Three-mode squeezed vacuum state in Fock space as an entangled state

Hongyi Fan[1,2] and Guichuan Yu[2]
[1]CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China
[2]Department of Material Science and Engineering, University of Science and Technology of China Hefei, Anhui, 230026, China
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We show that the operator $U=\exp[-ir(Q_1P_2+Q_2P_3+Q_3P_1)]$ is a three-mode squeezing operator for the three-mode quadratures exhibiting the standard squeezing, the corresponding squeezed vacuum state in three-mode Fock space is derived by virtue of the technique of integration within ordered product of operators. The entanglement involved in such a state is explained. The optical network for producing the three-mode squeezed state is proposed.

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I. INTRODUCTION

In recent years the entangled states and entanglement [1,2] have brought much attention and interests of physicists because of their applications in quantum communication [3]. The two-mode squeezed state, which is composed by idler mode and signal mode resulting from a parametric down conversion amplifier [4], is a typical entangled state of continuous variable. Theoretically, it is constructed by the two-mode squeezing operator S acting on the vacuum state $|00\rangle$,

$$S|00\rangle=\exp[\lambda(a_1a_2-a_1^\dagger a_2^\dagger)]|00\rangle$$
$$=\sec\mathrm{h}\lambda\,\exp(-a_1^\dagger a_2^\dagger\tanh\lambda)|00\rangle, \quad (1)$$

where λ is a squeezing parameter [5]. Using the relation between the Bose operators $(a_i,a_i^\dagger)$ and the coordinate, momentum operators

$$Q_i=\frac{1}{\sqrt{2}}(a_i+a_i^\dagger), \quad P_i=\frac{1}{i\sqrt{2}}(a_i-a_i^\dagger), \quad (2)$$

one can recast S into the form

$$S=\exp[i\lambda(Q_1P_2+Q_2P_1)]. \quad (3)$$

It is now clear that the two-mode squeezing operator S actually squeezes the entangled state $|\eta\rangle$ [6,7],

$$|\eta\rangle=\exp\left(-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^* a_2^\dagger+a_1^\dagger a_2^\dagger\right)|00\rangle_{12},$$
$$\eta=\eta_1+i\eta_2, \quad (4)$$

i.e.,

$$S|\eta\rangle=\frac{1}{\mu}|\eta/\mu\rangle, \quad \mu=e^\lambda. \quad (5)$$

The $|\eta\rangle$ state was constructed in Ref. [6] according to the idea of entanglement originated from Einstein, Podolsky, and Rosen in their argument that quantum mechanics is incomplete [8]. $|\eta\rangle$ obeys the eigenvector equation,

$$(Q_1-Q_2)|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle, \quad (P_1+P_2)|\eta\rangle=\sqrt{2}\eta_2|\eta\rangle \quad (6)$$

and the orthonormal-complete relation

$$\int\frac{d^2\eta}{\pi}|\eta\rangle\langle\eta|=1, \quad \langle\eta'|\eta\rangle=\pi\delta(\eta-\eta')\delta(\eta^*-\eta'^*). \quad (7)$$

Thus the two-mode squeezing operator has a neat and natural representation on the $\langle\eta|$ basis,

$$S=\frac{1}{\mu}\int\frac{d^2\eta}{\pi}|\eta/\mu\rangle\langle\eta|, \quad \mu=e^\lambda. \quad (8)$$

By introducing the two-mode quadrature operators as in Ref. [5],

$$x_1=\frac{1}{2}(Q_1+Q_2), \quad x_2=\frac{1}{2}(P_1+P_2), \quad (9)$$

we use Eq. (8) and Eq. (6) immediately derives

$$S^\dagger x_2S=\frac{1}{\mu^2}\int\frac{d^2\eta'}{\pi}|\eta'\rangle\langle\eta'/\mu|\int\frac{d^2\eta}{\pi}\frac{\eta_2}{\sqrt{2}\mu}|\eta/\mu\rangle\langle\eta|$$
$$=\int\frac{d^2\eta}{\pi}\frac{\eta_2}{\sqrt{2}\mu}|\eta\rangle\langle\eta|=x_2/\mu.$$

