J_{BUVN}. \tag{3.63}$$

3. $\eta_T(\epsilon)$ term

This calculation proceeds in much the same manner as the $\eta_C(\epsilon)$ calculation of Sec. III A 5. After expanding in ϵ and α, $\eta_T(\epsilon)=-1/3-(17/9)\epsilon+(2\pi/9)\alpha\epsilon-(106/27)\epsilon^2\Gamma(\epsilon)$. Combining this with Eq. (3.34),

$$\frac{\alpha}{4\pi}\Omega^\epsilon\eta_T(\epsilon)A_C^m=-\frac{1}{3}L_1A_C^m+\left\{-\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}\frac{17}{108}\frac{1}{\epsilon}-\frac{17}{36}\frac{\alpha}{\pi}+\frac{1}{18}\alpha^2+\frac{115}{324}\left(\frac{\alpha}{\pi}\right)^2\right\}B^m. \tag{3.64}$$

4. A_{tc}^m results

After doing the numerical integrations, we get

$$A_{tc}^m=\left(-\frac{1}{3}L_1\right)A_C^m+B^m\left\{-\frac{5}{9}\left(\frac{\alpha}{\pi}\right)+\frac{\alpha^2}{2}\ln\left(\frac{1}{\alpha}\right)+\left(\frac{\alpha}{\pi}\right)^2[-4.302905(8)]+\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}\left[\frac{1}{72}\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{1}{2}\zeta(2)+\frac{89}{108}\right)\right]\right\}. \tag{3.65}$$

C. Coulomb-transverse ladder

For the Coulomb-transverse amplitude,

$$A_{ct}^m=B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}I_{ct} \tag{3.66}$$

with

$$I_{ct}=\frac{e^{-2\epsilon\gamma_E}}{64}\int (dq)''_n(d\not{\ell})''_n(dp)'_3\frac{T_{ct}}{(\not{\ell}-\vec{q})^2[-(\not{\ell}-p)^2]D(W,q)D(-W,q)D(W,\not{\ell})D(-W,\not{\ell})}\left(\frac{8\pi\gamma}{D_p^2}\right) \tag{3.67}$$

and

$$T_{ct}=T_{rr}[\gamma_i(\gamma(\not{\ell}-Wn)+1)\gamma_0(\gamma(q-Wn)+1)\gamma^m(\gamma(q+Wn)+1)\gamma_0(\gamma(\not{\ell}+Wn)+1)\gamma_j\Phi(\vec{p})]\delta_{ij}^T(\not{\ell}-\vec{p}). \tag{3.68}$$

This integral is of order $\ln(\alpha)$ and α^0, with the log term coming from the zero and one q terms of the trace. We call this part $K_{\rm LO}$, while the $\gamma q\gamma^m\gamma q$ part, of order α^0, is labeled $K_{\rm UV}$.

1. K_{LO}

For $K_{\rm LO}$ we can write

$$K_{\rm LO}=\frac{1}{64}\int (dq)''_4(d\not{\ell})''_4(dp)'_3\frac{T_{\rm LO}}{(\not{\ell}-\vec{q})^2(-(\not{\ell}-p)^2)D(W,q)D(-W,q)D(W,\not{\ell})D(-W,\not{\ell})}\left(\frac{8\pi\gamma}{D_p^2}\right). \tag{3.69}$$

The trace term is the full trace minus $\gamma q \gamma^m \gamma q$:

$$T_{\rm LO} = T_{rr}(\gamma_i[\gamma(\not\ell - Wn)+1]\gamma_0\{[\gamma(q-Wn)+1] \times \gamma^m[\gamma(q+Wn)+1] - \gamma q \gamma^m \gamma q\}\gamma_0 \times [\gamma(\not\ell + Wn)+1]\gamma_j \Phi(\vec{p}))\delta^T_{ij}(\vec{\ell}-\vec{p}).$$

The q integration is done next, using the formulas in Eq. (3.6) and then the trace is computed. We find the trace can be divided into factors having either four or six powers of momentum. Those with six powers are $O(\alpha)$, as can be seen by using the counting rules. This leaves us with

$$T_{\rm LO} = -\frac{128}{3}[l^4 + p^4 - 4l^2\vec{p}^2 - 4\vec{\ell}\cdot\vec{p}\vec{p}^2 - 4\vec{\ell}\cdot\vec{p}l^2 + 10(\vec{\ell}\cdot\vec{p})^2]. \tag{3.70}$$

To get this expression, we have made the nonrelativistic approximation for $\Phi(\vec{p})$ and averaged over spins. To compute Eq. (3.69), the ℓ_0 integration is completed by closing the contour in the upper half plane. So,

$$\int \frac{d\ell_0}{\pi i}\frac{1}{(\not\ell - p)^2 D(W,\ell) D(-W,\ell)} = -\frac{1}{2}\frac{2\omega_l + r}{r\omega_l D_l((\omega_l + r)^2 - W^2)}, \tag{3.71}$$

where we have defined $\vec{r} = \vec{\ell} - \vec{p}$ and $r = |\vec{r}|$. Using the scaling arguments, we see that if $\vec{\ell}$ and $\vec{p}$ are confined to the nonrelativistic region, then Eq. (3.71) goes as $-1/(2r^2 D_l)$, this being the most singular term in the ℓ_0 integral. In this region, all the terms in $T_{\rm LO}$ are the same order. Consequently, we find a power count of either $\ln(\alpha)$ or α^0 when l and p are simultaneously restricted to the nonrelativistic region. However, it is also true that if $p \sim \gamma$ while $l \sim 1$, the first term of $T_{\rm LO}$ is $O(\alpha^0)$, while the other terms have extra factors of α. These terms can be neglected when the integration variables are restricted to this region of momentum space.

These are the only regions that have contributions of the order of interest. Proceeding, we write

$$-\frac{1}{2}\frac{2\omega_l + r}{r\omega_l D_l((\omega_l + r)^2 - W^2)} = -\frac{1}{(2r)\omega_l D_l}\left[\frac{1}{r} + \left(\frac{2\omega_l + r}{(\omega_l + r)^2 - W^2} - \frac{1}{r}\right)\right] = -\frac{1}{2\omega_l r^2 D_l} + \frac{1}{2\omega_l r^2((\omega_l + r)^2 - W^2)}. \tag{3.72}$$

Associated with this form are two subintegrals, $K_{\rm LO1}$ and $K_{\rm LO2}$.

For $K_{\rm LO1}$ we have

$$K_{\rm LO1} = -\frac{1}{12\pi}\int d^3l (dp)'_3 \frac{dx}{\sqrt{x h_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\left(\frac{8\pi\gamma}{D_p^2}\right) \times \frac{[l^4 + p^4 - 4l^2\vec{p}^2 - 4\vec{\ell}\cdot\vec{p}\vec{p}^2 - 4\vec{\ell}\cdot\vec{p}l^2 + 10(\vec{\ell}\cdot\vec{p})^2]}{\omega_l D_l r^4}. \tag{3.73}$$

The p integral can be done, leaving us with

$$K_{\rm LO1} = \frac{4}{3}\int dl \frac{dx}{\sqrt{x h_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{l^2}{\omega_l D_l} - \gamma\int dl \frac{dx}{\sqrt{x h_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{l^2}{\omega_l D_l} \times\left[-2\tan^{-1}\left(\frac{l}{\gamma}\right) + \frac{l}{\gamma}\frac{\gamma^2 - l^2}{D_l}\right]. \tag{3.74}$$

The l^2 terms in both integrals have a $\ln(\alpha)$ part. Combining these terms and using $1/\omega_l = 1 + (1-\omega_l)/\omega_l$, $K_{\rm LO1}$ takes the form

$$K_{\rm LO1} = -\frac{1}{3}\int dl \frac{dx}{\sqrt{x h_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\frac{l^2}{D_l} - \frac{1}{3}\int dl\, dx\left(\frac{1-\omega_l}{\omega_l}\right)\frac{1}{\sqrt{x(1-x)}}\tan^{-1}\left(\frac{1}{l\sqrt{1-x}}\right) - \pi\int dl\, dx \frac{1}{x\sqrt{l^2(1-x)+1}}\frac{l}{l^2+1} \times\left(\frac{l}{l^2+1} - \tan^{-1}(l)\right), \tag{3.75}$$

where $l \to \gamma l$ in the last integral. The $(1-\omega_l)$ integral is done numerically, while the others can be done analytically. Calling the numerically integrated part $K_{\rm LO1N}$, we get the expression

$$K_{\rm LO1} = -\frac{\pi^2}{6}\ln\left(\frac{1}{\alpha}\right) - \frac{\pi^2}{6}\ln(2) - \pi^2\left(\frac{\pi^2}{12} - \frac{1}{4}\right) + K_{\rm LO1N}. \tag{3.76}$$

The relativistic part of $K_{\rm LO}, K_{\rm LO2}$, is

$$K_{\rm LO2}=\frac{1}{12\pi}\int d^3l(dp)'_3\frac{dx}{\sqrt{xh_c}}\tan^{-1}\left(\frac{1}{\sqrt{h_c}}\right)\left(\frac{8\pi\gamma}{D_p^2}\right)\times\frac{[l^4+p^4-4l^2\vec{p}^2-4\vec{\not{l}}\cdot\vec{p}\vec{p}^2-4\vec{\not{l}}\cdot\vec{p}l^2+10(\vec{\not{l}}\cdot\vec{p})^2]}{\omega_l r^4((\omega_l+r)^2-W^2)}$$

$$\to\frac{1}{6}\int dl\,dx\,\frac{1}{\sqrt{x(1-x)}}\,\frac{1}{\omega_l(l+\omega_l)}\tan^{-1}\left(\frac{1}{l\sqrt{1-x}}\right). \tag{3.77}$$

To get this, all terms in the numerator that depend on p are dropped and the approximation $r\to l$ is made. This integral is done numerically. So, delaying the numerical integrations until the rest of the amplitude is calculated, we have

$$K_{\rm LO}=-\frac{\pi^2}{6}\ln\left(\frac{1}{\alpha}\right)-\frac{\pi^2}{6}\ln(2)-\pi^2\left(\frac{\pi^2}{12}-\frac{1}{4}\right)+K_{\rm LO1}{}_N+K_{\rm LO2}.$$

2. K_{UV}

The part of I_{ct} with the ultraviolet divergence is

$$K_{\rm UV}=\frac{e^{2\epsilon\gamma_E}}{64}\int(dq)''_n(d\not{l})''_n(dp)'_3\frac{T_{\rm UV}}{(\vec{\not{l}}-\vec{q})^2[-(\not{l}-p)^2]D(W,q)D(-W,q)D(W,\not{l})D(-W,\not{l})} \tag{3.78}$$

with

$$T_{\rm UV}=T_{rr}[\gamma_i(\gamma(\not{l}-W_n)+1)\gamma_0\gamma q\gamma^m\gamma q\gamma_0(\gamma(\not{l}+Wn)+1)\gamma_j\Phi(\tilde{p})]\delta^T_{ij}(\vec{\not{l}}-\vec{p}). \tag{3.79}$$

We use Eq. (3.28) to do the q integration, and, as in the previous calculations, get two terms for the ultraviolet part of the amplitude

$$A^m_{ct\rm UV}=\frac{\alpha}{4\pi}\Omega^\epsilon\eta_C(\epsilon)A^m_T+B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}K'_{\rm UV}. \tag{3.80}$$

