Calculation of mode coupling for quadrupole excitations in a Bose-Einstein condenstate

G. Hechenblaikner, S. A. Morgan, E. Hodby, O. M. Maragò, and C. J. Foot
Clarendon Laboratory, Department of Physics, University of Oxford, Parks Road, Oxford, OX1 3PU, United Kingdom
(Received 10 May 2001; published 12 February 2002)

In this paper, we give a theoretical description of resonant coupling between two collective excitations of a Bose-condensed gas on, or close to, a second-harmonic resonance. Using analytic expressions for the quasiparticle wave functions, we show that the coupling between quadrupole modes is strong, leading to a coupling time of a few milliseconds (for a TOP trap with radial frequency ~ 100 Hz and $\sim 10^4$ atoms). Using the hydrodynamic approximation, we derive an analytic expression for the coupling matrix element. These can be used with an effective Hamiltonian (that we also derive) to describe the dynamics of the coupling process and the associated squeezing effects.

DOI: 10.1103/PhysRevA.65.033612 PACS number(s): 03.75.Fi, 05.45.−a, 42.65.Ky

I. INTRODUCTION

In two recent experiments [1,2], we observed resonant coupling between the low-energy modes of oscillation in a Bose-condensed gas. In the first experiment [1], we excited an even parity quadrupole mode (the $m=0$ low-lying mode) and observed transfer of energy to a mode at twice the original frequency (the $m=0$ high-lying mode). The oscillations at the second harmonic were observed as soon as the excitation period ended and stayed constant in amplitude. This indicates strong coupling between the modes so that energy is transferred between them at a rate comparable to the mode oscillation frequency of a few hundred Hz, i.e., this an allowed transition between the vibrational modes. In contrast, the coupling between a scissors mode and a mode at half the initial frequency was found to be a much slower process [2]. This paper shows that the simple downconversion process is forbidden, i.e., the matrix element for the direct conversion of one quantum of the higher-lying scissors mode into two quanta of a lower-lying mode, is zero. This means that some more complicated process is required to explain the experimental results. We also show how to calculate the coupling rates between various modes analytically. We do not want to give a direct comparison with our experimental data, where the oscillations of the condensate widths were measured in time of flight (and not quasiparticle amplitudes), but we focus on a description of the coupling process in terms of quasiparticle amplitudes. For resonant coupling between the quadrupole excitations, we present a simple expression for the radial integrand of the matrix element that shows that the coupling mostly takes place in the boundary regions of the condensate. Finally, we show that the coupling is well described by a simple Hamiltonian that can be used for quantitative studies of the squeezing effects related to the harmonic generation processes.

The paper is structured as follows. Section II presents the nonlinear Schrödinger equation (NLSE) and the derivation of the Bogoliubov–de Gennes (BdG) equation from the many-body Hamiltonian. These equations form the basis of the following sections. In section III, we summarize the derivation of solutions to the BdG equations in the hydrodynamic limit following the method given in Ref. [3]. The assumptions and approximations made in that derivation are important for understanding the calculations in Sec. V. Section IV gives the derivation of the Hamiltonian describing second-harmonic generation (SHG) or degenerate down-conversion from the NLSE, closely following the approach given in Ref. [4]. The coupling matrix elements governing the nonlinear processes are calculated in Sec. V. A simple expression is found for resonant coupling and the results are compared to an exact numerical calculation. We show that symmetry arguments forbid the direct down-conversion of the scissors mode and discuss our results with respect to two recent experiments [1,2] by our group.

II. CONDENSATE EXCITATIONS

Our treatment of the coupling between the modes starts with the Gross-Pitaevskii equation (GPE) for the macroscopic wave function $\Psi(\mathbf{r},t)$ (also called the order parameter),

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+V_{ext}+g|\Psi|^2\right]\Psi. \tag{1}$$

The external potential for a harmonic trap is $V_{ext}(\mathbf{r})=m\Sigma_i\omega_i^2x_i^2/2$, and $g=4\pi\hbar^2a_s/m$ characterizes the nonlinearity that depends on the particle interaction strength through the scattering length a_s. The ground state Ψ_g is the lowest-energy eigenstate of the condensate and a solution to the time-independent NLSE,

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V_{ext}+g|\Psi_g|^2\right)\Psi_g=\mu\Psi_g, \tag{2}$$

where the energy of the ground state μ the chemical potential of the system.

One way to derive the collective excitations is to linearize the GPE for small perturbations around the ground state with the ansatz

$$\Psi(\mathbf{r},t)=e^{-i\mu t}\left[\Psi_g(\mathbf{r})+\sum_i\ (u_i(\mathbf{r})b_i e^{-i\omega_i t}+v_i^*(\mathbf{r})b_i^* e^{+i\omega_i t})\right]. \tag{3}$$

Substitution into the GPE and linearization with respect to the small amplitudes b_i yields the BdG equations

$$\mathcal{L}u_i+g\Psi_g^2 v_i=\hbar\omega_i u_i,$$

$$\mathcal{L}v_i+g\Psi_g^{*2}u_i=-\hbar\omega_i v_i. \tag{4}$$

The operator $\mathcal{L}$ is given by

$$\mathcal{L}=-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\mathbf{r})-\mu+2g|\Psi_g|^2. \tag{5}$$