The Fourier transformation of the state $|\eta\rangle$ is

$$|\xi\rangle=\int\frac{d^2\eta}{2\pi}|\eta\rangle e^{(\eta\xi^*-\eta^*\xi)/2}$$
$$=\exp\left(-\frac{1}{2}|\xi|^2+\xi a_1^\dagger+\xi^* a_2^\dagger-a_1^\dagger a_2^\dagger\right)|00\rangle_{12}, \quad (10)$$

which obeys the eigenvector equations,

$$(Q_1+Q_2)|\xi\rangle=\sqrt{2}\xi_1|\xi\rangle, \quad (P_1-P_2)|\xi\rangle=\sqrt{2}\xi_2|\xi\rangle.$$

In the $\langle\xi|$ representation S is

$$S=\mu\int\frac{d^2\eta}{\pi}|\mu\xi\rangle\langle\xi|.$$

One can easily show

$$S^\dagger x_1S=\mu x_1. \quad (11)$$

The variances of x_1 and x_2 in the state $S|00\rangle$ are in the *standard form*, i.e.,

$$\langle 00|S^\dagger x_2^2 S|00\rangle = \langle 00|(x_2/\mu)^2|00\rangle = e^{-2\lambda},$$
$$\langle 00|S^\dagger x_1^2 S|00\rangle = e^{2\lambda}, \tag{12}$$

recall that for single-mode squeezing case, the *standard squeezing* for the coordinate and the momentum operators is also $Q_1 \to e^{\lambda}Q_1$, $P_1 \to e^{-\lambda}P_1$. When one generalizes the form of S to Refs. [9,10]

$$S' = \exp[-i(\lambda_1 Q_1 P_2 + \lambda_2 Q_2 P_1)], \tag{13}$$

and using the technique of integration within an ordered product (IWOP) of operators as we did in Ref. [9], then one can get a one- and two-mode combined squeezed state,

$$S'|00\rangle = \frac{2}{\sqrt{L}}\exp\left\{\frac{1}{L}[(a_2^{\dagger 2} - a_1^{\dagger 2})\sinh^2\lambda \sinh 2\gamma + 2\sinh 2\lambda \cosh \gamma a_1^\dagger a_2^\dagger]\right\}|00\rangle, \tag{14}$$

where

$$\lambda_1 = \lambda e^{\gamma}, \quad \lambda_2 = \lambda e^{-\gamma}, \quad L = 4\cosh^2\lambda(1 + \sinh^2\gamma \tanh^2\lambda).$$

Clearly, when $e^{\gamma} = 1$, $\sinh\gamma = 0$, Eq. (14) reduces to the ordinary two-mode squeezed state (1). It is proved in Ref. [9] that when

$$0 < \tanh\lambda < \frac{1}{1 + \cosh\gamma}, \quad \lambda > 0, \tag{15}$$

the state $S'|00\rangle$ exhibits more stronger squeezing than the ordinary two-mode squeezed state. Then a question naturally arises: Is the following unitary operator,

$$U = \exp[ir(Q_1 P_2 + Q_2 P_3 + Q_3 P_1)], \tag{16}$$

also a squeezing operator in the three-mode Fock space? If yes, what is its corresponding squeezed vacuum state? Is it also an entangled state? Because Eq. (16) is more complicated than Eq. (13), and Q_1P_3, Q_3P_2, Q_2P_1 terms do not make up a closed Lie algebra, it is hardly to use Lie algebra to analyze U. To answer these questions we must first derive $U's$ normal product form and then analyze if the squeezing exists, and how behaves the state $U|000\rangle$. The work is arranged as: in Secs. II we use the IWOP technique to normally ordered expand U. In Secs. III–IV we examine the properties of the state $U|000\rangle$, we find that it just makes the variances of the three-mode quadrature operators behave as the same rule as shown in Eq. (12) for the two-mode case. In Sec. V we discuss how to design an optical network to realize the new three-mode squeezed vacuum state.

II. NORMAL PRODUCT FORM OF *U*

Because operators Q_1P_2, Q_2P_3, and Q_3P_1 neither commute with each other, nor make up any close Lie algebra relation by themselves, it seems difficult to disentangle U. Thus we must appeal to the IWOP technique. Rewriting U as

$$U = \exp\left[ir(Q_1, Q_2, Q_3)A\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix}\right] = \exp[irQ_iA_{ij}P_j],$$
$$A = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \quad i,j = 1,2,3, \tag{17}$$

where the repeated indices represent the Einstein summation notation. Using the Baker-Hausdorff formula we see

$$U^{-1}Q_kU = Q_k - rQ_iA_{ik} + \frac{1}{2!}ir^2[Q_iA_{ij}P_j, Q_lA_{lk}] + \cdots$$
$$= Q_i(e^{-rA})_{ik} = (e^{-r\tilde{A}})_{ki}Q_i,$$
$$U^{-1}P_kU = P_k + rA_{kj}P_j + \frac{1}{2!}ir^2[A_{kj}P_j, Q_lA_{lm}P_m] + \cdots$$
$$= (e^{rA})_{ki}P_i. \tag{18}$$