3. K_{UV}

The calculation of this expression is straightforward. After the q integration, we have

$$K_{\rm UV}=-\frac{e^{2\epsilon\gamma_E}}{64}\Gamma(1+\epsilon)\int(d\not{l})''_n\,dx\,du\,\frac{[x(1-x)]^{-\epsilon}}{(1-x)} \tag{3.81}$$

$$\frac{T_{rr}[\gamma_i(\gamma(\not{l}-W_n)+1)\gamma_0 NN^m_{C0}\gamma_0(\gamma(\not{l}+Wn)+1)\gamma_j(1+\gamma_0)\gamma\epsilon]\delta^T_{ij}(\vec{\not{l}})}{[l^2+a^2/(1-x)]^{1+\epsilon}(-\not{l}^2)(-\not{l}^2+2\not{l}_0)(-\not{l}^2-2\not{l}_0)}. \tag{3.82}$$

The divergent part of this integral can be done analytically and is (N_2 is the two $\not{l}$ contraction term and s, t, and y are the Feynman parameters used to do the $\not{l}$ integration)

$$-\frac{e^{2\epsilon\gamma_E}\Gamma(2\epsilon)}{256}\int ds\,dt\,dy\,dx\,du\,\frac{s^{3/2}t}{(1-x)^{1-\epsilon}}\,\frac{N_2}{(x(1-s)a^4)^\epsilon}=\frac{1}{72}\frac{1}{\epsilon^2}+\frac{29}{216}\frac{1}{\epsilon}+\frac{\pi^2}{144}+\frac{191}{324}, \tag{3.83}$$

while the remaining part of $K'_{\rm UV}$ is done numerically. We then get

$$K'_{\rm UV}=\frac{1}{72}\frac{1}{\epsilon^2}+\frac{29}{216}\frac{1}{\epsilon}+\frac{\pi^2}{144}+\frac{191}{324}+K_{\rm UV}{}_N. \tag{3.84}$$

4. $\eta_C(\epsilon)$ term

For this part of the calculation we need the transverse part of the one-rung ladder to order α. This is

$$A^m_T=-\frac{1-2\epsilon}{3-2\epsilon}L_1I_{\rm LO}B^m-\frac{2\alpha}{3\pi}\Omega^\epsilon B^m, \tag{3.85}$$

while $\eta_C(\epsilon)$ to sufficient accuracy is

$$\frac{4}{3}\Gamma(\epsilon)+\left[\frac{38}{9}+\frac{328}{27}\epsilon\right]\Gamma(1+\epsilon), \tag{3.86}$$

giving us

$$\frac{\alpha}{4\pi}\eta_C(\epsilon)\Omega^{\epsilon}A_T^m=\frac{4}{3}L_1A_T^m-\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^2\left(\frac{19}{216}\frac{1}{\epsilon}\right)-\frac{68}{81}\left(\frac{\alpha}{\pi}\right)^2. \tag{3.87}$$

5. A_{ct}^m results

For A_{ct}^m we finally get

$$A_{ct}^m=\left(\frac{4}{3}L_1\right)A_T^m+B^m\left\{-\frac{1}{6}\alpha^2\ln\left(\frac{1}{\alpha}\right)+\left(\frac{\alpha}{\pi}\right)^2[6.265\,238\,1(13)]+\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}\left[\frac{1}{72}\frac{1}{\epsilon^2}+\frac{5}{108}\frac{1}{\epsilon}\right]\right\}. \tag{3.88}$$

D. Transverse-transverse ladder

The double transverse graph is leading order $O(\alpha^0)$. There are no $\ln(\alpha)$ terms. The amplitude is given by

$$A_{tt}^m=B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}I_{tt}, \tag{3.89}$$

where

$$I_{tt}=\frac{e^{2\epsilon\gamma_E}}{64}\int(dq)_n''(d\not{\ell})_n''(dp)_3'\frac{T_{tt}}{(-(\not{\ell}-q)^2)(-(\not{\ell}-p)^2)D(W,q)D(-W,q)D(W,\not{\ell})D(-W,\not{\ell})}\left(\frac{8\pi\gamma}{D_p^2}\right) \tag{3.90}$$

and

$$T_{tt}=T_{rr}[\gamma_i(\gamma(\not{\ell}-n)+1)\gamma_s(\gamma(q-n)+1)\times\gamma^m(\gamma(q+n)+1)\gamma_t(\gamma(\not{\ell}+n)+1)\gamma_j\Phi(\tilde{p})]\times\delta_{ij}^T(\vec{\ell}-\vec{p})\,\delta_{st}^T(\vec{\ell}-\vec{p}). \tag{3.91}$$

With the absence of any log terms, it is reasonable to try to calculate the entire amplitude via parameters. However, this approach results in numerical instabilities, forcing us to isolate the numerically troublesome terms and calculate them separately. The ill-behaved parts are found to occur in the ultraviolet finite part of the amplitude. Hence, we separate the transverse-transverse amplitude into a UV finite and UV divergent part, $I_{TT}=L_{\rm fin}+L_{\rm UV}$.

1. L_{fin}

One way of getting a numerically stable integral is to use parameters for the inner loop integration and then do a Wick rotation on the remaining momentum variable (no poles are crossed). Also, we can set $\vec{p}$ to 0 in the fermion propagators, γ to 0 in the denominators and use $\int(dp)_3'(8\pi\gamma)/D_p^2=1$. This simplifies matters considerably and we are left with the following to calculate:

$$A_{\rm fin}^m=B^m\left(\frac{\alpha}{\pi}\right)^2L_{\rm fin}, \tag{3.92}$$

where

$$L_{\rm fin}=\frac{1}{64}\int(dq)_4''(d\not{\ell})_4'\frac{T_{\rm fin}}{(-\not{\ell}^2)[-(q-\not{\ell})^2](-\not{\ell}^2+2\not{\ell}_0)(-\not{\ell}^2-2\not{\ell}_0)(-q^2+2q_0)(-q^2-2q_0)}. \tag{3.93}$$

In this approximation, the trace becomes

$$T_{\rm fin}=T_{rr}[\gamma_i(\gamma(\not{\ell}-n)+1)\gamma_s\{(\gamma(q-n)+1)\times\gamma^m(\gamma(q+n)+1)-\gamma q\gamma^m\gamma q\}\gamma_t(\gamma(l+n)+1)\times\gamma_j(1+\gamma_0)\times(-\gamma\epsilon)]\delta_{ij}^T(\vec{\ell}-\vec{q})\,\delta_{st}^T(\vec{\ell}). \tag{3.94}$$

Using parameters for the q integral, we get

$$L_{\rm fin}=\frac{1}{64}\int(d\not{\ell})_4''\,dz\,dx\,du\,\frac{z^{3/2}x}{(-\not{\ell}^2)(-\not{\ell}^2+2\not{\ell}_0)(-\not{\ell}^2-2\not{\ell}_0)}\times\left[\frac{N_0}{\Delta^2}-\frac{1}{2}\frac{N_1}{\Delta}\right], \tag{3.95}$$

where $\Delta = zxH_T$ and H_T is as defined in the transverse-Coulomb ladder calculation. Interestingly enough, the trace of the term that is linear in q in the fermion line vanishes. This simplifies the contraction terms which become

$$N_0 = -\frac{64}{3}(zx)^2 \ell_0^2 l^2 \quad \text{and} \quad N_1 = \frac{64}{3}\ell_0^2.$$

Next, a $zx(1-x)$ is factored from the Δ's. Each term in the denominator has a 1 as the coefficient of its ℓ^2 term. The integration contour is then rotated counterclockwise to run along the imaginary ℓ_0 axis. A change of variable $\ell_0 \to i\ell_0$ is made, resulting in a numerically stable integral.

2. UV divergent terms

The UV divergent part of the amplitude is

$$A^m_{\rm UV} = -\frac{B^m}{64}\left(\frac{\alpha}{\pi}\right)^2 (\Omega e^{-\gamma_E})^{2\epsilon} \int (dq)''_n (d\ell)''_n \frac{T_{\rm UV}}{[-(\ell-q)]^2(-\ell^2)(-q^2+2q_0)(-q^2-2q_0)(-\ell^2+2\ell_0)(-\ell^2-2\ell_0)} \tag{3.96}$$

with the trace

$$T_{\rm UV} = T_{rr}[\gamma_i(\gamma(\ell-n)+1)\gamma_s \gamma q \gamma^m \gamma q \gamma_t(\gamma(\ell+n)+1) \times \gamma_j(1+\gamma_0)(-\gamma\epsilon)]\delta^T_{st}(\vec{\ell}-\vec{q})\,\delta^T_{ij}(\vec{\ell}). \tag{3.97}$$

The integral of the $\gamma q \gamma^m \gamma q$ term is

$$\tilde{\Lambda}^m_T(l-n,l+n) = \int (dq)''_n \frac{\gamma_s \gamma q \gamma^m q \gamma_t}{(-q^2+2q_0)(-q^2-2q_0)} \times \delta^T_{st}(\vec{\ell}-\vec{q}) \tag{3.98}$$

$$= \int dz\,dx\,du\,\frac{z^{3/2}x}{(zx)^\epsilon}\left[\Gamma(2+\epsilon)\frac{\tilde{N}_{T0}}{H_T^{2+\epsilon}} - \frac{1}{2}\Gamma(1+\epsilon)\frac{\tilde{N}_{T1}}{H_T^{1+\epsilon}} + \frac{1}{4}\Gamma(\epsilon)\frac{N_{T2}}{H_T^{\epsilon}}\right]. \tag{3.99}$$

$\tilde{N}_{T0}$ and $\tilde{N}_{T1}$ are the 0 and 1 contraction terms and are different than those calculated in the transverse-Coulomb section. However, the two contraction term, N_{T2}, and H_T, are as before.

The divergent part of this expression is made manifest by an integration by parts, so we get

$$\tilde{\Lambda}^m_T = \eta_T(\epsilon)\gamma^m + \tilde{\Lambda}^m_{TA} + \tilde{\Lambda}^m_{TB} \tag{3.100}$$

with

$$\tilde{\Lambda}^m_{TA} = \int dz\,dx\,du\,\frac{z^{3/2}x}{(zx)^\epsilon}\left[\Gamma(2+\epsilon)\frac{\tilde{N}N_{T0}}{H_T^{2+\epsilon}} - \frac{1}{2}\Gamma(1+\epsilon)\frac{\tilde{N}N_{T1}}{H_T^{1+\epsilon}}\right]. \tag{3.101}$$

and the following definitions:

$$\tilde{N}N_{T0} = \frac{\tilde{N}_{TO}}{(zx)^2},$$

$$\tilde{N}N_{T1} = \frac{\tilde{N}_{T1}}{zx} + 2(1-\epsilon)l^2 x\left[\frac{n-3}{n-1} - (5-n)z\right]\gamma^m.$$

$\eta_T(\epsilon)$ and $\bar{\Lambda}^m_{TB}$ are the same as in the transverse-Coulomb section.