By solving the BdG equations we find the eigenmodes with energies $\hbar\omega_i$, and wave functions v_i,u_i that satisfy the orthogonality and symmetry relations

$$\int\ d^3\mathbf{r}(u_i u_j^*-v_i v_j^*)=\delta_{ij},$$

$$\int\ d^3\mathbf{r}(u_i v_j^*-v_i u_j^*)=0. \tag{6}$$

The small complex amplitude coefficients b_i,b_i^* in Eq. (3) can be replaced by annihilation and creation operators $\hat{b}_i,\hat{b}_i^\dagger$, respectively. This is justified by the standard approach of second quantization, where the eigenmodes of a classical system are found and then the complex amplitudes are replaced by mode operators. Alternatively, one can start with the grand-canonical many-body Hamiltonian for the field operator $\hat{\Psi}(\mathbf{r},t)$,

$$\hat{H}=\int\ d^3\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r},t)\left[-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\mathbf{r})-\mu+\frac{g}{2}\hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\right]\hat{\Psi}(\mathbf{r},t), \tag{7}$$

and make the ansatz

$$\hat{\Psi}(\mathbf{r},t)=\Psi_g+\sum_i\ [u_i(\mathbf{r})\hat{b}_i(t)+v_i^*(\mathbf{r})\hat{b}_i^\dagger(t)]. \tag{8}$$

In this approach, the field operator is split into its expectation value (the condensate part) and a fluctuating part that accounts for collective excitations and the thermal cloud. Substitution of Eq. (8) into the Hamiltonian of Eq. (7), and neglecting terms of order three or four in the excitation operators $\hat{b}_i,\hat{b}_i^\dagger$ gives a quadratic Hamiltonian that is diagonalized exactly if ψ_g satisfies the GPE of Eq. (2) and the wave functions u_i,v_i are solutions of the BdG Eqs. (4). The Hamiltonian can, therefore, be written as

$$\hat{H}=E_g+\sum_i\ \hbar\omega_i\hat{b}_i^\dagger\hat{b}_i+C. \tag{9}$$

Here C is the zero-point energy of the noncondensate and E_g is the energy of the condensate given by

$$E_g=\int\ d^3\mathbf{r}\Psi_g^*\left[-\frac{\hbar^2}{2m}\nabla^2+V_{ext}-\mu+\frac{g}{2}|\Psi_g|^2\right]\Psi_g. \tag{10}$$

So far, in all experiments on collective excitations, the eigenmodes have been excited strongly into a coherent state. For these conditions, one can assume that the mode operators commute and can replace them by complex numbers so that Eq. (8) reduces to Eq. (3) except for a factor $e^{-i\mu t}$ that amounts to a shift in the zero of energy.

III. CALCULATING THE QUASIPARTICLE WAVE FUNCTIONS IN THE HYDRODYNAMIC LIMIT

In this section, we give a brief overview of how to calculate the excited-state wave functions u_i,v_i directly following the approach given in Ref. [3]. The starting point are the BdG Eqs. (4). These can be rewritten in dimensionless units by introducing the following coordinate transforms: $y_j=r_j/l_j, j=1,\ldots,3$, where $l_j=(2\mu/m\omega_j^2)^{1/2}$ are the characteristic lengths of the condensate in the Thomas-Fermi regime. As in Ref. [3], we define the small dimensionless parameter $\xi=\hbar\overline{\omega}/2\mu$ and the dimensionless energy of mode i, $\epsilon_i=E_i/\hbar\overline{\omega}$, where E_i is the energy of mode i and $\overline{\omega}=(\omega_1\omega_2\omega_3)^{1/3}$ is the geometric mean oscillator frequency. We also introduce the mean characteristic length $l_c=(l_1 l_2 l_3)^{1/3}$ and the dimensionless Laplace operator $\tilde{\triangle}=\Sigma_{j=1}^3(\omega_j/\overline{\omega})\partial^2/\partial y_j^2$ and define $y^2=\Sigma_{j=1}^3 y_j^2$. The resulting equations are

$$-\xi^2\tilde{\triangle}u_i+y^2u_i+(2u_i+v_i)\overline{n_0}=(1+2\xi\epsilon_i)u_i, \tag{11}$$

$$-\xi^2\tilde{\triangle}v_i+y^2v_i+(2v_i+u_i)\overline{n_0}=(1-2\xi\epsilon_i)v_i, \tag{12}$$

$$-\xi^2\tilde{\triangle}\psi_g+y^2\psi_g+\overline{n_0}\psi_g=\psi_g, \tag{13}$$

where $\overline{n_0}=|\psi_g|^2 g/\mu$. These equations can be combined to form fourth-order equations for the functions $f_i^\pm=u_i\pm v_i$. In the hydrodynamic limit the expression $\psi_g=\sqrt{n_0(1-y^2)}$, where $n_0=\mu/g$ is the maximum condensate density, is then substituted for the ground-state wave function ψ_g, and terms of second order in the small parameter ξ are omitted.

