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Entanglement induced by a single-mode heat environment

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A thermal field, which frequently appears in problems of decoherence, provides us with minimal information about the field. We study the interaction of the thermal field and a quantum system composed of two qubits and find that such a chaotic field with minimal information can nevertheless entangle qubits that are prepared initially in a separable state. This simple model of a quantum register interacting with a noisy environment allows us to understand how memory of the environment affects the state of a quantum register.

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A thermal field is emitted by a source in thermal equilibrium at temperature T. The thermal field is a field about which we have minimal information, as we know only the mean value of the energy [1]. Recently, Bose *et al.* [2] showed that entanglement can always arise in the interaction of a single qubit in a pure state with an arbitrarily large system in any mixed state and illustrated this using a model of the interaction of a two-level atom with a thermal field. Using this model, they studied the possibility of entangling a qubit with a large system defined in an infinite dimensional Hilbert space. The entanglement between the system and the thermal field reduces the system to a mixed state when the field variables are traced over. In this paper we address the question, "Is it possible for a thermal field, which is a highly chaotic field, to induce entanglement between qubits?" The entanglement of an atom and a field may be a totally different problem from the entanglement of two atoms (or qubits) by their mutual interaction with a chaotic field.

We consider a quantum register composed of two two-level atoms interacting with a single-mode thermal field, whose annihilation and creation operators are denoted by $\hat{a}$ and $\hat{a}^\dagger$. We will investigate the entanglement between the two atoms, which are initially separable. This simple interaction model of a quantum register with its environment allows us to understand how the memory of the environment affects the state of a quantum register.

When a quantum system of two qubits prepared in $\hat{\rho}_s$ interacts with an environment represented by the density operator $\hat{\rho}_E$, the system and environment evolve for a finite time, governed by the unitary time evolution operator $\hat{U}(t)$. The density operator for the system and environment at time t is

$$\hat{\rho}(t)=\hat{U}(t)(\hat{\rho}_E\otimes\hat{\rho}_s)\hat{U}^\dagger(t). \tag{1}$$

After performing a partial trace over environment variables, we find the final density matrix $\hat{\rho}_s(t)$ of the quantum system in the following Kraus representation [3]:

$$\hat{\rho}_s(t)=\mathrm{Tr}_E\,\hat{\rho}(t)=\sum_\mu \hat{K}_\mu\hat{\rho}_s\hat{K}_\mu^\dagger, \tag{2}$$

where the Kraus operators $\hat{K}_\mu$ satisfy the property $\Sigma_\mu\hat{K}_\mu^\dagger\hat{K}_\mu=\mathbb{1}_s$. Unitary evolution of the quantum system is a special case in which there is only one nonzero term in the operator sum (2). If there are two or more terms, the pure initial state becomes mixed. For the mixed thermal environment, not only entanglement but also classical correlation of the system and environment can make the system evolve from the initial pure state into a mixed one. The mutual information between the system and the environment is nonzero if the information for the total system is not the same as the total sum of information for each subsystem. A nonzero mutual information is due to classical correlation and/or entanglement [4]. Thus there being nonzero mutual information between the system and environment is a sufficient condition for the initial pure system to have evolved into a mixed state. The action of $\hat{K}_\mu$ projects the system into a pure state of $\hat{K}_\mu\hat{\rho}\hat{K}_\mu^\dagger$ when the initial state is pure.

In the context of cavity quantum electrodynamics, the study of the interaction with atoms has been used to look for the generation of entanglement between two atoms when the two atoms are present simultaneously in the cavity [5] and when the two atoms interact consecutively with the cavity [6]. In these studies, the cavity is normally prepared in the vacuum and by the superposition of processes involving depositing and not depositing one photon into the cavity, the two atoms can evolve into their entangled state. However, we are interested in the possibility of entanglement via a chaotic thermal field about which we have only minimum information.

For simplicity, we consider identical two-level atoms 1 and 2 which are coupled to a single-mode thermal field with the same coupling constant γ. The ground and excited states for the atom i ($i=1,2$) are, respectively, denoted by $|g\rangle_i$ and $|e\rangle_i$. The cavity mode is assumed to be resonant with the atomic transition frequency. Under the rotating wave approximation, the Hamiltonian in the interaction picture is

$$\hat{H}_I=\hbar\gamma\sum_{i=1,2}(\hat{a}\hat{\sigma}_i^+ + \hat{a}^\dagger\hat{\sigma}_i^-) \tag{3}$$

where the atomic transition operators are $\hat{\sigma}_i^-=|g\rangle_i\langle e|$ and $\hat{\sigma}_i^+=|e\rangle_i\langle g|$.