Putting these in Eq. (3.89), we find

$$A^m_{\rm UV} = \frac{\alpha}{4\pi}\Omega^\epsilon \eta_T(\epsilon) A^m_T + \left(\frac{\alpha}{\pi}\right)^2 B^m (\Omega e^{-\gamma_E})^{2\epsilon} L'_{\rm UV} \tag{3.102}$$

with

$$L'_{\rm UV} = L_A + L_B = \frac{e^{2\epsilon\gamma_E}}{64}\int (d\ell)''_n \frac{T_{rr}[\gamma_s(\gamma(\ell-n)+1)(\tilde{\Lambda}^m_{TA}+\bar{\Lambda}^m_{BT})(\gamma(\ell+n)+1)\gamma_t(1+\gamma_0)\gamma\epsilon]}{(-\ell^2)(-\ell^2+2\ell_0)(-\ell^2-2\ell_0)}\,\delta^T_{st}(\vec{l}). \tag{3.103}$$

3. L_A and L_B calculation

To do this calculation, we introduce three Feynman parameters, s, t, and y. The UV divergences for each are separated and calculated analytically, while the remaining parts of L_A and L_B are numerically integrated. The result is

$$L_A = \left(-\frac{481}{864} + \frac{\pi^2}{18}\right)\frac{1}{\epsilon} + L_{AN} \tag{3.104}$$

and

$$L_B=-\frac{1}{288}\frac{1}{\epsilon^2}-\frac{65}{1728}\frac{1}{\epsilon}-\frac{5\pi^2}{1728}-\frac{23}{128}+L_{BN} \tag{3.105}$$

and

$$L'_{\rm UV}=L_A+L_B =-\frac{1}{288}\frac{1}{\epsilon^2}+\left(\frac{\pi^2}{18}-\frac{1027}{1728}\right)\frac{1}{\epsilon}-\frac{5\pi^2}{1728}-\frac{23}{128}+L_{AN} +L_{BN}. \tag{3.106}$$

4. $\eta_T(\epsilon)$ term

For this term we use the previously given expressions for $\eta_T(\epsilon)$ and A_T^m and get

$$\left(\frac{\alpha}{4\pi}\right)\Omega^\epsilon\eta_T(\epsilon)A_T^m=-\frac{1}{3}L_1A_T^m+\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}\left(\frac{17}{432}\frac{1}{\epsilon}\right)+\frac{223}{648}\left(\frac{\alpha}{\pi}\right)^2. \tag{3.107}$$

5. A_{tt}^m results

Adding the numerical integrations to the analytic parts,

$$A_{tt}^m=\left(-\frac{1}{3}L_1\right)A_T^m+B^m\left\{\left(\frac{\alpha}{\pi}\right)^2[0.385\,704\,0(21)] +\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}\left[-\frac{1}{288}\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{1}{3}\zeta(2)-\frac{959}{1728}\right)\right]\right\}. \tag{3.108}$$

E. Double-ladder results

The total double-ladder amplitude is gotten by adding Eqs. (3.36), (3.65), (3.88), and (3.108). This gives

$$A_{LL}^m=\left(\frac{\alpha}{4\pi}\right)\Omega^\epsilon\Gamma(\epsilon)A_1^m+B^m\left\{1-2\left(\frac{\alpha}{\pi}\right)+\frac{\alpha^2}{3}\ln\left(\frac{1}{\alpha}\right) +\left(\frac{\alpha}{\pi}\right)^2 5.226\,397(11)+\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon} \times\left[-\frac{1}{32}\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{1}{6}\zeta(2)+\frac{89}{192}\right)\right]\right\}, \tag{3.109}$$

where A_1^m is the one-loop amplitude.

The hardest part of the calculation is now done. The remaining amplitudes have leading order α^0. A brief synopsis of their computation follows.

IV. SELF-ENERGY AMPLITUDE

The amplitude for the self-energy correction to the vertex corrected lowest-order amplitude is

$$A_{\rm SE}^m=\frac{B^m}{2}\frac{\alpha}{4\pi}\Omega^\epsilon\int(d\not{\ell})''_n(dp)'_3\,T_{rr}[\gamma^\mu S_F(\not{\ell}-Wn+p)\gamma^m \times S_F(\not{\ell}+Wn+P)\Sigma_C(\not{\ell}+Wn+p)S_F(\not{\ell}+Wn+p) \times\gamma^\nu\Phi(\vec{p})]\left(\frac{8\pi\gamma}{D_p^2}\right)D_{\mu\nu}^C(\not{\ell}). \tag{4.1}$$

Σ_C is the mass renormalized self-energy operator in Coulomb gauge. This must be put in a form where the renormalization constant is explicit. There are several equivalent expressions. One that is particularly convenient is to write Σ_C as the Yennie gauge self-energy operator plus a gauge correction term. One advantage of this form is that all terms are separately infrared finite. Other forms for Σ_C have individual terms that are infrared divergent. These cancel upon summation, but it makes these forms more difficult to work with. Specifically, we have

$$\Sigma_C(\not{\ell}+Wn+p)=(B_1-B_{1Y})(\gamma(\not{\ell}+Wn+p)-1) +\Sigma_Y(\not{\ell}+Wn+p)+\Sigma_G(\not{\ell}+Wn+p), \tag{4.2}$$

where $B_1=-L_1$ is the Coulomb gauge wave function renormalization constant, B_{1Y} is the corresponding Yennie gauge quantity, and Σ_G, the gauge correction piece, is the sum of five terms. This gives for $A_{\rm SE}^m$,

$$A_{\rm SE}^m=2B_1A_1^m+B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}I_{\rm SE} \tag{4.3}$$

with

$$I_{\rm SE}=\frac{e^{2\epsilon\gamma_E}}{32}\int(d\not{\ell})'_n\frac{T_{\rm SE}}{(-\not{\ell}^2-2\not{\ell}N)^2(-\not{\ell}^2+2\not{\ell}n)} \tag{4.4}$$

and

$$T_{\rm SE}=T_{rr}[\gamma^\mu(\gamma(\not{\ell}-n)+1)\gamma^m(\gamma(n+\not{\ell})+1)(\bar{\Sigma}_Y(n+\not{\ell}) +\bar{\Sigma}_G(n+\not{\ell}))(\gamma(n+\not{\ell})+1)\gamma^\nu(1+\gamma_0)\gamma_\epsilon]. \tag{4.5}$$

To get this, the definitions $\Sigma_i=\alpha/(4\pi)\Omega^\epsilon\bar{\Sigma}_i$ are used. Then, since $I_{\rm SE}$ is leading order α^0, $\vec{p}$ is set to 0 in the fermion lines, the substitutions $W\to1,\Phi(\vec{p})\to-(1+\gamma_0)\gamma\epsilon$ are made, and $\int(dp)'_3(8\pi\gamma)/D_p^2=1$ is used to do the remaining integration over the relative momentum.

In this nonrelativistic limit, the expressions for the self-energy terms become

$$\overline{\Sigma}_G(n+\not{\ell}) = \overline{\Sigma}_1(n+\not{\ell}) + \overline{\Sigma}_2(n+\not{\ell}) + \overline{\Sigma}_3(n+\not{\ell}) + \overline{\Sigma}_4(n+\not{\ell}) + \overline{\Sigma}_5(n+\not{\ell}),$$

$$\overline{\Sigma}_1(n+\not{\ell}) = \frac{1}{2}\Gamma(1+\epsilon)\int dx\, dz\, z^{-1/2-\epsilon}x^{-\epsilon} \times \frac{[(\gamma(n+\not{\ell})-1)N_R + N_L(\gamma(n+\not{\ell})-1)]}{H^{1+\epsilon}},$$

$$\overline{\Sigma}_2(n+\not{\ell}) = l^2(\gamma(n+\not{\ell})-1)\Gamma(2+\epsilon) \times \int dx\, dz\, dt\, x^{1-\epsilon}z^{-1/2-\epsilon}\frac{(1-z)(2-x)}{\overline{H}^{2+\epsilon}},$$

$$\overline{\Sigma}_3(n+l) = \frac{l^2}{2}(\gamma(n+\not{\ell})-1)\Gamma(1+\epsilon) \times \int dx\, dz\, dt\, x^{1-\epsilon}z^{-1/2-\epsilon} \times \frac{(1-z)[1-(3-2\epsilon)z]}{\overline{H}^{1+\epsilon}},$$

$$\overline{\Sigma}_4(n+\not{\ell}) = -(\gamma(n+\not{\ell})-1)(\not{\ell}^2+2\not{\ell}_0)\frac{\beta^2}{2}\Gamma(2+\epsilon) \times \int dx\, x^{1-\epsilon}\frac{1}{G^{2+\epsilon}},$$

$$\overline{\Sigma}_5(n+\not{\ell}) = (\gamma(n+\not{\ell})-1)^2\beta\Gamma(1+\epsilon) \times \int dx\, x^{1-\epsilon}\frac{1}{G^{1+\epsilon}}\left[(\gamma(n+\not{\ell})+2)(1-x) + (1-\epsilon)\frac{\beta}{2}(\gamma(n+\not{\ell})+1)\right],$$

and

$$\overline{\Sigma}_Y(n+\not{\ell}) = -(\gamma(n+\not{\ell})-1)^2(\beta+1)\Gamma(1+\epsilon) \times \int dx\, \frac{x^{1-\epsilon}}{(1-x)^{1+\epsilon}}\frac{\gamma(r+\not{\ell})}{[-\not{\ell}^2-2\not{\ell}n+x/(1-x)]^{1+\epsilon}}, \tag{4.6}$$

where β, the gauge parameter that defines the Yennie gauge [27], equals $2/(1-2\epsilon)$ and $q=(1+\not{\ell}_0, z\vec{\not{\ell}})$. The denominators are $G=x+(1-x)(-\not{\ell}^2-2n\not{\ell})$, $H=G+x(1-z)l^2$, $\overline{H}=G+x(1-z)tl^2$, while the spinor factors in $\overline{\Sigma}_1$ are

$$N_R=(\gamma[(n+\not{\ell})-xq]+1)[\gamma(n+\not{\ell})-2q_0\gamma_0]+(2-x) \tag{4.7}$$

and

$$N_L=[\gamma(n+\not{\ell})-2q_0\gamma_0]\{\gamma[(n+\not{\ell})-xq]+1\}+(2-x). \tag{4.8}$$

Another advantage of expressing $\overline{\Sigma}_C$ in this manner is that, with the exception of $\overline{\Sigma}_1$, there are canceling terms in the numerators and denominators.