Now we introduce the operator $\hat{G}$ with the definition

$$\hat{G}=(1-y^2)\tilde{\triangle}-2\sum_i\ y_i(\omega_i/\overline{\omega})^2\partial/\partial y_i \tag{14}$$

and define new functions $W_i(y_1,y_2,y_3)$ by $f_i^\pm(y)=C_i^\pm(1-y^2)^{\mp 1/2}W_i(y_1,y_2,y_3)$, where the relation between the coefficients $C_i^\pm$ is given by $C_i^+=\epsilon\xi C_i^-$. Using these definitions one finally obtains from Eqs. (11), (12) the compact expression [3]

$$\hat{G}W + 2\epsilon^2 W = 0, \tag{15}$$

where we have omitted the mode-index i for simplicity. For a spherical trap, this equation can be solved exactly. It is a hypergeometric differential equation with Jacobi polynomials as the general solution. The quantization of the energies comes from the condition that the function must converge at the condensate boundaries, which yields an analytic expression for the mode spectrum [5,3].

In the most general case of an anisotropic trap with three different trap frequencies, one can make a polynomial ansatz. The symmetry of the Hamiltonian means that parity is a good quantum number for any spatial coordinate. We shall find the solutions for the quadrupolar modes where the order of the polynomial is 2. We can make the ansatz [3]

$$W \propto y_i y_j, \quad i \neq j, \tag{16}$$

to find the three odd-parity eigenfunctions with eigenfrequencies $\Omega = (\omega_i^2 + \omega_j^2)^{1/2}$. These so-called scissors modes have been studied extensively by our group [6] and we use Eq. (16) to derive some of their coupling properties in Sec. V. We have to make a different ansatz to find the three even-parity eigenfunctions (which are also referred to as diagonal quadrupolar modes [7]),

$$W \propto 1 + \sum_{j=1}^{3} b_j (\overline{\omega}/\omega_j)^2 y_j^2. \tag{17}$$

The polynomial coefficients b_j completely characterize the mode geometry and will be very important later on in our expression for the coupling matrix element. In the following, we use the abbreviation $\overline{b_j} = b_j(\overline{\omega}/\omega_j)^2$. Substitution of Eq. (17) into Eq. (15) shows that W is a solution, provided that the following equations hold:

$$S \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0, \tag{18}$$

$$\sum_{j=1}^{3} b_j + \frac{\Omega^2}{\overline{\omega}} = 0, \tag{19}$$

where the matrix S is defined as

$$S = \begin{pmatrix} 3 - \frac{\Omega^2}{\omega_1^2} & 1 & 1 \\ 1 & 3 - \frac{\Omega^2}{\omega_2^2} & 1 \\ 1 & 1 & 3 - \frac{\Omega^2}{\omega_3^2} \end{pmatrix}. \tag{20}$$

The eigenfrequencies Ω are found by demanding that $\det(S) = 0$. The resulting equation is

$$\Omega^6 - 3\Omega^4(\omega_1^2 + \omega_2^2 + \omega_3^2) + 8\Omega^2(\omega_2^2\omega_3^2 + \omega_1^2\omega_3^2 + \omega_1^2\omega_2^2) - 20(\omega_1^2\omega_2^2\omega_3^2) = 0. \tag{21}$$

This general expression simplifies for the case of an axially symmetric trap ($\omega_1 = \omega_2$). In this case the solutions are $\Omega = \sqrt{2}\omega_1$ for the $m=2$ mode, and

$$\Omega^2/\Omega_1^2 = 2 + \frac{3}{2}\lambda^2 \mp \frac{1}{2}\sqrt{9\lambda^4 - 16\lambda^2 + 16}, \tag{22}$$

for the $m=0$ low lying and the $m=0$ high-lying mode. Here, the trap anisotropy is given by $\lambda = \omega_z/\omega_r$, where ω_z, ω_r are the axial and radial trap frequencies, respectively. Equations (21) and (22) were derived in the review paper on Bose-Einstein Condensates by Dalfovo *et al.* [8]. In the next step, the polynomial coefficients b_j are found from any two of the three equations in Eq. (18) and Eq. (19).

So far, we have summarized important known results that enable us to calculate the quasiparticle wave functions for all six quadrupole modes. We will use these in Sec. V to calculate the matrix elements that describe the coupling of these modes.

IV. A MODEL FOR SECOND-HARMONIC COUPLING BETWEEN TWO MODES

We follow the approach given in Ref. [4] to derive a set of coupled nonlinear equations (describing second-harmonic generation) from the NLSE. For convenience, we normalize the condensate wave function to unity and change the parameter g in Eq. (1) to $N_0 g$, where N_0 is the number of particles in the condensate. We introduce a set of excitations that is normal to the condensate and also diagonalizes the many-body Hamiltonian of Eq. (7). This is achieved by projecting out the overlap with the condensate from the solutions to the BdG equations to give quasiparticle wave functions defined by

$$\tilde{u}_i = u_i - c_i \Psi_g, \tag{23}$$

$$\tilde{v}_i^* = v_i^* + c_i^* \Psi_g, \tag{24}$$

where $c_i = \int d^3\mathbf{r}[\Psi_g^* u_i] = -\int d^3\mathbf{r}[\Psi_g v_i]$. These wave functions still diagonalize the many-body Hamiltonian (7) and the orthogonality relations (6) hold as well. The advantage of introducing excitations orthogonal to the ground state is that it makes it easier to extract the amplitudes of various excitations from a given wave function. In terms of the orthogonal excitations a general wave function can be written as [4]

