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Universal quantum computation and simulation using any entangling Hamiltonian and local unitaries

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What interactions are sufficient to simulate *arbitrary* quantum dynamics in a composite quantum system? We provide an *efficient* algorithm to simulate any desired two-body Hamiltonian evolution using any fixed two-body entangling *n*-qubit Hamiltonian and local unitary operations. It follows that universal quantum computation can be performed using *any* entangling interaction and local unitary operations.

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A central goal of quantum physics is to understand and control quantum dynamics. Recently, the emergence of fields such as quantum control [1], laser cooling [2], quantum communication, and quantum computation [3,4] has focused efforts to understand and control quantum dynamics at the single quantum level.

Our interest is in the dynamics of composite quantum systems. An especially important example of such a system is a quantum computer, which is a composite of a large number of two-level quantum systems (qubits). We wish to determine which interactions are sufficient for the simulation of *arbitrary* quantum dynamics in such a system. Our results demonstrate equivalence between this property of *universality* and the ability to entangle all components of the system.

More precisely, we consider the following problem: what dynamics can we produce with a specified two-body, *n*-qubit Hamiltonian, given the ability to perform arbitrary local unitary operations on individual qubits? Under these conditions, we exhibit an explicit algorithm which shows that *any* Hamiltonian which produces entanglement can be used to efficiently simulate an arbitrary two-body dynamical operation. This holds even if the Hamiltonian *alone* is only capable of producing a small amount of entanglement.

It follows that any entangling interaction, together with local unitary operations, is sufficient to perform universal quantum computation. Our results thus confirm the ''folklore'' belief that the ability to entangle is a crucial element in quantum computation.

Substantial prior work has been done on universal operations, and many specific sets of universal gates are known [5,3]. Our work differs from previous work on the general requirements for universality in several regards. Closest is the work in [6] and [7], where it was shown that almost any two-qubit quantum gate is universal for quantum computation. This work focused on unitary gates rather than continuous-time Hamiltonian evolution, and did not explicitly determine which sets of unitary gates are universal. Our work explicitly determines which two-body Hamiltonians, together with the additional requirement of local unitary operations, are universal. Furthermore, in [6] and [7] it was assumed that gates could be independently applied to *any* pair of qubits in the computer, and thus required the ability to turn on and turn off interactions between different pairs of qubits. By contrast, we assume only a fixed entangling operation, although we do require the ability to turn on and turn off arbitrary local unitary operations.

Our techniques make use of generalizations of standard nuclear-magnetic-resonance (NMR) techniques for decoupling and refocusing [8,9]. Similar ideas have been applied by [10,11] to the problem of efficiently implementing coupled logic gates using a restricted class of Hamiltonians which arises naturally in NMR.

The structure of this Rapid Communication is as follows. We begin with a precise formulation of our goals and results. A specific two-qubit example is given to illustrate our techniques, and the general algorithm is described for the case of an arbitrary two-qubit system. The efficiency of the algorithm and the effect of errors are then discussed. We conclude by generalizing the algorithm to *n*-qubit systems.

An arbitrary Hamiltonian on *n* qubits can be given the *operator expansion*

$$H=\sum_{j_1,\ldots,j_n=0}^{3} h_{j_1\cdots j_n}\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_n}, \tag{1}$$

where the $h_{j_1\cdots j_n}$ are real numbers and $\sigma_1,\sigma_2,\sigma_3$ are the usual Pauli sigma matrices, with $\sigma_0\equiv I$ the identity. Our discussion is restricted to the case of time-independent Hamiltonians containing only one- and two-body terms, that is, if $h_{j_1\cdots j_n}\neq 0$ then only one or two of the $j_1,\ldots,j_n$ are not equal to zero. If the Hamiltonian contains a nonzero contribution to $\sigma_k\otimes\sigma_l$ then we say the Hamiltonian *couples* systems *k* and *l*. This focus on two-body Hamiltonians is a mild restriction as most candidate systems for quantum information processing are of this type.

Under what circumstances is it possible to produce entanglement between an *arbitrary* pair of systems, even ones that are not directly coupled by the Hamiltonian *H*? Not surprisingly, Hamiltonians which have terms coupling systems *k* and *l* can produce entanglement between these systems. We say that systems *k* and *k′* are *connected* if there is a sequence $(k,k_1,\ldots,k_m,k')$ such that each adjacent pair in the sequence is coupled by *H*. It is clear that if *k* and *k′* are not connected then no entanglement can be created between them, and thus it is not possible to perform an arbitrary unitary operation on the system. Conversely, it follows from our later discussion (and is *a priori* plausible) that if a pair of systems is connected then it is possible to create entanglement between them (cf. [12,13]). This motivates our defini-

tion of a *two-body entangling Hamiltonian* as a two-body Hamiltonian such that all pairs of systems are connected.