The density operator for the combined atom-field system follows a unitary time evolution generated by the evolution operator $\hat{U}(t)=\exp(-i\hat{H}t/\hbar)$. For the interaction of two two-level atoms with a single-mode field, using Hamiltonian (3) and Taylor expansions of sine and cosine functions, we find that the analytical form of the evolution operator is given in the atomic basis $\{|ee\rangle,|eg\rangle,|ge\rangle,|gg\rangle\}$ by

$$\hat{U}(t)=\begin{pmatrix} 2\gamma^2\hat{a}(\hat{C}-\hat{\Theta})\hat{a}^\dagger+1 & -i\gamma\hat{a}\hat{S} & -i\gamma\hat{a}\hat{S} & 2\gamma^2\hat{a}(\hat{C}-\hat{\Theta})\hat{a} \\ -i\gamma\hat{S}\hat{a}^\dagger & \frac{1}{2}(\cos\hat{\Omega}t+1) & \frac{1}{2}(\cos\hat{\Omega}t-1) & -i\gamma\hat{S}\hat{a} \\ -i\gamma\hat{S}\hat{a}^\dagger & \frac{1}{2}(\cos\hat{\Omega}t-1) & \frac{1}{2}(\cos\hat{\Omega}t+1) & -i\gamma\hat{S}\hat{a} \\ 2\gamma^2\hat{a}^\dagger(\hat{C}-\hat{\Theta})\hat{a}^\dagger & -i\gamma\hat{a}^\dagger\hat{S} & -i\gamma\hat{a}^\dagger\hat{S} & 2\gamma^2\hat{a}^\dagger(\hat{C}-\hat{\Theta})\hat{a}+1 \end{pmatrix} \quad (4)$$

where $\hat{\Omega}^2=\hat{\Theta}^{-1}=2\gamma^2(2\hat{a}^\dagger\hat{a}+1)$ and the time-dependent operators $\hat{C}$ and $\hat{S}$ are defined by

$$\hat{C}=\hat{\Theta}\cos\hat{\Omega}t \quad \text{and} \quad \hat{S}=\hat{\Omega}^{-1}\sin\hat{\Omega}t. \quad (5)$$

The thermal radiation field with its mean photon number $\bar{n}$ is a weighted mixture of Fock states and its density operator is represented by $\hat{\rho}_E=\Sigma_n P_n|n\rangle\langle n|$ where the weight function P_n is

$$P_n=\frac{\bar{n}^n}{(1+\bar{n})^{n+1}}. \quad (6)$$

We are interested in the evolution of the quantum system; hence the time-dependent density operator $\hat{\rho}_s(t)$ is obtained by tracing over the field variables as in Eq. (2): $\hat{\rho}_s(t)=\Sigma_{nm}P_n\langle m|\hat{U}(t)|n\rangle\hat{\rho}_s(0)\langle n|\hat{U}^\dagger(t)|m\rangle$. We denote the ijth element of the matrix $\langle m|\hat{U}|n\rangle$ by U_{ij}^{nm}. Using the orthogonality of the Fock state, $\hat{\rho}_s(t)$ is obtained in the Kraus representation:

$$\hat{\rho}_s(t)=\sum_{n=0}^{\infty}\sum_{\mu=1}^{5}P(n)\hat{K}_\mu^n\hat{\rho}_s(0)\hat{K}_\mu^{n\dagger} \quad (7)$$

where the operators are

$$\hat{K}_1^n=\mathrm{diag}(U_{11}^{nn},U_{22}^{nn},U_{33}^{nn},U_{44}^{nn})+U_{23}^{nn}(|eg\rangle\langle ge|+\text{H.c.}),$$

$$\hat{K}_2^n=\sqrt{2}U_{12}^{nn-1}|ee\rangle\langle s|+\sqrt{2}U_{24}^{nn-1}|s\rangle\langle gg|,$$

$$\hat{K}_3^n=\sqrt{2}U_{21}^{nn+1}|s\rangle\langle ee|+\sqrt{2}U_{42}^{nn+1}|gg\rangle\langle s|,$$

$$\hat{K}_4^n=U_{14}^{nn-2}|ee\rangle\langle gg|,$$

$$\hat{K}_5^n=U_{41}^{nn+2}|gg\rangle\langle ee|, \quad (8)$$

with H.c. denoting the Hermitian conjugate and $|s\rangle=(|eg\rangle+|ge\rangle)/\sqrt{2}$. Note that $\hat{K}_1^n$ determines a process that does not change the mean energy of the field while $\hat{K}_2^n$ and $\hat{K}_4^n$ determine, respectively, one- and two-photon absorption processes. $\hat{K}_3^n$ and $\hat{K}_5^n$, respectively, describe atomic transitions through one- and two-photon emission processes.