$$\Psi(\mathbf{r},t) = e^{-i\mu t}\left\{(1+b_g)\Psi_g(\mathbf{r}) + \sum_{i>0}[\tilde{u}_i(\mathbf{r})b_i(t) + \tilde{v}_i^*(\mathbf{r})b_i^*(t)]\right\}, \tag{25}$$

where the coefficient b_g describes the change in the condensate. It is easy to show that for the orthogonal excitations the following relationships hold:

$$\int d^3\mathbf{r}\psi_g^*\Psi e^{+i\mu t}=1+b_g,$$

$$\int d^3\mathbf{r}[\tilde{u}_i^*\Psi e^{+i\mu t}-\tilde{v}_i^*\Psi^* e^{-i\mu t}]=b_i. \tag{26}$$

The population of the condensate ground state is given by $|1+b_g|^2N_0$ and the population of the excited states by $|b_i|^2N_0$.

In the next step, we obtain the equations of evolution for the complex coefficients $b_i(t)$ by substituting the expansion of the wave function (25) into the GPE (1), and carrying out the projections described by Eqs. (26). We so obtain the Heisenberg equations for the *c*-number equivalents of the mode operators b_i. We then transform these equations for the mode amplitudes into the interaction picture by making the ansatz $b_i(t)=b_i^R(t)e^{-i\omega_i t}$. This gives rise to a large number of terms oscillating at frequencies $\omega_i\pm\omega_k\pm\omega_j$. If we focus on second-harmonic processes, where $\omega_k=\omega_i$ and $\omega_j\simeq 2\omega_i$, we can neglect all the rapidly oscillating terms and retain only the term oscillating at $\Delta_{ij}=\omega_j-2\omega_i$. This is called the rotating-wave approximation. If we neglect any variation in the population of the condensate mode we obtain the following coupled equations of motion for the two modes $i=1$ and $j=2$:

$$i\hbar\frac{db_1^R}{dt}=N_0gM_{12}b_1^{R*}b_2^Re^{-i\Delta_{12}t}, \tag{27}$$

$$i\hbar\frac{db_2^R}{dt}=\frac{1}{2}N_0gM_{12}^*b_1^Rb_1^Re^{i\Delta_{12}t}, \tag{28}$$

where the matrix element M_{12} is given by

$$M_{12}=2\int d^3\mathbf{r}[\psi_g^*(2\tilde{u}_1^*\tilde{v}_1^*\tilde{u}_2+\tilde{v}_1^*\tilde{v}_1^*\tilde{v}_2)+\psi_g(2\tilde{u}_1^*\tilde{v}_1^*\tilde{v}_2+\tilde{u}_1^*\tilde{u}_1^*\tilde{u}_2)]. \tag{29}$$

These equations describe the transfer of excitation between the two modes via annihilation (creation) of two quanta in mode 1 and creation (annihilation) of one quantum in mode 2, which is also called a second harmonic process. The matrix element M_{12} contains all the information on the geometry of the two modes that are coupled. If we excite the lower mode at resonance ($\Delta_{12}=0$) and there is no initial population in the upper mode, then all the excitation is transferred to the upper mode. The opposite is not true, i.e., there is no transfer from an initially excited upper mode to the lower mode if we start off with zero population in the lower mode. We will see later in this section that in a quantum-mechanical description, where the lower mode is described by operators rather than *c*-numbers, down-conversion does occur. The strength of the processes depends on the spatial overlap between the respective quasiparticle wave functions. The characteristic time scale for the transfer from mode 1 to mode 2 is given by the expression

$$T=\left|\frac{\sqrt{2}\hbar}{NgM_{12}b_1(0)}\right|. \tag{30}$$

This model for the second-harmonic coupling between two collective excitations allows us to find an explicit expression for the matrix element governing the process.

A. Quantum-mechanical model for the coupling

Alternatively, the two coupled nonlinear equations (28) can be derived from the Hamiltonian (31), which gives a full quantum-mechanical description and clearly shows the underlying physical processes

$$H=\hbar\omega_1\hat{a}_1^\dagger\hat{a}_1+2\hbar\omega_1\hat{a}_2^\dagger\hat{a}_2+\frac{\hbar\kappa}{2}(\hat{a}_1^{\dagger 2}\hat{a}_2+\hat{a}_1^2\hat{a}_2^\dagger), \tag{31}$$

where $\hat{a}_1^\dagger\hat{a}_1,\hat{a}_2^\dagger\hat{a}_2$ give the quasiparticle populations of mode 1 and 2, respectively. We derive the Heisenberg equations for the mode operators by the relation

$$\dot{\hat{a}}_i=\frac{i}{\hbar}[H,\hat{a}_i]. \tag{32}$$

To remove the fast oscillation at the mode frequencies ω_1,ω_2 from the operators $\hat{a}_1,\hat{a}_2$, we introduce the slowly varying operators $\hat{b}_1,\hat{b}_2$: $\hat{b}_1=\hat{a}_1(1/\sqrt{N_0})e^{i\omega_1 t},\hat{b}_2=\hat{a}_2(1/\sqrt{N_0})e^{i\omega_2 t}$, where the $\sqrt{N_0}$ factor arises because of the different normalization.