This implies that the action of U on the three-mode coordinate eigenstate $|\vec{q}\rangle$ is

$$U|\vec{q}\rangle = |\Lambda|^{1/2}|\Lambda\vec{q}\rangle, \quad \Lambda = e^{-r\tilde{A}}, \quad |\Lambda| \equiv \det\Lambda, \tag{19}$$

and U has the representation

$$U = \int d^3q U|\vec{q}\rangle\langle\vec{q}| = |\Lambda|^{1/2}\int d^3q|\Lambda\vec{q}\rangle\langle\vec{q}|, \quad U^\dagger = U^{-1}. \tag{20}$$

In fact, using Eq. (14) we have

$$U^{-1}Q_kU = |\Lambda|\int d^3q|\vec{q}\rangle\langle\Lambda\vec{q}|Q_k\int d^3q'|\Lambda\vec{q}'\rangle\langle\vec{q}'|$$
$$= (\Lambda Q)_k,$$

which is consistent with Eq. (18). Thus

$$U = \exp[irQ_iA_{ij}P_j] = \sqrt{\det e^{-r\tilde{A}}}\int d^3q|e^{-r\tilde{A}}\vec{q}\rangle\langle\vec{q}|. \tag{21}$$

Using

$$|\vec{q}\rangle = \pi^{-3/4}{:}\exp\left[-\frac{1}{2}\tilde{\vec{q}}\vec{q} + \sqrt{2}\tilde{\vec{q}}a^\dagger - \frac{1}{2}\tilde{a}^\dagger a^\dagger\right]{:}|0\rangle; \tag{22}$$

here $\tilde{a}^\dagger = (a_1^\dagger, a_2^\dagger, a_3^\dagger)$ and $\tilde{\vec{q}} = (q_1, q_2, q_3)$, and

$$|0\rangle\langle 0| = {:}\exp[-\tilde{a}^\dagger a]{:},$$

as well as the IWOP technique we can put U into normal ordering form

$$U = \pi^{-3/2}|\Lambda|^{1/2}\int_{-\infty}^{\infty}d^3q{:}\exp\left[-\frac{1}{2}\tilde{\vec{q}}(1 + \tilde{\Lambda}\Lambda)\vec{q} + \sqrt{2}\tilde{\vec{q}}(\tilde{\Lambda}a^\dagger + a) - \frac{1}{2}(\tilde{a}a + \tilde{a}^\dagger a^\dagger) - \tilde{a}^\dagger a\right]{:}. \tag{23}$$

Using the mathematical formula

$$\int_{-\infty}^{\infty} d^r x \exp[-\tilde{x}Fx+\tilde{x}v]=\pi^{n/2}(\det F)^{1/2}\exp\left[\frac{1}{4}\tilde{v}F^{-1}v\right], \tag{24}$$

we perform the integration in Eq. (23) and obtain the explicit normal ordering form of U

$$U=\left[\det\Lambda \Big/ \det\left(\frac{1+\tilde{\Lambda}\Lambda}{2}\right)\right]^{1/2}:\exp\left\{(\tilde{\Lambda}a^\dagger+a)(1+\tilde{\Lambda}\Lambda)^{-1}(\tilde{\Lambda}a^\dagger+a)-\frac{1}{2}(\tilde{a}^\dagger a^\dagger+\tilde{a}a)-\tilde{a}^\dagger a\right\}:.$$

Let $N=\frac{1}{2}(\tilde{\Lambda}\Lambda+I)$, U is simplified as

$$U=|\Lambda|^{1/2}|N|^{-1/2}\exp\left[\frac{1}{2}\tilde{a}^\dagger(\Lambda N^{-1}\tilde{\Lambda}-I)a^\dagger\right]:\exp[\tilde{a}^\dagger(\Lambda N^{-1}-I)a]:\exp\left[\frac{1}{2}\tilde{a}(N^{-1}-I)a\right]. \tag{25}$$

Thus we see that the IWOP plays a decisive role in obtaining Eq. (25).