Putting the various pieces together, we can write

$$I_{\rm SE} = I_Y + \sum_{i=1}^{5} I_i \tag{4.9}$$

with the following expressions for the I_i's:

$$I_Y = -\frac{(1+\beta)\Gamma(1+\epsilon)}{32}e^{2\epsilon\gamma_E}\int (d\not{\ell})_n'' dx \frac{x^{1-\epsilon}}{(1-x)^{1+\epsilon}}\frac{T_{rr}\{\gamma_\mu[\gamma(\not{\ell}-n)+1]\gamma^m\gamma(\not{\ell}+n)\gamma_\nu(1+\gamma_0)\gamma\epsilon\}}{[-\not{\ell}^2-2\not{\ell}n+x/(1-x)]^{1+\epsilon}(-\not{\ell}^2+2\not{\ell}n)}D^C_{\mu\nu}(\not{\ell}), \tag{4.10}$$

$$I_1 = -\frac{\Gamma(1+\epsilon)}{64}e^{2\epsilon\gamma_E}\int (d\not{\ell})_n'' dx\, dz\, z^{-1/2-\epsilon}x^{-\epsilon} \times \frac{T_{rr}[\gamma_\mu(\gamma(\not{\ell}-n)+1)\gamma^m[(\gamma(\not{\ell}+n)-1)N_R + N_L(\gamma(\not{\ell}-n))]\gamma_\nu(1+\gamma_0)\gamma\epsilon]}{H^{1+\epsilon}(-\not{\ell}^2-2\not{\ell}n)^2(-\not{\ell}^2+2\not{\ell}n)}D^C_{\mu\nu}(\not{\ell}), \tag{4.11}$$

$$I_2 = -\frac{\Gamma(2+\epsilon)}{32}e^{2\epsilon\gamma_E}\int (d\not{\ell})_n'' dx\, dz\, z^{-1/2-\epsilon}x^{1-\epsilon}(1-z)(2-x)l^2 \times \frac{T_{rr}[\gamma_\mu(\gamma(\not{\ell}-n)+1)\gamma^m(\gamma(\not{\ell}+n)+1)\gamma_\nu(1+\gamma_0)\gamma\epsilon]}{\overline{H}^{2+\epsilon}(-\not{\ell}^2-2\not{\ell}n)(-\not{\ell}^2+2\not{\ell}n)}D^C_{\mu\nu}(\not{\ell}), \tag{4.12}$$

$$I_3=-\frac{\Gamma(1+\epsilon)}{64}e^{2\epsilon\gamma_E}\int (d\not{\ell})''_n dx\,dz\,dt\,z^{-1/2-\epsilon}x^{1-\epsilon}(1-z)[1-(3-2\epsilon)z]l^2$$
$$\times\frac{T_{rr}[\gamma_\mu(\gamma(\not{\ell}-n)+1)\gamma^m(\gamma(\not{\ell}+n)+1)\gamma_\nu(1+\gamma_0)\gamma\epsilon]}{\bar{H}^{1+\epsilon}(-\not{\ell}^2-2\not{\ell}n)(-l^2+2\not{\ell}n)}D^C_{\mu\nu}(\not{\ell}), \tag{4.13}$$

$$I_4=-\frac{\Gamma(2+\epsilon)\beta^2}{64}e^{2\epsilon\gamma_E}\int (d\not{\ell})''_n dx\,x^{1-\epsilon}\frac{T_{rr}[\gamma_\mu(\gamma(\not{\ell}-n)+1)\gamma^m(\gamma(\not{\ell}+n)+1)\gamma_\nu(1+\gamma_0)\gamma\epsilon]}{G^{2+\epsilon}(-\not{\ell}^2+2ln)}D^C_{\mu\nu}(\not{\ell}), \tag{4.14}$$

and

$$I_5=-\frac{\Gamma(1+\epsilon)\beta}{32}e^{2\epsilon\gamma_E}\int (d\not{\ell})''_n dx\,x^{-\epsilon}T_{rr}\Bigg[\gamma_\mu(\gamma(\not{\ell}-n)+1)\gamma^m$$
$$\times\frac{(\{[\gamma(\not{\ell}+n)+2](1-x)+(1-\epsilon)\beta/2\}\gamma(\not{\ell}+n)+1)\gamma_\nu(1+\gamma_0)\gamma\epsilon}{G^{1+\epsilon}(-\not{\ell}^2+2\not{\ell}n)}\Bigg]D^C_{\mu\nu}(\not{\ell}). \tag{4.15}$$

Each I_i is evaluated using parameters, with the divergent terms, if any, separated and computed analytically. The remaining finite parts are done numerically, giving us

$$A^m_{\rm SE}=2B_1A^m_1-B^m\left(\frac{\alpha}{\pi}\right)^2[5.683\,940\,3(11)]$$
$$+B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}\left[\frac{1}{16}\frac{1}{\epsilon^2}+\left(\frac{5}{6}\zeta(2)-\frac{139}{96}\right)\frac{1}{\epsilon}\right]. \tag{4.16}$$

V. SIDE VERTEX

The amplitude for the vertex correction to the annihilation vertex is

$$A^m_{\rm SV}=\frac{B^m}{2}\frac{\alpha}{4\pi}\Omega^\epsilon\int (d\not{\ell})''_n(dp)'_3T_{rr}[\gamma^\alpha S_F(\not{\ell}-Wn+p)$$
$$\times\gamma^m S_F(\not{\ell}+Wn+p)\Lambda^\beta_{(1)}(\not{\ell}+Wn+,Wn+p)\Phi(\vec{p})]$$
$$\times\left(\frac{8\pi\gamma}{D_p^2}\right)D^{\alpha\beta}_C(\not{\ell}). \tag{5.1}$$

Λ^β_1 is the one-loop vertex operator in Coulomb gauge. When writing this operator in a form suitable for calculation, it is best to write a separate expression for the Coulomb and transverse parts. This results in the following lengthy expression for the vertex operator:

$$\Lambda^\beta_1(p'',p')=L_1\gamma^\beta+\left(\frac{\alpha}{4\pi}\right)\Omega^\epsilon\bar{\Lambda}^\beta_S(p'',p'), \tag{5.2}$$

where

$$\bar{\Lambda}^\beta_S(p'',p')=\bar{\Lambda}^\beta_{CS}(p'',p')+\bar{\Lambda}^\beta_{TS}(p'',p'), \tag{5.3}$$

and $L_1=-B_1$. The subtracted Coulomb part of the vertex correction function is

$$\bar{\Lambda}^\beta_{CS}(p'',p')=-\int dx\,du\Bigg\{x^{-1/2-\epsilon}$$
$$\times\left[A^\beta_{C2}\frac{\Gamma(2+\epsilon)}{H_C^{2+\epsilon}}+A^\beta_{C1}\frac{\Gamma(1+\epsilon)}{H_C^{1+\epsilon}}\right]$$
$$+\left[B^\beta_{C2}\frac{\Gamma(2+\epsilon)}{D_C^{2+\epsilon}}+B^\beta_{C1}\frac{\Gamma(1+\epsilon)}{D_C^{1+\epsilon}}\right]\Bigg\}, \tag{5.4}$$

where

$$H_C=1-uv[k_0^2-x\vec{k}^2]+(1-x)[u\vec{p}''^2+vp'^2], \tag{5.5a}$$

$$D_C=1-xuvk^2, \tag{5.5b}$$

and

$$A^\beta_{C2}=\frac{2}{n-3}x\gamma^\beta H_{C1}, \tag{5.6a}$$

$$A^\beta_{C1}=(N^\beta_{C0}-\gamma^\beta)+x\Bigg\{\left[\frac{-1}{n-3}+\frac{x}{n-1}\right]\gamma^\beta$$
$$+x\frac{n-2}{n-1}\gamma^0\gamma^\beta\gamma^0\Bigg\}H_{C1}, \tag{5.6b}$$

$$B^\beta_{C2}=\frac{2}{n-3}\gamma^\beta uvk^2, \tag{5.6c}$$

with

$$B^\beta_{C1}=\left\{\left[\frac{-1}{n-3}+\frac{1}{n-1}\right]\gamma^\beta+\frac{n-2}{n-1}\gamma^0\gamma^\beta\gamma^0\right\}uvk^2, \tag{5.6d}$$

$$N^{\beta}_{C0}=\gamma^0(\gamma(p''-Q_C)+1)\gamma^{\beta}[\gamma(p'-Q_C)+1]\gamma^0, \tag{5.7}$$

and the definitions $k=p''-p'$, $v=1-u$, $Q_C=(q^0,x\vec{q})$, $q=up''+vp'$, and $H_{C1}=(\partial/\partial x)H_C$. The subtracted transverse part of the vertex correction function is

$$\overline{\Lambda}^{\beta}_{TS}(p'',p')=\int dz\,dx\,du\Bigg\{z^{1/2-\epsilon}x^{-\epsilon}\times\left[A^{\beta}_{T2}\frac{\Gamma(3+\epsilon)}{H_T^{2+\epsilon}}+A^{\beta}_{T1}\frac{\Gamma(1+\epsilon)}{H_T^{1+\epsilon}}\right]+x^{-\epsilon}\left[B^{\beta}_{T2}\frac{\Gamma(2+\epsilon)}{D_T^{2+\epsilon}}+B^{\beta}_{T1}\frac{\Gamma(1+\epsilon)}{D_T^{1+\epsilon}}\right]\Bigg\}, \tag{5.8}$$

where

$$H_T=x-xuv[k_0^2-z\vec{k}^2]+(1-x)[u(1-p_0''^2)+v(1-p_0'^2)]+(1-zx)[u\vec{p}''^2+v\vec{p}'^2], \tag{5.9a}$$

$$D_T=x-zxuvk^2+z(1-x)[u(1-p''^2)+v(1-p'^2)], \tag{5.9b}$$

and

$$A^{\beta}_{T2}=zxN^{\beta}_{T0}-\frac{z}{n-1}\overline{N}^{\beta}_{T1}H_{T1}, \tag{5.10a}$$

$$A^{\beta}_{T1}=\frac{-1}{2}(N^{\beta}_{T1}-\overline{N}^{\beta}_{T1})+\frac{n-2}{2}zx\left\{\left[\frac{-1}{n-1}+(n-3)z\right]\gamma^{\beta}+\left[-\frac{n-2}{n-1}+2z\right]\gamma^0\gamma^{\beta}\gamma^0\right\}H_{T1}, \tag{5.10b}$$

$$B^{\beta}_{T2}=\frac{1}{n-1}\overline{N}^{\beta}_{T1}D_{T1}, \tag{5.10c}$$

$$B^{\beta}_{T1}=-\frac{n-2}{n-1}x\{(1-4\epsilon+2\epsilon^2)\gamma^{\beta}+(2-\epsilon)\gamma^0\gamma^{\beta}\gamma^0\}D_{T1}, \tag{5.10d}$$

with

$$N^{\beta}_{T0}=\gamma^i(\gamma(p''-Q_T)+1)\gamma^{\beta}(\gamma(p'-Q_T)+1)\times\gamma^j(\vec{q}^2\delta_{ij}-q_iq_j), \tag{5.11a}$$

$$N^{\beta}_{T1}=\gamma^i(\gamma(p''-l)+1)\gamma^{\beta}(\gamma(p'-l)+1)\times\gamma^j(\not{\ell}^2\delta_{ij}-\not{\ell}_i\not{\ell}_j)|_{1-\text{contraction}}, \tag{5.11b}$$

$$\overline{N}^{\beta}_{T1}=-(n-2)x^2[n\gamma^{\beta}-\gamma^0\gamma^{\beta}\gamma^0], \tag{5.11c}$$

and the definitions $Q_T=x(q^0,z\vec{q})$, $H_{T1}=(\partial/\partial z)H_T$, and $D_{T1}=(\partial/\partial z)D_T$. The contraction in (5.11b) is to be done over $\not{\ell}$ using the "metric" diag$(1/z,-\delta_{ij})$ and the replacement $\not{\ell}\to Q_T$. Both $\overline{\Lambda}^{\beta}_{CS}(p'',p')$ and $\overline{\Lambda}^{\beta}_{TS}(p'',p')$ vanish in the "particle at rest" limit, that is when p'', $p'\to n$ and $\gamma n\to 1$ on the left and right. For the present calculation, $p''=\not{\ell}+n$, $p'=n$, $k=\not{\ell}$ and $q=n+u\not{\ell}$.