Each Kraus operator projects the atomic state into a pure state if the atomic state is initially pure. If the atoms are initially in their excited states the only operation that projects the system into an entangled state is $\hat{K}_3^n$. For a two-qubit system described by the density operator $\hat{\rho}$, a measure of entanglement can be defined in terms of the negative eigenvalues of the partial transposition [4]:

$$\mathcal{E}=-2\sum_i \mu_i^-, \quad (9)$$

where μ_i^- are the negative eigenvalues of the partial transposition of $\hat{\rho}$. When $\mathcal{E}=0$ the two qubits are separable [7] and $\mathcal{E}=1$ indicates maximum entanglement between them. By substituting the initial condition of the atoms, i.e., $\hat{\rho}_s=\hat{\rho}_s(0)=|ee\rangle\langle ee|$, into Eq. (7), we find

$$\hat{\rho}_s(t)=A_e|ee\rangle\langle ee|+2A_s|s\rangle\langle s|+A_g|gg\rangle\langle gg|, \quad (10)$$

where $A_e=\Sigma_nP_n(U_{11}^{nn})^2$, $A_s=\Sigma P_n|U_{21}^{nn+1}|^2$, and $A_g=\Sigma_nP_n(U_{41}^{nn+2})^2$. The eigenvalues of its partial transposition are then $\mu_0=A_s$ and $2\mu_\pm=(A_e+A_g)\pm[(A_e+A_g)^2-4A_eA_g+4A_s^2]^{1/2}$. It is obvious that μ_0 and μ_+ are always positive. The eigenvalue μ_- becomes negative if and only if $A_s>\sqrt{A_eA_g}$. In order to analyze this case, let us first consider the possibility of entangling two atoms when the field is in a Fock state $|\ell\rangle$, i.e., $P_n=\delta_{n,\ell}$. Using the values of U_{ij}^{nm} in Eq. (4), it is straightforward to prove that $|U_{21}^{\ell\ell+1}|^2\leqslant|U_{11}^{\ell\ell}U_{41}^{\ell\ell+2}|$ regardless of ℓ and the interaction time. It is interesting to note that, when two atoms are initially in their excited states, they cannot be entangled by simultaneous interaction with a Fock state whose energy is definite. In fact,

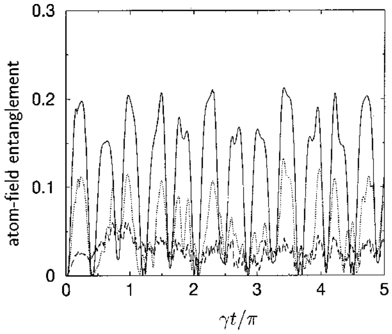

FIG. 1. Single atom-field entanglement against the interaction time when the pair of atoms is initially prepared in the state $|ee\rangle$ for $\bar{n}=0.1$ (solid), 1.0 (dotted), and 10.0 (dashed).

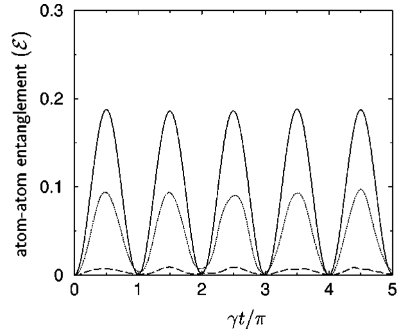

FIG. 2. Atom-atom entanglement induced by interaction with a thermal field when the atoms are initially prepared in $|eg\rangle$ for $\bar{n}=0.1$ (solid), 1.0 (dotted), and 10.0 (dashed).

the atoms are not entangled even via the vacuum ($\bar{n}=0$) in this case. For their interaction with a thermal field, $\sqrt{A_e A_g}$ has a lower bound $\Sigma_n P_n |U_{11}^{nn} U_{22}^{mm}|$ which, extending the analysis for the Fock state, is found to be larger than A_s. This proves that when Fock states cannot entangle two atoms by simultaneous interaction, neither can their classical mixture, for example a thermal state.