III. THE NEW THREE-MODE SQUEEZED VACUUM STATE

Note $A^3=I$, the unit 3×3 matrix, from the Cayley-Hamilton theorem we know that the expanding form of $\exp(-rA)$ must be

$$\tilde{\Lambda}=\exp(-rA)=a(r)I+b(r)A+c(r)A^2, \tag{26}$$

where $a(r)$ is

$$a(r)=\sum_{n=0}^{\infty}\frac{(-r)^{3n}}{(3n)!},\quad b(r)=\sum_{n=0}^{\infty}\frac{(-r)^{3n+1}}{(3n+1)!},$$

$$c(r)=\sum_{n=0}^{\infty}\frac{(-r)^{3n+2}}{(3n+2)!}.$$

To determine $a(r)$, $b(r)$, and $c(r)$, we take A being 1, $e^{i(2/3)\pi}$, and $e^{i(4/3)\pi}$, respectively, then we have

$$\begin{aligned}\exp(-r)&=a(r)+b(r)+c(r),\\ \exp(-re^{i(2/3)\pi})&=a(r)+b(r)e^{i(2/3)\pi}+c(r)e^{i(4/3)\pi},\\ \exp(-re^{i(4/3)\pi})&=a(r)+b(r)e^{i(4/3)\pi}+c(r)e^{i(2/3)\pi}.\end{aligned} \tag{27}$$

Its solution is

$$\begin{aligned}a(r)&=\frac{1}{3}[e^{-r}+e^{-re^{i(2/3)\pi}}+e^{-re^{i(4/3)\pi}}]\\ &=\frac{1}{3}\left[e^{-r}+2e^{r/2}\cos\left(\frac{\sqrt{3}}{2}r\right)\right],\\ b(r)&=\frac{1}{3}[e^{-r}+e^{i(4/3)\pi}e^{-re^{i(2/3)\pi}}+e^{i(2/3)\pi}e^{-re^{i(4/3)\pi}}]\\ &=\frac{1}{3}\left[e^{-r}+2e^{r/2}\cos\left(\frac{\sqrt{3}}{2}r+\frac{2}{3}\pi\right)\right],\\ c(r)&=\frac{1}{3}[e^{-r}+e^{i(2/3)\pi}e^{-re^{i(2/3)\pi}}+e^{i(4/3)\pi}e^{-re^{i(4/3)\pi}}]\\ &=\frac{1}{3}\left[e^{-r}+2e^{r/2}\cos\left(\frac{\sqrt{3}}{2}r+\frac{4}{3}\pi\right)\right].\end{aligned} \tag{28}$$

It then follows

$$N=\frac{1}{2}(\tilde{\Lambda}\Lambda+I)=\frac{1}{6}\begin{pmatrix} f & g & g\\ g & f & g\\ g & g & f\end{pmatrix},\quad f=3+e^{-2r}+2e^{r},$$
$$g=e^{-2r}-e^{r}. \tag{29}$$

Substituting Eqs. (26), (28), and (29) into Eq. (25) yields

$$\begin{aligned}U=C\exp&\left\{\frac{1}{6(1+e^r)(1+e^{2r})}\left[(e^r-1)^3\sum_{i=1}^{3}a_i^{\dagger 2}+4(1-e^{3r})\sum_{i<j}a_i^\dagger a_j^\dagger\right]\right\}\\ &\times:\exp\frac{1}{3(1+e^r)(1+e^{2r})}\left\{-\left[3+e^r+e^{2r}+3e^{3r}-4e^{r/2}(1+e^{2r})\cos\left(\frac{\sqrt{3}r}{2}\right)\right]\sum_{i=1}^{3}a_i^\dagger a_i\right\}\\ &+2e^{r/2}\left[e^{r/2}(1+e^r)+2(1+e^{2r})\cos\left(\frac{\sqrt{3}r}{2}+\frac{4}{3}\pi\right)\right](a_1^\dagger a_2+a_2^\dagger a_3+a_3^\dagger a_1)\\ &+2e^{r/2}\left[e^{r/2}(1+e^r)+2(1+e^{2r})\cos\left(\frac{\sqrt{3}r}{2}+\frac{2}{3}\pi\right)\right](a_1^\dagger a_3+a_3^\dagger a_2+a_2^\dagger a_1):\\ &\times\exp\left\{\frac{-1}{6(1+e^r)(1+e^{2r})}\left[(e^r-1)^3\sum_{i=1}^{3}a_i^2+4(1-e^{3r})\sum_{i<j}a_ia_j\right]\right\},\end{aligned} \tag{30}$$