Using Eq. (5.3), we can write Eq. (5.1) as

$$A^m_{\rm SV}=2L_{(1)}A^m_1+B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_e})^{2\epsilon}I_{\rm SV}. \tag{5.12}$$

The subtracted part of this amplitude is $O(\alpha^0)$, so we can make the appropriate nonrelativistic approximations, simplifying the expression for $I_{\rm SV}$ to

$$I_{\rm SV}=-\frac{e^{2\epsilon\gamma_E}}{32}\int(d\not{\ell})''_n\,T_{rr}[\gamma^{\alpha}S_F(\not{\ell}-n)\gamma^mS_F(\not{\ell}+n)\times\overline{\Lambda}^{\beta}_s(\not{\ell}+n,n)(1+\gamma_0)\gamma\epsilon]D^C_{\alpha\beta}(\not{\ell}), \tag{5.13}$$

which, from the equation for $\overline{\Lambda}^{\beta}_S$, can be written as the sum of 16 terms.

Each of these integrals is $O(\alpha^0)$, so Feynman parameters are used. However some of the resulting parameter integrals are numerically unstable. Fortunately for these, only small adjustments are needed to get numerical stability.

For instance, unstable behavior occurs in the calculation of

$$I_{CCA1}=\frac{\Gamma(1+\epsilon)}{32}e^{2\epsilon\gamma_E}\int(d\not{\ell})''_n dx\,du\,x^{-1/2-\epsilon}\frac{T_{rr}[\gamma_0(\gamma(\not{\ell}-n)+1)\gamma^m(\gamma(\not{\ell}+n)+1)A^0_{C1}(1+\gamma_0)\gamma\epsilon]}{\vec{\ell}^2(-\not{\ell}^2-2\not{\ell}_0)(-\not{\ell}^2+2\not{\ell}_0)H_C^{1+\epsilon}}. \tag{5.14}$$

(The first subscript refers to the spanning photon and the second subscript refers to the exchange photon. So, I_{CCA1} means the A^{β}_{C1} part of Λ^{β}_S and a Coulomb exchange photon.) The terms which cause the trouble are not hard to locate. They are the ones in the trace that are linear in l_0. But, these are odd functions of $\not{\ell}_0$ and integrate to 0. Upon eliminating these from the trace, the resulting numerical integration converges.

Another example of a numerically unstable integral is

$$I_{TCA1}=-\frac{\Gamma(1+\epsilon)}{32}e^{2\epsilon\gamma_E}\int(d\not{\ell})''_n dx\,du\,dz\,z^{1/2-\epsilon}x^{-\epsilon}\frac{T_{rr}[\gamma_0(\gamma(\not{\ell}-n)+1)\gamma^m(\gamma(\not{\ell}+n)+1)A^0_{T1}(1+\gamma_0)\gamma\epsilon]}{\vec{\ell}^2(-\not{\ell}^2-2\not{\ell}_0)(-\not{\ell}^2+2\not{\ell}_0)H_T^{1+\epsilon}}. \tag{5.15}$$

Here, again, the troublesome terms are the ones linear in $\not{\ell}_0$. However, these can not be neglected since these expressions are not odd functions of $\not{\ell}_0$. A strategy that works is to isolate the $\not{\ell}_0$ part of the integrand and use

$$\frac{1}{(-\not{\ell}^2-2\not{\ell}_0)}\frac{1}{(-\not{\ell}^2+2\not{\ell}_0)} = \frac{1}{4\not{\ell}_0}\left(\frac{1}{(-\not{\ell}^2-2\not{\ell}_0)}-\frac{1}{(-\not{\ell}^2+2\not{\ell}_0)}\right)$$

and cancel the l_0 in the numerator. This procedure yields a numerically stable integral.

For other similar integrals, relations such as $-\not{\ell}^2=(-\not{\ell}^2-2\not{\ell}_0-\not{\ell}^2+2\not{\ell}_0)/2$ can be used to eliminate denominators and stabilize the integrands.

Adding the results of all the integrations, we find

$$A^m_{\rm SV}=2L_1A^m_1+B^m\left(\frac{\alpha}{\pi}\right)^2\left\{(\Omega e^{-\gamma_E})^{2\epsilon}\left(-\frac{1}{16}\frac{1}{\epsilon^2}+\left[-\frac{1}{12}\zeta(2)+\frac{5}{24}\right]\frac{1}{\epsilon}\right)-2.093\,273\,3(12)\right\}. \tag{5.16}$$

VI. CROSSED LADDER

The crossed ladder diagram is leading order α^0, so we make the usual nonrelativistic simplifications and get an α independent integral whose finite part is computed numerically. The amplitude is

$$A^m_{\rm CL}=B^m\left(\frac{\alpha}{\pi}\right)^2(\Omega e^{-\gamma_E})^{2\epsilon}I_{\rm CL} \tag{6.1}$$

$$I_{\rm CL}=\frac{e^{2\epsilon\gamma_E}}{64}\int(d\not{\ell})''_n(dq)''_nT_{rr}[\gamma_\alpha S_F(q-n)\gamma_\mu S_F(q-\not{\ell}-n)\times\gamma^mS_F(q-\not{\ell}+n)\gamma_\beta S_F(n-\not{\ell})\gamma_\nu(1+\gamma_0)\gamma\epsilon]\times D_C^{\alpha\beta}(q)D_C^{\mu\nu}(\not{\ell}). \tag{6.2}$$

The calculation proceeds as follows. We write the photon propagators as sums of the Coulomb part and the transverse part. This partitions $I_{\rm SV}$ in a natural way into four parts, according to whether both photons are Coulomb, one Coulomb and the other transverse, etc. The Coulomb-transverse and the transverse-Coulomb should be equal so their separate calculation provides a partial check on our results. A straightforward application of Feynman parameters suffices, with one exception, to extract the UV divergent parts of each integral and to yield a finite integral that behaves nicely when computed numerically.

The exception to the above occurs when computing the Coulomb-Coulomb part. After separating the UV divergent parts, the remaining finite part is numerically unstable. The unstable part of the remaining finite part can be identified as the following integral:

$$I=\frac{1}{4}\int(d\not{\ell})''_4(dq)''_4\frac{(2-q_0)(2-\not{\ell}_0)}{\vec{q}^2l^2[(\not{\ell}_0-1)^2-\omega_l^2][(q_0-1)^2-\omega_q^2][(\not{\ell}_0-q_0+1)^2-\omega'^2][(\not{\ell}_0-q_0-1)^2-\omega'^2]}. \tag{6.3}$$

The ω's are the relativistic energies:

$$\omega_q=\sqrt{\vec{q}^2+1},\quad \omega'=\sqrt{(\vec{\not{\ell}}-\vec{q})^2+1} \tag{6.4}$$

and ω_l is as before.

For the numerical evaluation, we first integrate over $\not{\ell}_0$ and q_0 which yields the following surprisingly simple looking expression

$$I=-\int dl\,dq\,du\,\frac{(\omega_q\omega_l-1)(2\omega'+\omega_i+\omega_q)}{\omega_q\omega_l\omega'(\omega'^2-1)[(\omega'+\omega_l+\omega_q)^2-1]}, \tag{6.5}$$

where u is the cosine of the angle between $\vec{\not{\ell}}$ and $\vec{q}$ and l and q are the magnitudes of the momentum vectors. This integral can be integrated successfully and gives $-1.233\,703\,2(21)$.

This diagram has no inner loop divergence, so its divergent part is proportional to $1/\epsilon$. After separating this, the final value of the crossed ladder amplitude is

$$A^m_{\rm CL}=B^m\left(\frac{\alpha}{\pi}\right)^2\left\{(\Omega e^{-\gamma_E})^{2\epsilon}\left(\left[-\frac{5}{24}\zeta(2)+\frac{5}{48}\right]\frac{1}{\epsilon}\right)-1.796\,103(19)\right\}. \tag{6.6}$$

VII. VACUUM POLARIZATION

The amplitude originating from the vacuum polarization correction to the exchanged photon has been previously done [28–30]. It is the simplest to calculate and can be done analytically. For completeness, we will outline our calculation of this amplitude.

The unrenormalized Coulomb gauge photon propagator is

$$D'_{\mu\nu}(l)=\frac{1}{1+\Pi(\not{\ell})}D^C_{\mu\nu}(\not{\ell})=Z_3\frac{1}{1+\Pi_R(\not{\ell})}D^C_{\mu\nu}(\not{\ell}) \tag{7.1}$$

with $D^C_{\mu\nu}(\not{\ell})$ given in Eq. (1.6). Z_3 is the charge renormalization constant, $1/[1+\Pi(0)]$ and $\Pi_R(\not{\ell}^2)$ the renormalized

scalar vacuum polarization function, of which the one loop approximation is needed. A useful form for this is

$$\Pi_{(1)R}(\not{\ell}^2) = -\frac{\alpha}{3\pi}\Omega^{\epsilon}\Gamma(1+\epsilon)\not{\ell}^2 \times \int dz \frac{z^2(3-2z)(1-2z)}{(1-z(1-z)\not{\ell}^2)^{1+\epsilon}}. \quad (7.2)$$

Then, the vacuum polarization amplitude takes the form

$$A^m_{\rm VP} = B^m\left(\frac{\alpha}{\pi}\right)^2 (\Omega e^{-\gamma_E})^{2\epsilon} I_{\rm VP} \quad (7.3)$$

with

$$I_{\rm VP} = -\frac{\Gamma(1+\epsilon)}{48} e^{2\epsilon\gamma_E} \int (d\not{\ell})''_n\, dz \times \frac{z^2(3-2z)(1-2z)}{(1-z(1-z)\not{\ell}^2)^{1+\epsilon}} \frac{1}{D(W,\not{\ell})D(-W,\not{\ell})} \times T_{rr}[\gamma^{\mu}(\gamma(\not{\ell}\text{-}n)+1)\gamma^m(\gamma(\not{\ell}+n)+1)\gamma^{\nu} \times (1+\gamma_0)\gamma\epsilon][\not{\ell}^2 D^C_{\mu\nu}(\not{\ell})]. \quad (7.4)$$

The trace times $\not{\ell}^2 D^C_{\mu\nu}(\not{\ell})$ is

$$\frac{4}{l^2}\not{\ell}^2(\not{\ell}^2-4) + 8\epsilon\not{\ell}^2 - \frac{16(\epsilon-1)}{2\epsilon-3}\not{\ell}^2. \quad (7.5)$$

Symmetry arguments can be used to reduce the number of denominators so that only two Feynman parameters are needed. The integrals are relatively standard and we get

$$A^m_{\rm VP} = B^m\left(\frac{\alpha}{\pi}\right)^2 (\Omega e^{-\gamma^E})^{2\epsilon}\left\{\frac{1}{24}\frac{1}{\epsilon^2} - \frac{1}{144}\frac{1}{\epsilon} + \frac{\pi^2}{16} - \frac{1}{864}\right\}. \quad (7.6)$$

VIII. TOTAL TWO-LOOP VERTEX CORRECTION TO THE ONE ANNIHILATION PHOTON AMPLITUDE

We have completed all the necessary calculations to compute the energy shift of the two-loop vertex corrected one-photon annihilation amplitude. This shift is twice the difference of the shifts associated with Figs. 1(c) and 1(d), to which we must also add the shift due to Fig. 1(a).