The main result of this Rapid Communication is the following:

> Let H be a given two-body entangling Hamiltonian on n qubits, and let K be a desired two-body Hamiltonian on n qubits. Then we have an efficient algorithm to simulate evolution due to K, to any desired degree of accuracy, using only (a) the ability to evolve according to H and (b) the ability to perform local unitary operations on the individual qubits.

In particular, given such a Hamiltonian it follows that we can perform an arbitrary two-qubit unitary gate on any specified pair of qubits. Thus, by well-known universality results [5,3], we may efficiently perform any quantum computation.

Three elementary observations about Hamiltonian evolution form the key to our methods:

(a) Imagine we can evolve according to the Hamiltonian J, and perform unitary operations U and $U^\dagger$. Then it follows from the identity $e^{-itUJU^\dagger}=Ue^{-itJ}U^\dagger$ that we can exactly simulate evolution according to the Hamiltonian $UJU^\dagger$.

(b) Imagine we can evolve according to Hamiltonians J_1 and J_2. Then we can simulate evolution due to J_1+J_2 for small times Δ, due to the approximate identity

$$e^{-i\Delta(J_1+J_2)}\approx e^{-i\Delta J_1}e^{-i\Delta J_2}. \tag{2}$$

Initially we treat this identity as though it is exact, and analyze the effect of errors later.

(c) Imagine we can evolve according to a Hamiltonian J. Then, by appropriate timing, we can exactly simulate evolution according to λJ for any $\lambda>0$.

The basic idea can be illustrated using a two-qubit example. Suppose we have the ability to evolve according to the two-qubit Hamiltonian

$$H=Z\otimes I+2X\otimes Z+Z\otimes Z, \tag{3}$$

where X, Y, and Z are a convenient shorthand for the Pauli sigma matrices. H couples the two qubits, and is thus a two-body entangling Hamiltonian. The first step of our procedure is to show that H and local unitaries can be used to simulate evolution according to the largest coupling term in H, in this case $X\otimes Z$. We do this using (a), (b), and (c) to eliminate the other terms in H, according to the identity

$$X\otimes Z=\frac{1}{4}(X\otimes I)H(X\otimes I)^\dagger+\frac{1}{4}H. \tag{4}$$

This procedure eliminates the undesired couplings by averaging over the given interaction H and a rotated version of the interaction, $(X\otimes I)H(X\otimes I)^\dagger$.

Using the ability to simulate evolution by the Hamiltonian $X\otimes Z$ we can easily obtain the ability to simulate a Hamiltonian which is *any* product of Pauli matrices. Products of the form $I\otimes\sigma_j$ and $\sigma_j\otimes I$ follow immediately from our ability to do local unitaries. Products of the form $\sigma_j\otimes\sigma_k$ follow from observation (a) and the fact that $\sigma_j\otimes\sigma_k=(U\otimes V)X\otimes Z(U\otimes V)^\dagger$ for appropriate single-qubit rotations U and V. It is easy to see that observation (a) also allows us to simulate terms of the form $-\sigma_j\otimes\sigma_k$. An arbitrary two-qubit Hamiltonian K can be decomposed as a linear combination of products of Pauli matrices, and thus by observations (b) and (c) may be simulated using our ability to simulate $X\otimes Z$. Thus K may be simulated using H and local unitaries.

The general two-qubit case follows using similar techniques. Suppose H is an entangling Hamiltonian and choose $r,s\neq 0$ such that $|h_{r,s}|$ is maximized. It is easy to verify that

$$\mathrm{sgn}(h_{r,s})\sigma_r\otimes\sigma_s=\sum_{j\in\{0,r\},k\in\{0,s\}}\frac{(\sigma_j\otimes\sigma_k)H(\sigma_j\otimes\sigma_k)^\dagger}{4|h_{r,s}|}-\frac{h_{r,0}\sigma_r\otimes I+h_{0,s}I\otimes\sigma_s+h_{0,0}I\otimes I}{|h_{r,s}|}. \tag{5}$$

Using observations (a), (b), and (c) it follows that both $\sigma_r\otimes\sigma_s$ and $-\sigma_r\otimes\sigma_s$ can be simulated using H and local unitary operations, and thus the result follows for a general two-qubit Hamiltonian K.