where

$$C=|\Lambda|^{1/2}|N|^{-1/2}=2/[(1+e^{r})\sqrt{e^{-r}\cosh r}]. \tag{31}$$

Operating U in Eq. (30) on the three-mode vacuum state leads to the new three-mode squeezed vacuum state

$$U|000\rangle=C\exp\left\{\frac{1}{6(1+e^{r})(1+e^{2r})}\left[(e^{r}-1)^{3}\times\sum_{i=1}^{3}a_i^{\dagger 2}+4(1-e^{3r})\sum_{i<j}^{3}a_i^{\dagger}a_j^{\dagger}\right\}|000\rangle. \tag{32}$$

Especially, when $e^{r}\rightarrow 0$,

$$U|000\rangle|_{r\rightarrow -\infty}\sim\left\{\exp\frac{1}{6}\left[-\sum_{i=1}^{3}a_i^{\dagger 2}+4\sum_{i<j}^{3}a_i^{\dagger}a_j^{\dagger}\right]\right\}|000\rangle\equiv|\,\rangle_s. \tag{33}$$

We can check the validity of C in the following way. Let

$$\lambda_1=\frac{(1-e^{r})^{3}}{(1+e^{r})(1+e^{2r})},\quad \lambda_2=\frac{(1-e^{3r})}{(1+e^{r})(1+e^{2r})},$$

and

$$M=-\frac{1}{6}\begin{pmatrix}\lambda_1 & -2\lambda_2 & -2\lambda_2\\ -2\lambda_2 & \lambda_1 & -2\lambda_2\\ -2\lambda_2 & -2\lambda_2 & \lambda_1\end{pmatrix},$$

then C, the normalization coefficient, can be alternately calculated by

$$\begin{aligned}1&=\langle 000|U^{\dagger}U|000\rangle\\ &=|C|^{2}\langle 000|\exp\{aM\tilde{a}\}\exp\{a^{\dagger}M\tilde{a}^{\dagger}\}|000\rangle.\end{aligned}$$

Using the operator identity [15] which is derived by the IWOP technique

$$\begin{aligned}&\exp\{a\sigma\tilde{a}\}\exp\{a^{\dagger}\tau\tilde{a}^{\dagger}\}\\ &=\left[\det\begin{pmatrix}I & -2\tau\\ -2\sigma & I\end{pmatrix}\right]^{-1/2}:\left\{\exp\frac{1}{2}(a^{\dagger}a)\right.\\ &\quad\left.\times\begin{pmatrix}I & -2\tau\\ -2\sigma & I\end{pmatrix}^{-1}\begin{pmatrix}a\\ a^{\dagger}\end{pmatrix}-a^{\dagger}a\right\}:,\end{aligned}$$

we obtain

$$\begin{aligned}\langle 000|U^{\dagger}U|000\rangle&=|C|^{2}\left[\det\begin{pmatrix}I & -2M\\ -2M & I\end{pmatrix}\right]^{-1/2}\\ &=|C|^{2}\left[\frac{64e^{4r}}{(1+e^{r})^{4}(1+e^{2r})^{2}}\right]^{-1/2}=1,\end{aligned}$$

which coincides with Eq. (31).

It is interesting to observe that $|\,\rangle_s$ is just the common eigenvector [16] of the three compatible Jacobian operators in three-body case with zero eigenvalues, i.e.,

$$(P_1+P_2+P_3)|\,\rangle_s=0\quad\left(\frac{\mu_2Q_2+\mu_3Q_3}{\mu_2+\mu_3}-Q_1\right)|\,\rangle_s=0,$$
$$(Q_3-Q_2)|\,\rangle_s=0 \tag{34}$$

as

$$\left[(P_1+P_2+P_3),\ \frac{\mu_2Q_2+\mu_3Q_3}{\mu_2+\mu_3}-Q_1\right]=0,$$
$$[Q_3-Q_2,(P_1+P_2+P_3)]=0. \tag{35}$$

Since the common eigenvector of three compatible Jacobian operators is an entangled state, the state $|\,\rangle_s$ is also an entangled state.