It has been shown that a single two-level atom and a thermal field can be entangled due to their linear interaction, regardless of the temperature of the field [2]. If there are two atoms interacting with the thermal field, can we still see the entanglement between the field and any single atom? As discussed earlier, by interacting with the thermal field, a pure quantum system becomes mixed, so we know that the mutual information between the system and thermal field should become nonzero. However, this does not mean that there should be atom-field entanglement because the mutual information may include classical correlation as well. In order to consider the entanglement between a single atom and the field, we first find the single atom-field density operator $\hat{\rho}_{a-f}(t)$ by tracing the total density operator $\hat{\rho}(t)$ over the variables of the other atom. The atom is described as a two-dimensional system while the field is in infinite-dimensional space; the density operator $\hat{\rho}_{a-f}(t)$ is thus defined in a $2\times\infty$ dimensional space. An analytical form of entanglement is not available for the system in $2\times\infty$ space, so that Bose *et al.* project the atom-field state onto a 2×2 subspace and then compute the entanglement of formation for each of the outcomes. As this projection can be done by local actions, it does not increase the entanglement and the result gives a lower bound on the entanglement of the entire atom-field system. We have calculated the average of entanglements in this way for projected states, and this is plotted in Fig. 1 for various values of the mean photon number $\bar{n}$. We see that the atom and field are always entangled for $t>0$ despite the presence of the other atom. After a little algebra we can analytically prove this as a lower bound of entanglement occurs in the subspace of $|e,1\rangle, |e,0\rangle, |g,1\rangle$, and $|g,0\rangle$ when $P_0^2 |U_{21}^{01}|^2 (U_{11}^{00})^2 > 0$. The atom-field entanglement is a key to the decoherence process.

More interestingly, we find in Fig. 2 that atoms are entangled by a thermal field when one of the atoms is prepared in its excited state and the other in the ground state. If the initial state is $|eg\rangle$, the atomic system is represented by the density operator

$$\hat{\rho}_s(t)=\sum_n P_n\{(U_{22}^{nn}|eg\rangle+U_{23}^{nn}|ge\rangle)(\langle eg|U_{22}^{nn}+\langle ge|U_{23}^{nn}) + |U_{12}^{nn-1}|^2|ee\rangle\langle ee| + |U_{42}^{nn+1}|^2|gg\rangle\langle gg|\}. \quad (11)$$

One of the partial transposition eigenvalues may be negative when $4\Sigma_{nm}P_n P_m(-|U_{12}^{nn-1}|^2|U_{42}^{mm+1}|^2+U_{23}^{nn}U_{22}^{nn}U_{23}^{mm}U_{22}^{mm})>0$. Substituting U_{ij}^{nm} in Eq. (4), the left hand side of the inequality is found to be given by $(1/4)\Sigma_{nm}P_n P_m \sin^2\Omega_n t \sin^2\Omega_m t/[(2n+1)(2m+1)]$, which is always positive. It is surprising that a thermal state, which is a highly chaotic state in an infinite-dimensional Hilbert space, can entangle two qubits depending on their atomic initial preparation.

Let us now consider the situation when the atoms are initially both in their ground states. In this case, $\hat{K}_2^n$ in Eq. (8) is the only operator that projects the atoms into an entangled state. The density operator for the atoms has the same form as Eq. (10) but with different parameters: $A_e=\Sigma_n P_n (U_{14}^{nn-2})^2$, $A_s=\Sigma P_n|U_{24}^{nn-1}|^2$, and $A_g=\Sigma_n P_n (U_{44}^{nn})^2$. The measure of entanglement (9) is calculated and plotted in Fig. 3, which clearly shows entanglement between the two atoms for some interaction times. As we did for the initial state $|ee\rangle$, we again first consider the case of Fock-state interaction, in order to provide insights for the analysis of the more complicated case of thermal-field interaction. For the interaction of atoms with the Fock state $|\ell\rangle$, one of the eigenvalues for the partial transposition of $\hat{\rho}_s(t)$ is negative when $E_g^\ell \equiv |U_{24}^{\ell\ell-1}|^2 - |U_{44}^{\ell\ell}U_{14}^{\ell\ell-2}|>0$. Substituting the values U_{ij}^{nm} in Eq. (4) into the definition of E_g^ℓ, we find

$$E_g^\ell = \frac{1-c}{2\ell-1}\left[\ell(1+c)-\frac{1}{d}\left|\ell(1+c)-1\right|\right] \quad (12)$$

where $c=\cos\Omega_{n-1}t$ and $d=(2\ell-1)/\sqrt{\ell(\ell-1)}$. The atoms are not entangled by the vacuum interaction as $E_g^\ell=0$ for $\ell=0$. We find that the two atoms *are* entangled when $c>c_s\equiv -1+1/(\ell+d)$ for any Fock state ($\ell\geqslant 1$). This