The Heisenberg equations in terms of these operators are

$$i\hbar\frac{d\hat{b}_1}{dt}=\sqrt{N_0}\hbar\kappa\hat{b}_1^\dagger\hat{b}_2e^{-i\Delta_{12}t}, \tag{33}$$

$$i\hbar\frac{d\hat{b}_2}{dt}=\frac{1}{2}\sqrt{N_0}\hbar\kappa\hat{b}_1\hat{b}_1e^{i\Delta_{12}t}. \tag{34}$$

Equations (28) are obtained by replacing the mode operators by complex numbers and setting

$$\sqrt{N_0}\hbar\kappa=N_0gM_{12}. \tag{35}$$

However, quantum effects such as squeezing and sub-Poissonian statistics in the quasiparticle number can only be described by the operator equations (34) and are lost in making the classical approximation leading to Eqs. (28), where operators are replaced by complex numbers.

B. Squeezing in parametric down- and up-conversion

In this subsection we want to show briefly how a quantum description of the nonlinear processes leads to nonclassical effects such as squeezing. We apply the well-established theory of nonlinear effects in optics [9,10] to describe the phononic coupling in a condensate and demonstrate the direct dependence of squeezing on the coupling matrix element in both, up- and down-conversion processes. It is difficult to study squeezing for the full quantum-mechanical description of both modes given by the Hamiltonian (31) as the operator equations are nonlinear. We first investigate down-conversion for resonant coupling and transform the Hamiltonian (31) into the interaction picture to obtain

$$H_R = -i\frac{\hbar\kappa}{2}(\hat{b}_1^{\dagger 2}\hat{b}_2 - \hat{b}_1^2\hat{b}_2^{\dagger}), \quad (36)$$

where $\hat{b}_1,\hat{b}_1^\dagger$ denote the operators $\hat{a}_1,\hat{a}_1^\dagger$ transformed into the interaction picture. Then the mode operators $\hat{b}_2,\hat{b}_2^\dagger$ for mode 2 are replaced by the c number β_2 (we can, without loss of generality, assume that β_2 is real), but we retain the operators for mode 1. In addition, we will treat the mode amplitude of the upper level as a constant. This assumption does not account for the depletion of the upper mode and is only valid for small times when the occupation of the lower level is much smaller than the occupation of the upper level, i.e., for $N_1 \ll \beta_2^2$. The resulting Hamiltonian is quadratic in $\hat{b}_1,\hat{b}_1^\dagger$ and gives linear Heisenberg equations,

$$\frac{d\hat{b}_1}{dt} = [\hat{b}_1, H_R] = \kappa\beta_2\hat{b}_1^\dagger,$$

$$\frac{d\hat{b}_1^\dagger}{dt} = [\hat{b}_1^\dagger, H_R] = \kappa\beta_2\hat{b}_1. \quad (37)$$

Equations (37) can be diagonalized by expressing them in terms of the two quadrature phase amplitudes $\hat{Q}_x,\hat{Q}_p$ defined as

$$\hat{Q}_x = \hat{b}_1 + \hat{b}_1^\dagger,$$

$$\hat{Q}_p = \frac{\hat{b}_1 - \hat{b}_1^\dagger}{i}. \quad (38)$$

Simple integration yields $\hat{Q}_x(t) = e^{\kappa\beta_2 t}\hat{Q}_x(0), \hat{Q}_p(t) = e^{-\kappa\beta_2 t}\hat{Q}_p(0)$. These solutions can be used to calculate the evolution of $\hat{b}_1,\hat{b}_1^\dagger$ and the evolution for the number of down-converted quasiparticles, N_1, for which we find (assuming mode 2 was initially in a vacuum state $|0\rangle$)

$$N_1 = \langle 0|\hat{b}_1^\dagger\hat{b}_1|0\rangle = \sin h^2(\kappa\beta_2 t). \quad (39)$$

Equation (39) shows that in this quantum model down-conversion occurs even for zero initial population in mode 1. This is in contrast to the result of the semiclassical model discussed above. The evolution of the variances in the quadrature operators is found to be:

$$[\Delta\hat{Q}_x(t)]^2 = e^{2\kappa\beta_2 t}[\Delta\hat{Q}_x(0)]^2,$$

$$[\Delta\hat{Q}_p(t)]^2 = e^{-2\kappa\beta_2 t}[\Delta\hat{Q}_p(0)]^2. \quad (40)$$

This clearly demonstrates the squeezing in the $\hat{Q}_p$ quadrature component. However, it is important to keep in mind that Eqs. (40) are only valid for short times before the assumption that the upper mode is not depleted breaks down. A possible way to avoid depletion of the upper mode is to keep exciting it. The standard way to do this is to mechanically force the condensate into oscillation at the frequency and geometry corresponding to the upper mode.