IV. VARIANCES OF THE THREE-MODE QUADRATURES IN $|\,\rangle_s$

The quadratures in the three-mode case should be defined as

$$X_1=\frac{Q_1+Q_2+Q_3}{\sqrt{6}},\quad X_2=\frac{P_1+P_2+P_3}{\sqrt{6}},\quad [X_1,X_2]=\frac{i}{2}. \tag{36}$$

The expectation values of the quadratures in the state $|\,\rangle_s$ is $\langle X_1\rangle=\langle X_2\rangle=0$ and using Eqs. (28), (36), and (25)–(26) we see that the corresponding variance is

$$\begin{aligned}(\Delta X_1)^{2}&={}_s\langle|x_1^{2}|\rangle_s\\ &=\frac{1}{6}\langle 000|U^{\dagger}X_1^{2}U|000\rangle\\ &=\frac{1}{12}\sum_{ji}(\Lambda\tilde{\Lambda})_{ij}=\frac{1}{4}e^{-2r},\end{aligned} \tag{37}$$

$$(\Delta X_2)^{2}={}_s\langle|X_2^{2}|\rangle_s=\frac{1}{12}\sum_{ji}(\Lambda\tilde{\Lambda})_{ij}^{-1}=\frac{1}{4}e^{2r}, \tag{38}$$

which has the similar standard form to the two-mode case as shown in Eq. (12). Equations (37)–(38) clearly indicates that U is the correct three-mode squeezing operator for the three-mode quadratures (36).

V. OPTICAL NETWORK FOR PRODUCING THE STATE (25)

In Refs. [11,12] it is pointed out that the basic operations of optical devices (beam splitters, mirrors, optical fibre, and phase shifter) based on quantum optics components is the transformation of a set of incoming states into another set by a unitary transformation. Such transformations are performed by using optical networks. The linear networks can be realized by use of passive optical elements. In other cases, one may need to include active elements, amplifiers, or parametric mixers. We recall that a two-mode entangled state can be produced when the symmetric 50:50 beam-splitter operation on a pair of input modes [1]: one is the zero-momentum eigenstate $|p=0\rangle_1$ and the other is the zero-position eigenstate $|x=0\rangle_2$, (they can be considered as two light fields maximally squeezed in P and X direction, respectively). In this section we hope to design such an optical network that the light beams (one mode of zero-position eigenstate $|x=0\rangle_1$ and two modes of zero-momentum eigenstates $p=0\rangle_2\otimes|p=0\rangle_3$) entering the three input ports of this network will be changed into a tripartite entangled state. In another word, we hope that the network plays the role of transforming three single-mode squeezed states (two light fields maximally squeezed in P direction and one light field in X direction) incident on the network to the entangled state $|\ \rangle_s$ in Eq. (33). In Fock space $|x=0\rangle_i$ and $|p=0\rangle_i$ are expressed as

$$|x=0\rangle_i \sim \exp\left(\frac{-1}{2}a_i^{\dagger 2}\right)|0\rangle_i,$$

$$|p=0\rangle_i \sim \exp\left(\frac{1}{2}a_i^{\dagger 2}\right)|0\rangle_i. \tag{39}$$

Hence this optical network, whose function is represented by a unitary operator R, should meet the following requirement,

$$R|x=0\rangle_1\otimes|p=0\rangle_2\otimes|p=0\rangle_3 \rightarrow |\ \rangle_s = \left\{\exp\frac{1}{6}\left[-\sum_{i=1}^{3}a_i^{\dagger 2}+4\sum_{i<j}^{3}a_i^\dagger a_j^\dagger\right]\right\}|000\rangle, \tag{40}$$

as Eq. (33) indicated. Combining Eqs. (39) and (40), and letting

$$E=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix},$$

we see that R should engender the transformation,

$$R(a_1^{\dagger 2}-a_2^{\dagger 2}-a_3^{\dagger 2})R^{-1}=R\tilde{a}^\dagger E a^\dagger R^{-1} = \frac{1}{3}\left[-\sum_{i=1}^{3}a_i^{\dagger 2}+4\sum_{i<j}^{3}a_i^\dagger a_j^\dagger\right] = \tilde{a}^\dagger B a^\dagger, \tag{41}$$

where

$$B=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix}.$$

Supposing

$$R\tilde{a}^\dagger R^{-1}=\tilde{a}^\dagger\tilde{G}, \quad R a_i R^{-1}=G_{ij}a_j=a_i', \tag{42}$$

then from Eq. (41) we see that G must satisfies the matrix equation

$$\tilde{G}EG=B. \tag{43}$$

Its solution is an orthogonal matrix,

$$G=\begin{pmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ -\sqrt{2/3} & 1/\sqrt{6} & 1/\sqrt{6}\end{pmatrix}. \tag{44}$$