In more detail, suppose we wish to simulate K for a noninfinitesimal time $t>0$. We have shown that we can approximate evolution due to K for a small time Δ by applying an appropriate sequence of evolutions due to H and local unitary operations. Such a simulation requires, in general, 36 separate periods of evolution due to H, interleaved by single-qubit unitary gates applied to the two qubits. To simulate the evolution due to K over a time t we break the interval t into N increments of length $\Delta\equiv t/N$, and perform the simulation of K for each increment, repeating the small time step procedure N times, for a total of at most $36N$ separate periods of evolution due to H.

Let us turn to the sources of error in our simulation procedure. In practice, there will be experimental errors due to decoherence and discrepancies between the desired operations and those that are actually applied. We will not deal here with these types of errors, but these issues will be addressed in future work on the fault tolerance of our simulations. Here, we discuss errors that are *inherent* in our simulation technique. Although the procedure uses only the observations (a), (b), and (c), of which (a) and (c) are in principle exact, the identity Eq. (2) used in (b) only holds approximately. In order to perform a good simulation of K we therefore need to choose a time step Δ sufficiently small that Eq. (2) is a good approximation.

To do the error analysis, we introduce a measure quantifying how well our simulated evolution approximates the desired evolution due to K. That is, we wish to compare the unitary evolution W' achieved by our simulation with the unitary evolution $W=\exp(-iKt)$ that we wish to simulate. We use as our measure of error the operator norm of the difference between W and W', $\|W-W'\|$, defined by $\|A\|\equiv\max_{\psi:|\psi|=1}|A|\psi\rangle|$. This is physically well motivated since two operators W and W' are close according to this norm if

and only if the difference in their effects on an arbitrary state is bounded by a small number. The actual measure of error used is not all that important, but we find it useful to demand the following two properties, both of which are satisfied by the operator norm: (1) stability under tensor product with ancilla systems, that is, $\|A\|=\|I\otimes A\|$; and (2) invariance under unitary transformations, that is, $\|A\|=\|V'AV\|$ for any unitary operators V,V'. This latter property implies the *chaining inequality* for any unitary operators V_1,V_2,W_1,W_2

$$\|V_1W_1-V_2W_2\|\leqslant\|V_1-V_2\|+\|W_1-W_2\|. \tag{6}$$

We bound the errors induced by the approximation in Eq. (2) using the inequality [14]

$$\|e^{-i\tau(A_1+\cdots+A_m)}-e^{-i\tau A_1}\cdots e^{-i\tau A_m}\| \leqslant\frac{\tau^2}{2}\sum_{1\leqslant j\leqslant k\leqslant m}\|[A_j,A_k]\|, \tag{7}$$

where τ is a positive real number and the A_j are Hermitian operators. Applying this bound and the chaining property to the procedure we have described gives

$$\|W'-W\|\leqslant CD^2t\Delta, \tag{8}$$

where C is a constant which we can easily bound to be at most 10^4, and D is a parameter determined by the properties of H and K as follows. Let $h\equiv\max_{i,j}|h_{i,j}|$ and $k\equiv\max_{i,j}|k_{i,j}|$, where K has the operator expansion $\Sigma_{i,j}k_{i,j}\sigma_i\otimes\sigma_j$. Then $D\equiv|hk/h_{r,s}|$.

The error bound Eq. (8) can be improved substantially in several ways. The linear dependence on Δ in Eq. (8) is due to the technique used to simulate sums of Hamiltonians, namely, $e^{-i\Delta(J_1+J_2)}=e^{-i\Delta J_1}e^{-i\Delta J_2}+O(\Delta^2)$. Each simulation step thus contributes an error $O(\Delta^2)$, and there are t/Δ such steps for a total error $O(t\Delta)$. Higher-order approximation techniques [14] can be used to obtain more accurate simulations. For example, identities such as

$$e^{-i\Delta(J_1+J_2)}=e^{-i\Delta J_1/2}e^{-i\Delta J_2}e^{-i\Delta J_1/2}+O(\Delta^3) \tag{9}$$

yield a cumulative error which is $O(t\Delta^2)$. In general, an approximation analogous to Eq. (2) but accurate to order Δ^k leads to a cumulative error $O(t\Delta^{k-1})$. The tradeoff is such that higher-order approximations require the use of somewhat more complicated gate sequences for each time step. In practical applications, this additional complication must be balanced against the improvement in accuracy to achieve optimal results.