Similarly, squeezing occurs during SHG where two phonons from the lower mode are converted into one phonon of the upper mode. In this case, we cannot replace the lower mode by a c number as we have done for our investigation of down-conversion. We can again try to find an approximate solution valid only for small times. This is demonstrated in Ref. [10], where a Taylor series expansion is used to describe the time evolution of the mode operators. We assume that mode 1 is initially in a coherent state defined by $b_1|\beta_1\rangle = \beta_1|\beta_1\rangle$ with $\beta_1 = |\beta_1|e^{i|\Phi}$ and mode 2 is in the vacuum state. The result for the squeezing of the quadrature $\hat{Q}_x$ (of mode 1) to second order in time is then given by

$$[\Delta\hat{Q}_x(t)]^2 = 1 - \frac{1}{2}\kappa^2 t^2|\beta_1|^2\cos(2\Phi) + O(gt)^3. \quad (41)$$

This result only holds for small times for which $\frac{1}{4}|\beta_1|^2\kappa^2 t^2 \ll 1$. However, we can see how in both cases, down- and up-conversion, the squeezing of the quadrature components can be directly related to the nonlinear coupling strength.

V. CALCULATING THE NONLINEAR COUPLING RATE

We will now calculate the nonlinear coupling rates between two condensate excitations and compare them to those from recent experiments. For convenience, we take linear combinations of the normalized quasiparticle wave functions to give the new functions $\tilde{f}_i^+,\tilde{f}_i^-$ defined by

$$\tilde{f}_i^+ = \tilde{u}_i + \tilde{v}_i = f_i^+, \quad (42)$$

$$\tilde{f}_i^- = \tilde{u}_i - \tilde{v}_i = f_i^- - 2c_i\psi_g. \quad (43)$$

Written in terms of these functions, the matrix element in Eq. (29) has the form

$$M_{12} = 2\int d^3\mathbf{r}\ \ \psi_g\left\{\frac{1}{2}\tilde{f}_1^{+*}(\tilde{f}_1^{+*}\tilde{f}_2^+ + \tilde{f}_1^{-*}\tilde{f}_2^- + \frac{1}{4}\tilde{f}_2^+(\tilde{f}_1^{+*}\tilde{f}_1^{+*} - \tilde{f}_1^{-*}\tilde{f}_1^{-*})\right\}, \quad (44)$$

where we assumed that ψ_g is real. Alternatively, M_{12} can be written in terms of untilded functions (f_i^+,f_i^-) as the sum of two parts, $M_{12} = M_{12}^{(1)} + M_{12}^{(2)}$, which are defined as follows:

$$M_{12}^{(1)} = 2\int d^3\mathbf{r}\psi_g\left\{\frac{1}{2}f_1^{+*}(f_1^{+*} + f_2^+ + f_1^{-*}f_2^- + \frac{1}{4}f_2^+(f_1^{+*}f_1^{+*} - f_1^{-*}f_1^{-*})\right\}, \quad (45)$$

$$M_{12}^{(2)} = 2\int d^3\mathbf{r}\psi_g\{f_1^{+*}(2c_1^*c_2\psi_g^2 - c_1^*\psi_g f_2^- - c_2\psi_g f_1^{-*}) + f_2^+(c_1^*\psi_g f_1^{-*} - c_1^{*2}\psi_g^2)\}. \quad (46)$$

$M_{12}^{(2)}$ is zero if neither of the quasiparticle wave functions has any overlap with the condensate ground state. We can now use the functions f_i^+, f_i^-, which we found in Sec. III. We can write, in general,

$$f_1^+ = A_1 \epsilon_1 \xi (1-y^2)^{-1/2} W_1, \quad f_1^- = A_1 (1-y^2)^{1/2} W_1,$$

$$f_2^+ = A_2 \epsilon_2 \xi (1-y^2)^{-1/2} W_2, \quad f_2^- = A_2 (1-y^2)^{1/2} W_2, \tag{47}$$

where the A_1, A_2 are normalization constants determined from the normalization condition (6) and $W_1(y_1,y_2,y_3), W_2(y_1,y_2,y_3)$ are solutions to Eq. (15) for mode 1 and 2, respectively. Substituting these expressions into Eq. (45) gives

$$M_{12}^{(1)} = (\sqrt{n_0/N_0} A_1^2 A_2 \xi l_c^3/2) \int d^3\mathbf{y} W_1^2 W_2 [3\epsilon_1^2 \epsilon_2 \xi^2 \times (1-y^2)^{-1} - \Delta\epsilon_{12}(1-y^2)]. \tag{48}$$

The first term in the above integral proportional to $(1-y^2)^{-1}$ diverges at the condensate boundary $(y=1)$, but it must be dropped as it scales proportional to ξ^2 and in the derivation of the quasiparticle wave functions we omitted terms proportional to ξ^2 in the governing equations (hydrodynamic approximation) to obtain Eq. (15) for $W(y_1,y_2,y_3)$. We will see later in this paper that this is fully justified by comparison with exact numerical calculations. Note that the second term equals zero if the detuning $\Delta\epsilon_{12} = \epsilon_2 - 2\epsilon_1$ is zero.