It appears that we need to know the unrenormalized one-loop vertex correction to order α^6. However, this is not so since the α^6 part of the energy contribution of Fig. 1(a) minus twice the part of Fig. 1(d) that comes from the relative order α^2 part of the unrenormalized one-loop vertex correction A^m_1 cancel. Then, using

$$L_1 = \left(\frac{\alpha}{4\pi}\right)\Omega^{\epsilon}\Gamma(\epsilon),$$

$$L_2 = \left(\frac{\alpha}{\pi}\right)^2 [\Omega e^{-\gamma_E}]^{2\epsilon}\left\{\frac{7}{96}\frac{1}{\epsilon^2} + \left[\frac{3}{8}\zeta(2) - \frac{391}{576}\right]\frac{1}{\epsilon} + 1.258\,385(5) + O(\epsilon)\right\} \quad [21],$$

the formulas given earlier for the renormalized one and two-loop vertex amplitudes, $A^m_{1R} = A^m_1 - L_1 A^m_0$ and $A^m_{2R} = A^m_2 - L_1 A^m_1 - L_2 A^m_0 + L_1^2 A^m_0$ (see Appendix B) and taking into account the above mentioned cancellation, we get upon summing Eqs. (3.109), (4.16), (5.16), (6.6), and (7.6), a contribution to the energy shift of the hyperfine interval of

$$\Delta E_{1a+2(1c-1d)} = m\alpha^4\left\{\frac{1}{4} - \frac{\alpha}{\pi} + \frac{\alpha^2}{6}\ln\left(\frac{1}{\alpha}\right) - \alpha^2[0.083\,747\,2(12)]\right\}. \quad (8.1)$$

IX. PREVIOUSLY CALCULATED AMPLITUDES

The remaining amplitudes shown in Fig. 1 have been previously calculated and their results are compiled here for easy reference.

The one-loop vacuum polarization contribution, pictured in Fig. 1(b), is the renormalized scalar vacuum polarization function $\Pi_R(4W^2)$ times the lowest-order amplitude. This is

$$\Delta E_{1a} = \frac{m\alpha^4}{4}\left\{-\frac{8}{9}\left(\frac{\alpha}{\pi}\right) + \frac{\alpha^2}{4} + \cdots\right\} I_0^2 = m\alpha^4\left\{-\frac{2}{9}\left(\frac{\alpha}{\pi}\right) + \frac{\alpha^2}{16} - \frac{2}{27}\frac{\alpha^2}{\pi}\right\}. \quad (9.1)$$

The $\alpha^2/16$ term is from the expansion of the one-loop vacuum polarization function [29,31], while the $-2/27(\alpha^2/\pi)$ is from the relativistic expansion of the wave function.

The vacuum polarization contributions from Fig. 1(e) minus Fig. 1(f) is

$$\Delta E_{1e-1f} = m\alpha^4\left\{\frac{8}{9}\left(\frac{\alpha}{\pi}\right)^2 + \frac{2}{27}\frac{\alpha^2}{\pi}\right\}. \quad (9.2)$$

The $2/27(\alpha^2/\pi)$ term here also comes from the relativistic expansion of the wave function and cancels the similar term in ΔE_{1a}. The other part was worked out by Karshenboim [29].

The two-loop vacuum polarization contribution was done some time ago by Barbieri, Christillin, and Remiddi [23] and Samuel [24]. The result, plus the reducible product of the two one-loop vacuum polarization functions,

$$\Delta E_{1g+1h} = m\alpha^4\left\{-\frac{1}{8}\alpha^2\ln\left(\frac{1}{\alpha}\right) + 0.038\,332\,773\,8\right\}. \quad (9.3)$$

Note that this is does not include the ''Coulomb distortion'' part of [23] which we include in the many-potential amplitudes of Fig. 1(k).

The many-potential (MP) terms of Figs. 1(i), 1(j), and 1(k) are similar to the MP contributions in other equivalent formalisms. The corresponding contributions were worked out by Caswell and Lepage [15], where it was noted that the contributions having two annihilation photons and an annihilation photon with a transverse exchange photon should be

TABLE I. Contributions to the positronium hyperfine interval at order $m\alpha^6$.

Contribution	D	ΔE (MHz)
Three-photon-annihilation [36–38]	−0.051 94	−0.969
Two-photon-annihilation [39,40]	−0.032 48	−0.606
One-photon-annihilation (this work, [32])	−0.125 65	−2.344
One-photon-exchange [28,41,42]	−0.013 74	−0.256
Two-photon-exchange [43,44]	−0.545 35	−10.175
Three-photon-exchange [33,35,45–47]	0.376 32	7.021
Total	−0.392 83	−7.329

formalism independent, while the term involving the exchange of a Coulomb minus lowest order photon should be formalism dependent. We found this to be the case, although our result for the Coulomb minus lower order contribution agrees with that of Buchmüller and Remiddi [14], which underscores the closeness of the formalisms. For the individual MP contributions we have, $\Delta E_{1i}=m\alpha^6[1/16-1/16\zeta(2)]$, $\Delta E_{1j}=m\alpha^6[7/16-1/4\zeta(2)]$, and $\Delta E_{1k}=m\alpha^6[-3/16]$ to give a total MP contribution of $\Delta E_{1i+1j+1k}=m\alpha^6[5/16-5/16\zeta(2)]$.

The final contribution, from Fig. 1(l) is the derivative term. This is similar to the derivative term calculated by Caswell and Lepage [15], except for a formalism dependent sign and the fact that only the one-photon annihilation part contributes here. The derivative term is $\Delta E_{1l}=-m\alpha^6/32$.

The total result for the coefficient of $m\alpha^6$ coming from all one-photon annihilation contributions is

$$\Delta E_{1-\gamma-\mathrm{ann}}=-0.125\,648\,1(12). \tag{9.4}$$

This agrees with the analytic result

$$\Delta E^{\mathrm{analytic}}_{1-\gamma-ann}=\frac{m\alpha^6}{\pi^2}\left\{\frac{13}{32}\zeta(3)+\frac{27}{8}\zeta(2)\ln(2)-\frac{1183}{192}\zeta(2)+\frac{1477}{324}\right\} \tag{9.5}$$

of Hoang, Labelle, and Zebarjad [32], who used the effective field theory NRQED for their calculation. The numerical value of the analytic result is −0.125 648 7.

X. THE TOTAL HYPERFINE INTERVAL

In Table I, we list all order $m\alpha^6$ contributions to the hyperfine interval which gives for D a value of −0.392 83 and a contribution of −7.329 MHz to the energy difference.

Since our work was completed, the order $\alpha^7\ln(\alpha)$ contribution has been calculated by Kniehl and Penin [9], and Melnikov and Yelkhovsky [10]. This allows us, using the results of Hoang, Labelle and Zebarjad, Eq. (9.5), the three-photon exchange (pure recoil) results of Czarnecki, Melnikov and Yelkhovsky [33], the order $\alpha^7\ln^2(\alpha)$ calculation of Karshenboim [5], and the above mentioned contribution to write an analytic expression for ΔE correct to order $\alpha^7\ln(\alpha)$

$$\Delta E_{\mathrm{th}}=m\alpha^4\left\{\frac{7}{12}-\frac{\alpha}{\pi}\left(\frac{8}{9}+\frac{1}{2}\ln(2)\right)+\alpha^2\left(\frac{5}{24}\ln\left(\frac{1}{\alpha}\right)+\frac{1367}{648\pi^2}-\frac{5197}{3456}+\left(\frac{221}{144}+\frac{1}{2\pi^2}\right)\ln(2)-\frac{53}{32\pi^2}\zeta(3)\right)-\frac{7}{8}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right)+\left(\frac{217}{90}-\frac{17}{3}\ln(2)\right)\frac{\alpha^3}{\pi}\ln\left(\frac{1}{\alpha}\right)\right\}. \tag{10.1}$$

This gives the theoretical prediction $\Delta E_{\mathrm{th}}=203\,391.69$ MHz, which disagrees with the experimental values by 2.6 to 4.2 MHz.

XI. IS THERE A POSITRONIUM HYPERFINE PUZZLE?

Since the $\alpha^7\ln^2(\alpha)$ term contributes −0.92 MHz and the $\alpha^7\ln(\alpha)$ term contributes −0.32 MHz to the hyperfine splitting, it is reasonable to expect that the α^7 term G in Eq. (1.1) will not be sizable enough to bring theory and experiment into agreement. This raises the spectre of whether there is a ''positronium hyperfine puzzle'' analogous to the orthopositronium decay rate puzzle [34]. This cannot be answered until a more precise experiment is done. Also useful would be an estimate of at least the dominant contribution of G in order to rule out the possibility that G has a value (≈20 to 30) large enough to account for the present discrepancy.

With regard to the theoretical calculation, based upon our experiences with the present calculation, it is our opinion the next order term should be computed using more modern field theory methods, collectively called effective field theories (EFT) [35].

Positronium is a nonrelativistic system and EFT are constructed to take advantage of this by making a separation between the nonrelativistic and relativistic parts of the calculation. This distinction is determined by the various energy scales present in the theory. This separation allows for much simpler identification of terms that contribute to a particular order in α. This is to be contrasted with conventional relativistic field theory methods, as used in this paper, where integrals contribute to many orders in the fine structure constant. It is this that makes the analytic evaluation of these integrals extremely difficult.

EFT calculations are still highly nontrivial, particularly since the relativistic part of G will require the computation of the three-loop electron-positron scattering diagrams. This part of the calculation alone will require considerable resources, both human and computer. It is probably safe to say that this calculation will not be completed in the immediate future.