A question of interest is to ask which types of mode interactions are necessary to obtain the optical transfer evolution in Eqs. (41) and (42). We need to know the Hamiltonians characterizing the interaction between the modes of light entering an appropriate network realized by the use of passive optical elements. In another words, We want to extract an interacting Hamiltonian from the unitary transformation (41) or (42) of quantum states. Recall in Refs. [13,14] we have proposed a systematic prescription for obtaining Hamiltonian for preassigned unitary transformations of quantum states. That is by mapping the classical c-number transformation in a coherent state basis onto quantum-mechanical operators of Fock space and using the IWOP technique to find the Hamiltonian. Here we start from the transformation a_i (which possesses the coherent state $|z_i\rangle$ as its eigenvector)$\rightarrow a_i'\equiv R a_i R^{-1}=G_{ij}a_j$ (which possesses eigenvector $|G_{ij}z_j\rangle$), to construct the following ket-bra integral operator in the coherent state representation

$$R=\int\prod_{i=1}^{3}\frac{d^2 z_i}{\pi}|G_{ij}z_j\rangle\langle z_i|. \tag{45}$$

Then preforming the integration with the IWOP technique we have

$$R=\int\prod_{i=1}^{3}\frac{d^2 z_i}{\pi}:\exp\left\{\sum_i\left(-|z_i|^2+\sum_j a_i^\dagger G_{ij}z_j\right) + z_i^* a_i - a_i^\dagger a_i\right\}:=:\exp\{\tilde{a}^\dagger(G-I)a^\dagger\}:. \tag{46}$$

Using relation

$$e^{\tilde{a}^\dagger\Lambda a^\dagger}=:e^{\tilde{a}^\dagger(e^\Lambda-I)a^\dagger}:,$$

we can put Eq. (46) into

$$R=\exp\{\tilde{a}^\dagger(\ln G)a^\dagger\}. \tag{47}$$

Let $\ln G=itK$, with $K^\dagger=K$, then the time-evolution operator is $R(t)=\exp\{it\tilde{a}^\dagger K a^\dagger\}$, the corresponding Hermitian Hamiltonian is

$$H=-\tilde{a}^\dagger K a^\dagger. \tag{48}$$

Experimentally, we can design a multiport optical network that can transform the input mode a_i to output mode a_i'. Since G is an orthogonal matrix, from the general form of such matrix,

$$G(\alpha,\beta,\gamma)=\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma-\sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix} \tag{49}$$

and comparing Eq. (44) with Eq. (49) we identify

$$\cos\alpha=\sqrt{2/5}\quad \sin\alpha=-\sqrt{3/5},\quad \cos\beta=\sqrt{1/6},\quad \sin\beta=\sqrt{5/6},$$
$$\cos\gamma=2/\sqrt{5},\quad \sin\gamma=1/\sqrt{5}. \tag{50}$$

From the general form of orthogonal matrix $G(\alpha,\beta,\gamma)$ we can calculate its logarithm [16]

$$\ln G(\alpha,\beta,\gamma)=\frac{\phi}{\sin\frac{\phi}{2}}\begin{pmatrix} 0 & \pm\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2} & \pm\sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2} \\ \mp\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2} & 0 & \pm\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2} \\ \mp\sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2} & \mp\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2} & 0 \end{pmatrix}, \tag{51}$$

where

$$\cos\frac{\phi}{2}=\cos\frac{\beta}{2}\left|\cos\frac{\alpha+\gamma}{2}\right|,$$
$$\sin\frac{\phi}{2}=\left[1-\cos^2\frac{\beta}{2}\cos^2\frac{\alpha+\gamma}{2}\right]^{1/2}, \tag{52}$$

and the sign $\mp$, respectively, corresponds to whether $\cos(\alpha+\gamma)/2>0$, or <0. Then