A. Coupling between two even-parity quadrupolar excitations

So far, we have made no assumptions about the geometry of the two modes that are coupled and Eqs. (45), (46) are valid for any pair of modes. We now focus on the quadrupolar modes of a triaxial trap and investigate the coupling between any two diagonal modes, for which the function $W(y_1,y_2,y_3)$ is represented by the polynomial given in Eq. (17). We can calculate the matrix element M_{12} from Eqs. (46), (48). An explicit expression for M_{12} and its derivation is given in the Appendix. It is important to note that for on-resonant coupling between the two modes $(\omega_2 = 2\omega_1)$ the matrix element simplifies considerably. In that case we can give a simple expression for the radial integrand of the matrix element in terms of the dimensionless position y, where the condensate boundaries are given by $y=1$,

$$M_{12} = -2\sqrt{\frac{n_0}{N_0}} A_2 \int_0^1 y^6 (1-y^2) dy = -2\sqrt{\frac{n_0}{N_0}} A_2 \frac{2}{63}, \tag{49}$$

where A_2 denotes the normalization constant for the wave function of mode 2, given by Eq. (A8). It is important to note that Eq. (49) describes the resonant coupling between any two diagonal quadrupole modes in a triaxial trap. This allows a quick and easy calculation of the coupling strength, coupling rates, and squeezing effects associated with the nonlinear process. We can see from the radial integrand of Eq. (49) that the coupling is strongest in the outer regions of the condensate and reaches a maximum at a distance of $0.87 l_c$ from the center. In this region, the condensate is still well described by the hydrodynamic approximation that only breaks down at a distance of order the healing length from the condensate boundaries. The healing length in our recent experiments [1,2] was about $0.05 l_c$.

B. Coupling between two modes in a spherical trap

Now we want to give a quantitative comparison between the solutions we found to the coupling matrix element in a hydrodynamic approach and the exact solutions calculated from the numerical solutions to the BdG equations. To facilitate the numerical calculations we will look at the coupling between two quadrupolar modes in a spherical trap of frequency ω, where the total angular momentum l and the azimuthal angular momentum m are good quantum numbers. The two modes are the $l=2, m=0$ mode and the $l=0, m=0$ mode with frequencies of $\sqrt{2}\omega$ and $\sqrt{5}\omega$, respectively. In a trap with only axial symmetry these two modes get mixed and become the $m=0$ low lying and the $m=0$ high-lying (breathing) mode. Their quasiparticle wavefunctions can be presented by the ansatz (47), where $W_1 = (3\cos^2\theta - 1)y^2$ and $W_2 = (1 - 5y^2/3)$. These are the solutions to the hypergeometric differential equations discussed in Sec. III. The normalization constants are $A_1 = \sqrt{35/16\pi l_c^3 \epsilon_1 \xi}$ and $A_2 = (3/2)\sqrt{7/4\pi l_c^3 \epsilon_2 \xi}$.

One can see from Eq. (46) that in $M_{12}^{(2)}$ all terms containing the constant c_1 are zero, because the overlap between mode 1, which is proportional to Y_2^0, and the ground state, which is proportional to Y_0^0, is zero. The only remaining term is proportional to c_2 and it turns out to be $-2\sqrt{n_0/N_0} A_2 y^6 (1-y^2)$, which is exactly the expression we found for the resonant case $(\Delta\epsilon_{12}=0)$ in a general triaxial trap [see Eq. (49)]. But for these two modes $\Delta\epsilon_{12} \neq 0$ and we have to consider the contribution from $M_{12}^{(1)}$ as well, so that we obtain

$$M_{12} = -2\sqrt{\frac{n_0}{N_0}} A_2 \int_0^1 \left[\frac{7}{4\epsilon_1} \Delta\epsilon_{12} (1-y^2)(1-5/3y^2) + (1-y^2) \right] y^6 dy. \tag{50}$$

The wave functions for the ground state and the coupling matrix elements M_{12} are plotted in Fig. 1 and Fig. 2, respectively. It is important to note that for the resonant case the only contribution comes from Eq. (49). Also, for the not-quite-resonant case displayed in Fig. 2, the integrand is dominated by this contribution. This shows that the coupling between different quasiparticle excitations predominantly takes place in the boundary region of the condensate and an explicit analytic expression for the spatial probability of the nonlinear process is given by Eq. (49).

The integrated values of M_{12} in the hydrodynamic approximation and for the exact numerical calculation are $-0.161 l_c^{-3}$ and $-0.157 l_c^{-3}$, respectively. The error of the