A second way to improve the bound in Eq. (8) is to leverage specific knowledge of the given and desired Hamiltonians. For example, imagine that we have available the Hamiltonian of Eq. (3), and wish to simulate a controlled-NOT gate [3]. We can do this more efficiently than implied by the identity in Eq. (5) by examining the properties of the controlled-NOT gate. Up to an unimportant global phase, the controlled-NOT gate may be generated by applying the Hamiltonian $I\otimes X+Z\otimes I-Z\otimes X$ for a time $t=\pi/4$. The terms in this Hamiltonian commute, so the controlled-NOT operation is given by $e^{-i(I\otimes X)t}e^{-i(Z\otimes I)t}e^{i(Z\otimes X)t}$. Thus, to simulate the controlled-NOT gate for a time t, it suffices to simulate evolution according to the Hamiltonian $K=-Z\otimes X$, followed by local unitary operations. We observe that

$$K=(R\otimes RX)\frac{(X\otimes I)H(X\otimes I)^\dagger+H}{4}(R\otimes RX)^\dagger, \tag{10}$$

where R is the Hadamard gate [3], denoted here by R instead of the usual H to avoid confusion with the given Hamiltonian. Using the method outlined earlier gives a cumulative error $8t\Delta$. If we wish to have an accuracy of 10^{-3} this corresponds to roughly 10^4 periods of evolution according to H, interleaved with local unitary operations. This number of operations is probably too large to be practical, however it is substantially better than is obtained using the general bound Eq. (8).

Further improvement may be obtained by using the higher-order approximation Eq. (9). Using the operator norm, simple algebra shows that the correction in Eq. (9) may be bounded to order Δ^3 by $\frac{1}{6}\|J_1\|\|J_2\|(\|J_1\|+2\|J_2\|)\Delta^3$. In this specific example, this reduces to $\frac{1}{128}\|H\|^3\Delta^3$ for a cumulative error of at most $\frac{1}{128}\|H\|^3t\Delta^2$. Bounding $\|H\|$ by $\|Z\otimes I\|+2\|X\otimes Z\|+\|Z\otimes Z\|=4$ we see that the cumulative error is at most $\frac{1}{2}t\Delta^2$. Therefore, to achieve an accuracy of 10^{-3} in our simulation of the controlled-NOT gate we need approximately 10^2 periods of evolution due to H, interleaved with local unitary operations. Further improvements may be obtained by using better approximations than Eq. (9).

The number of operations required to simulate an arbitrary unitary operation can thus be substantial. In practice, this disadvantage may be offset by the advantages gained in using the natural coherent interactions present in a system. Furthermore, our results merely provide a lower bound on the efficiency with which it is possible to simulate an arbitrary unitary operation, and provide substantial impetus to search for better methods in specific cases.

We now turn to the n-qubit case. The basic idea is to reduce the problem to the two-qubit case already solved. We divide the system into two parts, a *principal system* P consisting of two qubits which are coupled by the Hamiltonian H, and the *remainder* of the system, denoted S. We use a technique generalizing the work in [11,15] that turns off all interactions between P and S and within S, leaving only the interactions present in P. These interactions can then be used, as before, to simulate arbitrary dynamics on the two qubits in P. Thus, it is possible to simulate arbitrary dynamics on *any* two qubits coupled by the Hamiltonian H. Finally, an arbitrary interaction between qubits k and k' may be effected by performing a sequence of SWAP gates between the qubits connecting k and k', applying the desired interaction, and then swapping back.

The first step is to decouple systems P and S. To do this, let X_S denote a tensor product of X operators applied bitwise to all the qubits in S. Define Y_S and Z_S similarly. Observe that forming the Hamiltonian

$$H'=\frac{1}{4}[H+X_SHX_S^\dagger+Y_SHY_S^\dagger+Z_SHZ_S^\dagger] \tag{11}$$

leaves the Hamiltonian on P invariant, but eliminates all coupling terms between P and S, and all single-system terms on S.