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APPENDIX A: NOTATION

The orthopositronium reference wave function is

$$\Psi^0(p)=2\pi\delta(p_0)\left(\frac{2\omega_p}{\omega_p+m}\right)\left(\frac{\omega_p+W}{2W}\right)^{1/2}\times[\Lambda_+(\vec{p})(1+\gamma_0)(-\gamma\epsilon)\Lambda_-(\vec{p})(-\gamma^0)]\times\left(\frac{\phi_0}{2\sqrt{2}}\right)\left(\frac{8\pi\gamma}{D_p^2}\right)=2\pi\delta(p_0)\left(\frac{\phi_0}{2\sqrt{2}}\right)\left(\frac{8\pi\gamma}{D_p^2}\right)\Phi(\vec{p}), \tag{A1}$$

where $\phi_0=\sqrt{(m\alpha)^3/(8\pi)}$ is the wave function at contact, $D_p=p^2+\gamma^2$, $p=|\vec{p}|$, $W=m\sqrt{(1-\alpha^2)/(4n^2)}$, $\epsilon=(0,\hat{\epsilon})$ is the positronium spin vector and

$$\hat{\epsilon}_0=(0,0,1), \tag{A2}$$

$$\hat{\epsilon}_{\pm1}=-\frac{1}{\sqrt{2}}(\pm1,i,0). \tag{A3}$$

The nonrelativistic approximation of the wave function is

$$\Psi^0(p)\to2\pi\delta(p_0)(1+\gamma_0)(-\gamma\epsilon)\left(\frac{\phi_0}{2\sqrt{2}}\right)\left(\frac{8\pi\gamma}{D_p^2}\right). \tag{A4}$$

Other definitions used in the text are

$$B^m=\sqrt{2}ie\phi_0\epsilon^m,$$

$$(d\not{\ell})'_n=(d^n\not{\ell})/(2\pi)^n,$$

$$(d\not{\ell})''_n=(d^n\not{\ell})/(i\pi^{n/2}),$$

$$D(\pm W,q)=-q^2\pm2Wnq+\gamma^2,$$

$$D_l=l^2+\gamma^2,l=|\vec{\ell}|,$$

$$n=(1,\vec{0}).$$

APPENDIX B: RENORMALIZED PERTURBATION SERIES

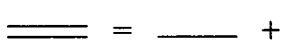
In this appendix, we will derive Eq. (1.9) graphically, using the following definitions. The vertices are

(B1)

e is the renormalized charge, e_0 the unrenormalized charge and $e=e_0Z_3^{1/2}$. Z_3 is the charge renormalization constant and to two loops is

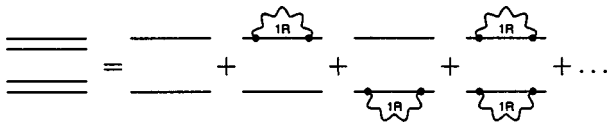
$$Z_3=\frac{1}{1+C}=[1+C_1+C_2+\cdots]^{-1}=1-C_1-C_2+C_1^2+O(\alpha^3). \tag{B2}$$

A renormalized fermion propagator is



(B3)

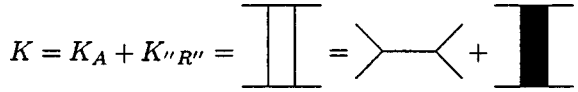
and the product of two renormalized fermion propagators is



(B4)

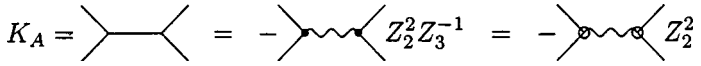
In the two last equations, the unadorned lines are free fermion propagators, *nR* stands for *n*-loop renormalized and the stubby lines attached to the self energy insertions are free propagators.

The two-particle irreducible kernel is



(B5)

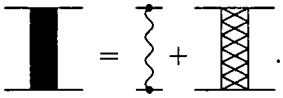
with[2]



(B6)

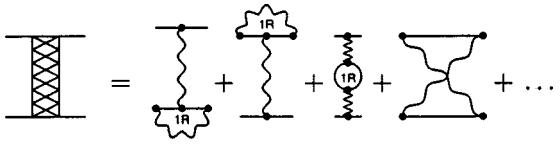
[2]The factor of -1 is a fermionic minus sign.

and



(B7)

The last figure on the right stands for the expansion

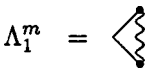


(B8)

The unrenormalized vertex function is

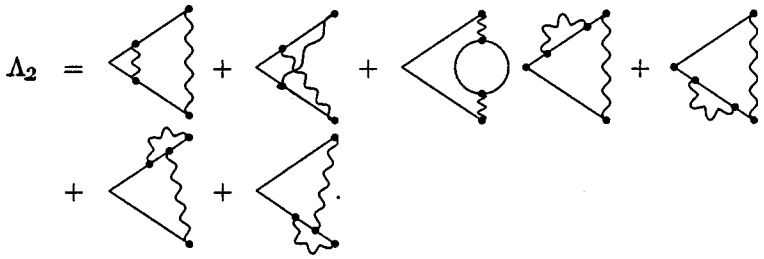
$$\Gamma'^{m} = \gamma^m + \Lambda_1^m + \Lambda_2^m + \cdots . \tag{B9}$$

In this, 1, 2, etc., refer to one-loop, two-loop, etc., so



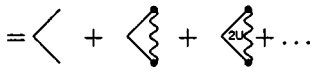
(B10)

and



(B11)

If we multiply Γ'^{m} by a wave function on the right and integrate we obtain



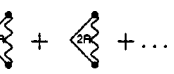
(B12)

In the graphical representation of the amplitudes, the wave function is always implicit on the right (left). A_0^m is represented by the first diagram, A_1^m by the second and A_2^m by the third diagram. $2U$ stands for two-loop unrenormalized.

The renormalized vertex is

$$\begin{aligned}\Gamma_R^m = Z_1(\Gamma')^m &= (1+L_1+L_2+\ldots)^{-1}(\gamma^m+\Lambda_1^m+\Lambda_2^m+\ldots)\\ &= \gamma^m+(\Lambda_1^m-L_1\gamma^m)+(\Lambda_2^m-L_1\Lambda_1^m-L_2\gamma^m\\ &\quad+\gamma^m L_1^2+\cdots)\\ &= \gamma^m+\Lambda_{1R}^m+\Lambda_{2R}^m+\cdots,\end{aligned} \tag{B13}$$

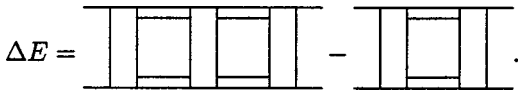
and when convoluted with a wave function,



(B14)

A_{1R}^m is represented by the second diagram and A_{2R}^m by the third. It is important to remember that each term in Eq. (B14) is a complicated function of α and can be expanded in a series of powers of α and $\ln(\alpha)$.

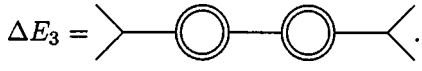
We are now in a position to derive Eq. (1.9). The diagrams we are interested in come from the second and third terms of Eq. (1.8), i.e., $K_{\rm BS}SK_{\rm BS}SK_{\rm BS}-K_{\rm BS}SK_{\rm BS}$, which we represent pictorially as



(B15)

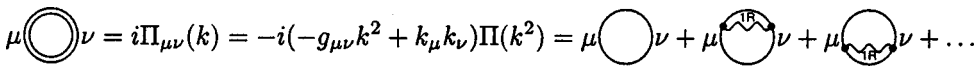
Implicit on the left and right are wave functions and integrations over relative momenta. The double horizontal lines and the double vertical lines are the previously defined renormalized fermion propagators and two particle irreducible kernels.

We need those parts of ΔE that contain the single annihilation photon kernel. To order α^6, these are the ones with one, two or three factors of the annihilation kernel K_A. Call these ΔE_1, ΔE_2, and ΔE_3, and their sum $\Delta E_{123}=\Delta E_1+\Delta E_2+\Delta E_3$. For the term with three factors of the annihilation kernel, we have



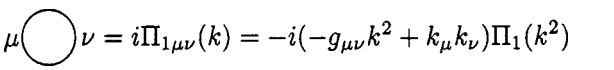
(B16)

The vacuum polarization bubble is



(B17)

To the required order of accuracy, only the first term of Eq. (B17) is needed. Then, we can use



(B18)

and

$$\Pi_1(k^2)=\Pi_1(0)+\Pi_{1S}(k^2)=C_1+\Pi_{1R}(k^2). \quad \text{(B19)}$$

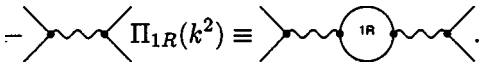
In this last equation, we have replaced the subtracted one-loop vacuum bubble by the renormalized one-loop vacuum bubble, that is,

$$\Pi_{1R}(k^2)=Z_3\Pi_{1S}(k^2)\rightarrow\Pi_{1S}(k_2),$$

since this involves no loss of accuracy. Then

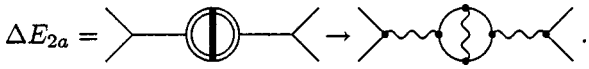
(B20)

To get this form, we have made use of the definition



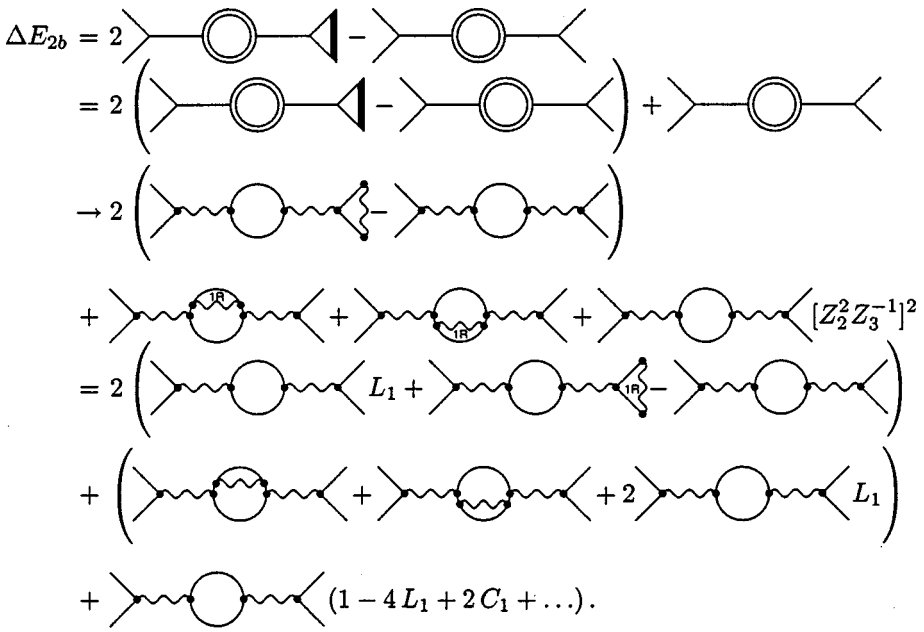
(B21)

For ΔE_2, we separate it into two subexpressions. The first, ΔE_{2a}, is particularly simple:

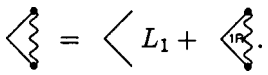


(B22)

The remaining subexpression is

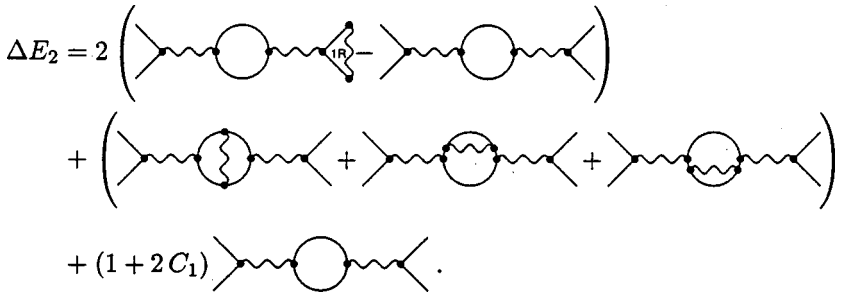


To get this, we have expressed the renormalized self-energy insertions in the vacuum polarization loops in terms of the unrenormalized self-energy operator. Also, we used $A_1^m=\gamma^m L_1+A_{1R}^m$, which pictorially is