$$\sin\frac{\alpha}{2}=\sqrt{\left(1-\sqrt{\frac{2}{5}}\right)\Big/2},$$
$$\cos\frac{\alpha}{2}=-\sqrt{\left(1+\sqrt{\frac{2}{5}}\right)\Big/2},$$
$$\sin\frac{\beta}{2}=\sqrt{\left(1-\frac{1}{\sqrt{6}}\right)\Big/2},$$
$$\cos\frac{\beta}{2}=\sqrt{\left(1+\frac{1}{\sqrt{6}}\right)\Big/2},\quad \sin\frac{\gamma}{2}=\sqrt{\left(1-\frac{2}{\sqrt{5}}\right)\Big/2},$$
$$\cos\frac{\gamma}{2}=\sqrt{\left(1+\frac{2}{\sqrt{5}}\right)\Big/2}.$$

It is easily seen that $(\ln G)^\dagger=-\ln G$, an antisymmetric matrix.

When r in Eq. (16) is time dependent, $r\to r(t)$, $U\to U(t)$, we seek the interaction Hamiltonian that can generate the time evolution of the standard three-mode squeezing transformation $|\vec{q}\rangle|_{t=0}\to|\Lambda(t)|^{1/2}|\Lambda(t)\vec{q}\rangle|_t$ as indicated by Eq. (19). For this purpose we differentiate $U(t)$ with respect to time t,

$$\frac{dU(t)}{dt}=iQ_iA_{ij}P_jU(t)\frac{dr}{dt}. \tag{53}$$

Recasting Eq. (53) in the standard form for the equation of motion in an interaction picture

$$i\partial U(t)/\partial t=H_{IP}(t)U(t),$$

we obtain the Hamiltonian in the interaction picture that governs the time evolution,

$$\begin{aligned} H_{IP}(t)&=-Q_iA_{ij}P_j\frac{dr}{dt}\\ &=\frac{i}{2}\frac{dr}{dt}[(a_1a_2-a_1^\dagger a_2^\dagger)+(a_3a_2-a_3^\dagger a_2^\dagger)\\ &\quad+(a_3a_1-a_3^\dagger a_1^\dagger)+(a_1^\dagger a_2-a_1a_2^\dagger)+(a_2^\dagger a_3-a_2a_3^\dagger)\\ &\quad+(a_3^\dagger a_1-a_3a_1^\dagger)]. \end{aligned} \tag{54}$$

Thus the system that undergoes the required squeezing has the Hamiltonian in the Schrödinger picture

$$H=\sum_{i=1}^{3}\omega_i a_i^\dagger a_i+\frac{i}{2}\frac{dr}{dt}[(a_1a_2e^{i(\omega_1+\omega_2)t}-\text{H.c.})+(a_3a_2e^{i(\omega_3+\omega_2)t}-\text{H.c.})+(a_3a_1e^{i(\omega_3+\omega_1)t}-\text{H.c.})]+\frac{i}{2}\frac{dr}{dt}[(a_1^\dagger a_2e^{i(\omega_2-\omega_1)t}-\text{H.c.})+(a_2^\dagger a_3e^{i(\omega_3-\omega_2)t}-\text{H.c.})+(a_3^\dagger a_1e^{i(\omega_1-\omega_3)t}-\text{H.c.})], \tag{55}$$

where H.c. denotes the Hermitian conjugate, ω_i are the uncoupled modes frequencies, and $dr/2dt$ represents the coupling constant. The Hamiltonian describes two-photon parametric process (with creation or annihilation of the pairing of modes i and j, $i\neq j$) through the interaction with classical pumping mode, simultaneously, as well as the process of linear interaction between modes i and j, $i\neq j$. This dynamic mechanism may be realized in a combined setup including both parametric amplifiers fabricated from second-order $\chi^{(2)}$ susceptibility materials and nonlinear symmetric directional coupler that is composed of optical waveguides.

In summary, we have shown that the operator $U=\exp[-ir(Q_1P_2+Q_2P_3+Q_3P_1)]$ is a three-mode squeezing operator for the three-mode quadratures exhibiting the standard squeezing, the corresponding squeezed vacuum state in three-mode Fock space is derived by virtue of the IWOP technique. The entanglement involved in this state is analyzed. The optical network for producing such an ideal squeezed state is constructed and the system Hamiltonian for generating the squeezing evolution is derived. The three-mode squeezed state and entangled state may have potential uses in theoretically analyzing tripartite quantum teleportation.

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