We now explain a recursive construction to eliminate all remaining couplings in S. First, we break the block S into two blocks S_0 and S_1 of approximately equal size. We decouple S_0 and S_1 by forming the Hamiltonian

$$H'' = \frac{1}{4}[H' + X_{S_0} H' X_{S_0}^\dagger + Y_{S_0} H' Y_{S_0}^\dagger + Z_{S_0} H' Z_{S_0}^\dagger]. \tag{12}$$

Next, we break S_0 into two blocks S_{00} and S_{01} of approximately equal size, and break S_1 into two blocks S_{10} and S_{11} of approximately equal size. We can decouple S_{00} from S_{10}, and S_{01} from S_{11} in a *single step* by forming the Hamiltonian

$$H''' = \frac{1}{4}[H'' + X_{S_{00}} X_{S_{10}} H'' X_{S_{00}}^\dagger X_{S_{10}}^\dagger + Y_{S_{00}} Y_{S_{10}} H'' Y_{S_{00}}^\dagger Y_{S_{10}}^\dagger + Z_{S_{00}} Z_{S_{10}} H'' Z_{S_{00}}^\dagger Z_{S_{10}}^\dagger]. \tag{13}$$

We repeat this blocking procedure $\lceil \log_2(n-2) \rceil$ times to decouple all the terms in S, leaving a sum over $O(4^{\log_2 n}) = O(n^2)$ terms involving the conjugation of H by local unitary operations.

Thus, simulating a Hamiltonian K applied to P for a time t requires the use of $O(n^2)$ periods of evolution due to H, interleaved with local unitary operations. Using a similar error analysis to that described earlier, and the stability property of the operator norm, we find an error $O(n^2 t\Delta)$. In practice it may be possible to do substantially better by leveraging our knowledge of specific systems, and using better approximations.

A number of problems will be addressed in future work, including: (a) the extension of our results beyond the qubit model to higher-dimensional systems; (b) the *fault tolerance* [16,3] of our simulation techniques; (c) the optimization of our techniques for specific systems; and (d) the further study of the general requirements for universal computation (cf. [17–20]). For example, it is likely interesting to impose restrictions on the class of local unitary operations that may be applied during the computation, perhaps adopting a *cellular automata* model in which operations are applied nearly homogeneously across the entire system.

The results presented in this Rapid Communication demonstrate that all two-body, n-qubit entangling Hamiltonians are *equivalent* in the sense that any such Hamiltonian can be used to efficiently simulate any other with the aid of local unitary operations. We conjecture that the same result is *not* true for k-body Hamiltonians where $k>2$. It would be of interest to determine, in general, what characteristics of two sets of Hamiltonians determine whether they are equivalent.

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[1] H. Rabitz, R. de Vivie-Riedle, M. Motzkus, and K. Kompa, Science **288**, 824 (2000).
[2] C.E. Wieman, D.E. Pritchard, and D.J. Wineland, Rev. Mod. Phys. **71**, S253 (1999).
[3] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[4] J. Preskill, *Physics 229: Advanced Mathematical Methods of Physics—Quantum Computation and Information* (California Institute of Technology, Pasadena, 1998), http://www.theory.caltech.edu/people/preskill/ph229/
[5] A. Barenco *et al.*, Phys. Rev. A **52**, 3457 (1995).
[6] D. Deutsch, A. Barenco, and A. Ekert, Proc. R. Soc. London, Ser. A **449**, 669 (1995).
[7] S. Lloyd, Phys. Rev. Lett. **75**, 346 (1995).
[8] C.P. Slichter, *Principles of Magnetic Resonance* (Springer, Berlin, 1990).
[9] R.R. Ernst, G. Bodenhausen, and A. Wokaun, *Principles of Nuclear Magnetic Resonance in One and Two Dimensions* (Oxford University Press, Oxford, 1994).
[10] J.A. Jones and E. Knill, J. Magn. Reson. **141**, 322 (1999).
[11] D.W. Leung, I.L. Chuang, F. Yamaguchi, and Y. Yamamoto, Phys. Rev. A **61**, 042310 (2000).
[12] J.I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2000).
[13] W. Dür *et al.*, e-print quant-ph/0006034.
[14] H. De Raedt, in *Annual Reviews of Computational Physics IV*, edited by D. Stauffer (World Scientific, Singapore, 1996), pp. 107–146.
[15] N. Linden, H. Barjat, R.J. Carbajo, and R. Freeman, Chem. Phys. Lett. **305**, 28 (1999).
[16] J. Preskill, Proc. R. Soc. London, Ser. A **454**, 385 (1998).
[17] D.P. DiVincenzo *et al.*, Nature (London) **408**, 339 (2000).
[18] D. Bacon *et al.*, e-print quant-ph/0102140.
[19] E. Knill, R. Laflamme, and G.J. Milburn, Nature (London) **409**, 46 (2001).
[20] R. Raussendorf and H.J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).