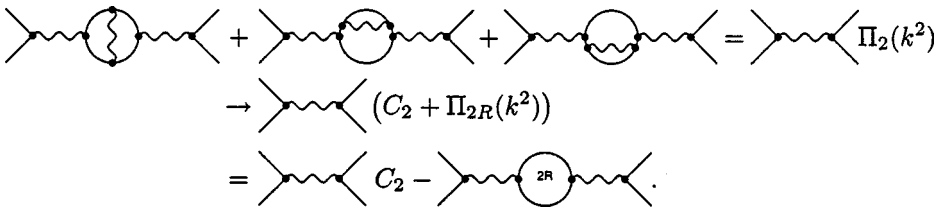
(B23)

So, upon adding ΔE_{2a} and ΔE_{2b} we get



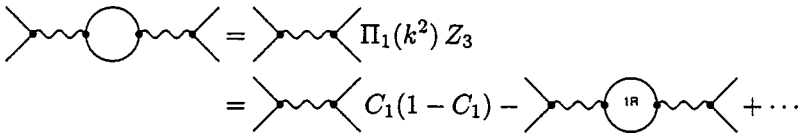
(B24)

This can be further simplified using



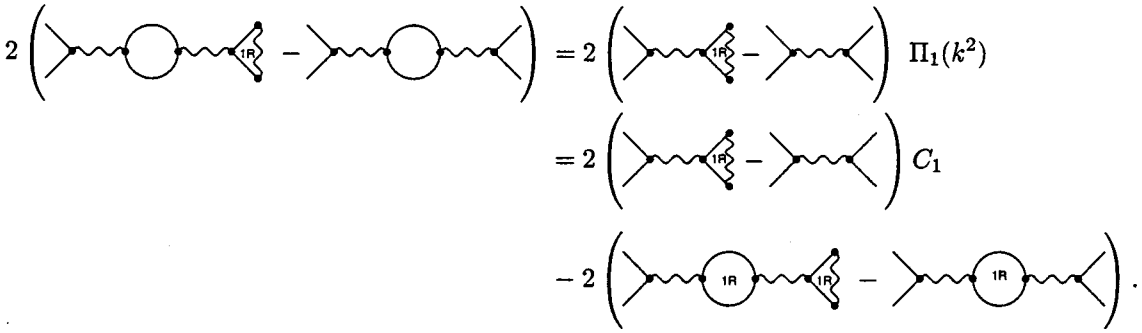
(B25)

Next, we need to write the unrenormalized one-loop vacuum polarization bubble in Eq. (B24) in terms of the renormalized bubble. To do this we need to express the two factors of the renormalized charge where a photon meets the vacuum polarization bubble in terms of the bare charges, using $e^2 = Z_3 e_0$. Doing this,



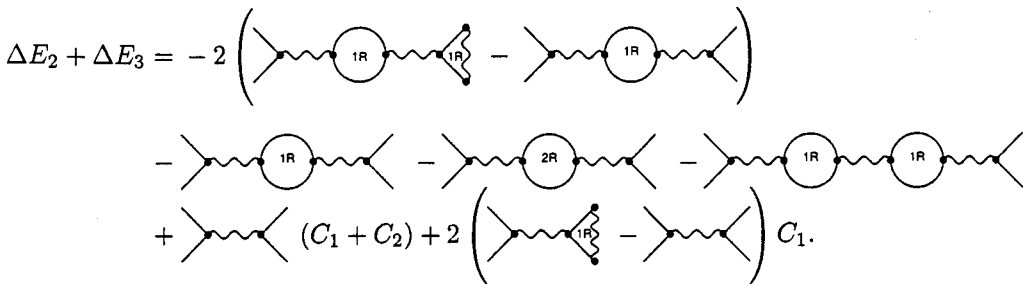
(B26)

For the terms in the first line of Eq. (B24) we can write



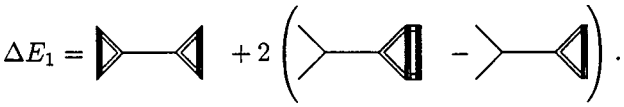
(B27)

Putting everything together, we get for the sum of the two- and three-annihilation kernels



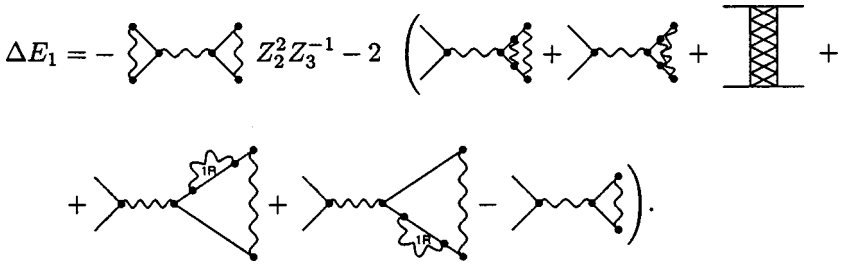
(B28)

The one-annihilation term is



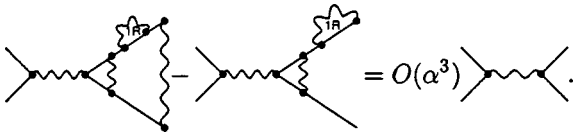
(B29)

For this, we make use of Eq. (B4) and Eq. (B7), giving



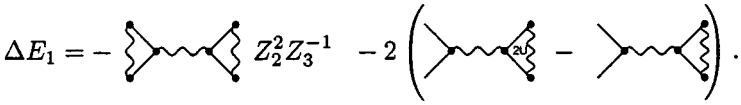
(B30)

To get this expression many cancellations and approximations have been used. An example of such an approximation is the following which uses the unperturbed wave equation to show that the difference of the two diagrams is of higher order.



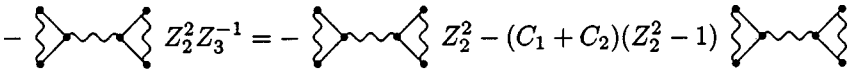
(B31)

Such terms can be dropped from Eq. (B29). In Eq. (B30), we used Eq. (B8) to expand the third diagram in the parentheses and then wrote the renormalized self-energy and vertex insertions in terms of their unrenormalized counterparts and appropriate renormalization constants. The renormalization constants cancel and we are left with

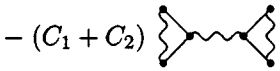


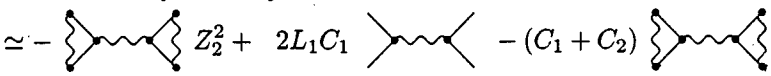
(B32)

Next, we want to put the first term in Eq. (B32) in a form with factors of C_1 and C_2 explicit before adding this to Eq. (B28). So, using $Z_3^{-1}=1+C_1+C_2+\cdots$, we get



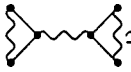
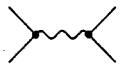
(B33)



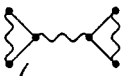
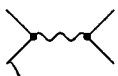
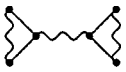


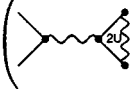
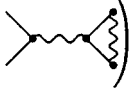
(B34)

To get this, we used

$$-(C_1+C_2)(Z_2^2-1)$$  $$\simeq 2C_1L_1$$  $$+O(\alpha^7).$$

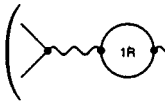
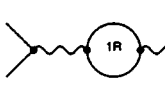
For ΔE_1, we now have

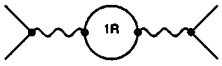
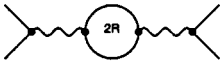
$$\Delta E_1 = -$$  $$Z_2^2 + 2L_1C_1$$  $$- (C_1+C_2)$$ 

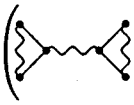
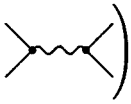
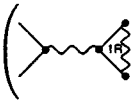
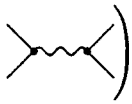
$$- 2 \left(\right.$$  $$-$$  $$\left. \right).$$ (B35)

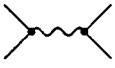
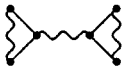
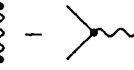
Adding this to Eq. (B28), we have

$$\Delta E_{123} = \Delta E_1 + \Delta E_2 + \Delta E_3$$

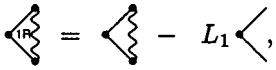
$$= -2 \left(\right.$$  $$-$$  $$\left. \right)$$

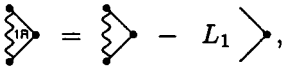
$$-$$  $$-$$  $$-$$ 

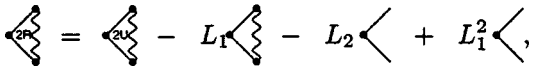
$$- (C_1+C_2) \left(\right.$$  $$-$$  $$\left. \right) + 2C_1 \left(\right.$$  $$-$$  $$\left. \right)$$

$$+ 2L_1C_1$$  $$-$$  $$Z_2^2 - 2 \left(\right.$$  $$-$$ $$\left. \right).$$ (B36)

We were able to write the last two terms in Eq. (B36) in terms of renormalized quantities by making use of the identities

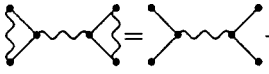
 (B37)

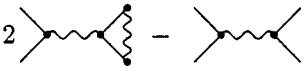
 (B38)

 (B39)

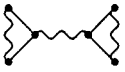
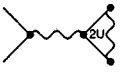
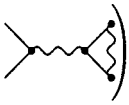
and the approximations

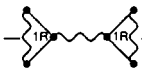
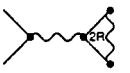
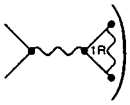
$$Z_2^2 = 1 - 2L_1 - 2L_2 + 3L_1^2 + O(\alpha^3),$$ (B40)

 $$+ O(\alpha^5)$$

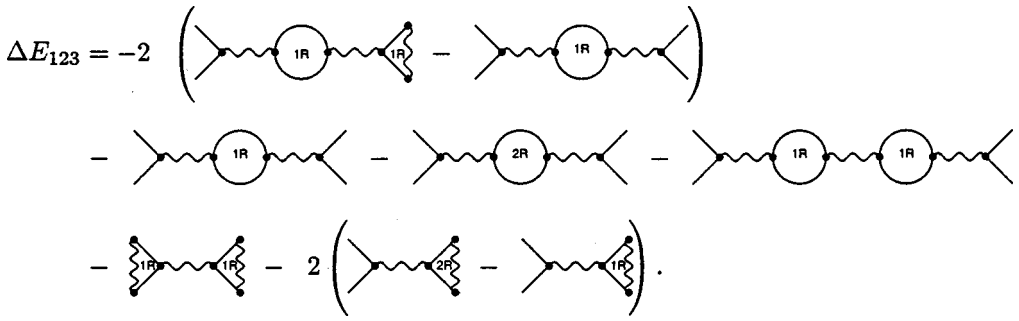
$$= 2$$  $$+ O(\alpha^6).$$ (B41)

We found that

$$-Z_2^2$$  $$- 2 \left(\right.$$  $$-$$  $$\left. \right)$$

$$= -$$  $$- 2 \left(\right.$$  $$-$$  $$\left. \right).$$ (B42)

We also note that the term proportional to C_2 is of $O(\alpha^7)$, as it is the sum of the terms proportional to C_1. Our final result for the one-photon-annihilation contribution to the energy shift is

$\Delta E_{123} = -2$ 

(B43)

The sum of the last two terms of ΔE_{123} is $\Delta E_{1a+2(1c-1d)}$, Eq. (1.9) of the main text.

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