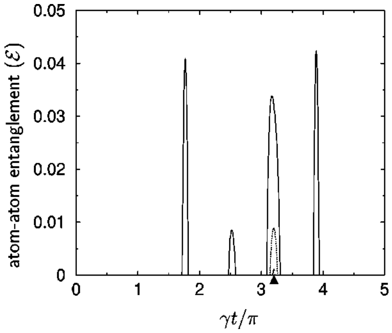

FIG. 3. Atom-atom entanglement induced by interaction with a thermal field of $\bar{n}=1$ when the atoms are initially in a pure state ground state (solid), i.e., $\lambda=0$ in Eq. (13), and in a mixed state of $\lambda=0.05$ (dotted). In the case of $\lambda=0.065$, entanglement is shown by a dashed line at the spot indicated by a triangle.

means that it is possible to entangle two atoms by linear interaction with any Fock state of the cavity field if the atoms are initially prepared in their ground states, in sharp contrast to the case of the initial preparation of atoms in the excited states. In particular, for the Fock state of $|1\rangle$, the atoms are always entangled except when $\cos\Omega_{\ell-1}t=1$. The critical parameter c_s takes the minimum value -1 at $\ell=1$ and maximizes to around -0.76 at $\ell=2$, and then gradually decreases as ℓ gets larger. The maximum value of E_g^{ℓ} is $1/\ell$ at $c=-1+1/\ell$ so that entanglement is smaller when ℓ gets larger.

We now consider the two atoms interacting with a thermal field. When the atoms are initially in $|eg\rangle$, entanglement is induced by a thermal field regardless of its mean photon number. Is it still true for the atoms initially in $|gg\rangle$? We find that the answer is no. The thermal state is a weighted mixture of Fock states with the weight P_ℓ given by Eq. (6). The weight factor P_ℓ is a decreasing function of ℓ. When $\bar{n}$ is very small, because of the vacuum dominance the atoms are not very much entangled. As $\bar{n}$ gets moderately larger, P_1 becomes more pronounced. Because $|1\rangle\langle 1|$ gives strong atomic entanglement, the atoms are entangled more in this case. However, as $\bar{n}\gg 1$, the weight function becomes flat and cancellation between the component Fock states washes out the entanglement feature.

So far we have considered the case when the atoms are in their pure states. Bose *et al.* [2] comment that the purity of the qubit is an important ingredient to see entanglement between the qubit and a massive system, but did not show the dynamics of the entanglement when the qubit is initially prepared in its mixed state. It is not possible to analyze the atom-field entanglement when the atoms are in a mixed state because of the lack of tools to analyze such a mixed $2\times\infty$ system. However, we can analyze how the initial mixedness of the atoms affects the entanglement between the atoms. We assume that each atom is initially prepared in a thermal mixture so that their initial density operators are

$$\hat{\rho}_s=\Pi_{i=1,2}[\lambda|e\rangle_i\langle e|+(1-\lambda)|g\rangle_i\langle g|] \qquad (13)$$

where λ depends on the temperature of the atoms. Using the Kraus representation (7) for the initial condition Eq. (13), the evolution of the density operator for the two atoms is found in the form $\hat{\rho}_s(t)=A_e|ee\rangle\langle ee|+A_s|s\rangle\langle s|+A_a|a\rangle\langle a|+A_g|gg\rangle\langle gg|$ where $|a\rangle=(|eg\rangle-|ge\rangle)/\sqrt{2}$ and $A_{e,s,a,g}$ are time-dependent coefficients. The atoms are then entangled when $(A_s-A_a)-2\sqrt{A_eA_g}>0$. The measure of entanglement is plotted for the interaction of atoms with a low temperature field with $\bar{n}=1$. It is remarkable to see that, with even a very small amount of mixture with $\lambda=0.065$, atomic entanglement is nearly washed out.

In conclusion, we have demonstrated the very interesting result that two atoms can become entangled through their interaction with a highly chaotic system depending on the initial preparation of the atoms. This study provides a degree of analytical understanding of the decoherence mechanism for a quantum system composed of a few qubits when the reservoir with which they interact retains some memory [8]. In our case, the memory is due to the fact that the Rabi frequency, which determines the dynamics of one atom, is increased or decreased by deexcitation or excitation of the other atom.

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