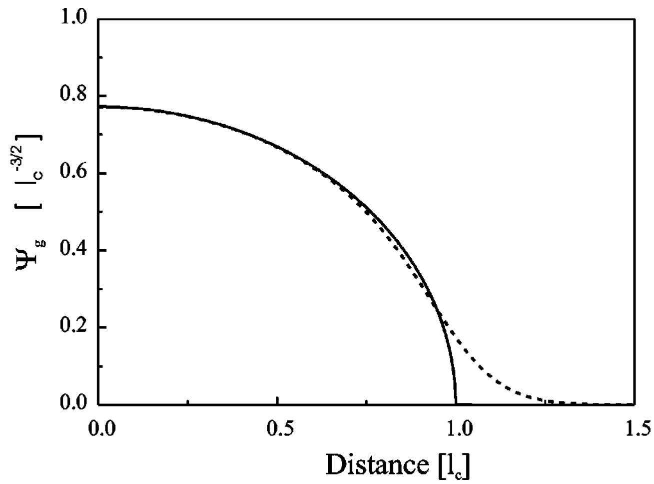

FIG. 1. The wave functions for the ground state in the hydrodynamic limit (solid line) and the exact solution to the GPE (dotted line) plotted in units of $1/\sqrt{l_c^3}$ against the distance from the center of the trap in units of the characteristic length l_c. The trap is spherical with 1.5×10^4 atoms and a frequency $\omega=120$ Hz . The healing length $\xi=0.05l_c$.

approximate analytical result with respect to the exact numerical calculation is in the order of ξ^2, as we expect. The good agreement between the two, even for a relatively small number of atoms, justifies the hydrodynamic approximations made in calculating the quasi particle wave functions and the coupling matrix element.

C. Coupling between two even parity modes in a TOP trap

In our experiments, we use a TOP trap that is axially symmetric and has an anisotropy defined by the parameter $\lambda=\omega_z/\omega_r$, where ω_z and ω_r are the axial and radial trap frequencies, respectively. In a recent experiment, we studied

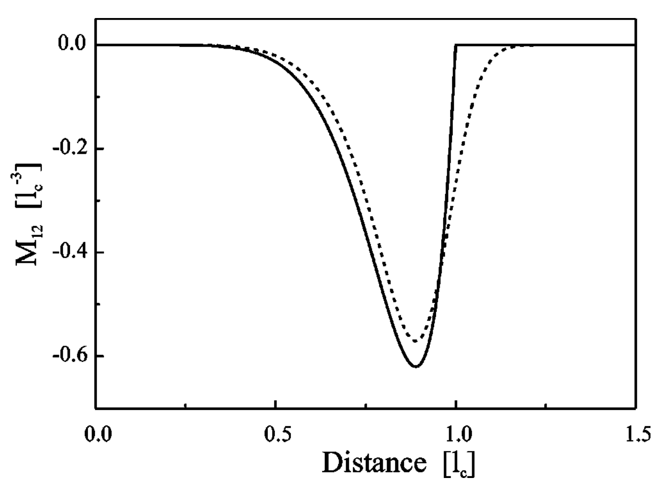

FIG. 2. The radial integrands for the matrix elements M_{12} in the hydrodynamic limit (solid line) and for the exact numerical calculation (dotted line) plotted in units of l_c^{-3} against the distance from the condensate center in units of l_c. The coupling is between the $l=0,m=0$ and the $l=2,m=0$ mode for the same trap conditions as in Fig. 1.

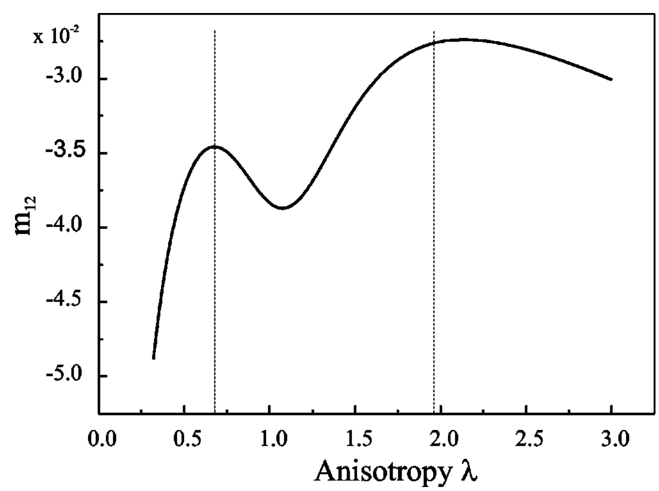

FIG. 3. The quantity $m_{12}=M_{12}l_c^3\sqrt{\xi}$ is plotted against the trap anisotropy λ. The two vertical lines show where there is resonant nonlinear coupling between the $m=0$ low lying and the $m=0$ high-lying modes. The resonances are determined by the matching of mode frequencies ($2\omega_1=\omega_2$) and they are located at $\lambda=0.68$ and $\lambda=1.95$.

the coupling between the $m=0$ low lying and the $m=0$ high-lying mode (which arise from the $l=0,m=0$ and the $l=2,m=0$ mode of the spherical trap when $\lambda\neq1$). We can use the formula for M_{12} for the general triaxial trap, given by Eqs. (A10), (A11) in the Appendix, to calculate the matrix element for any trap geometry. The result is shown in Fig. 3 in terms of the dimensionless quantity $m_{12}=M_{12}l_c^3\sqrt{\xi}$. The dependence of M_{12} on number and mean frequency is contained in $l_c^{-3}\xi^{-1/2}$ and thus m_{12} only depends on the mode geometry, i.e., it is only a function of λ. In order to get M_{12} for a specific trap, one has to read the dimensionless matrix element m_{12} from Fig. 3 and multiply it by $l_c^{-3}\xi^{-1/2}$.

From Eq. (22) one can find two values for the anisotropy λ where the two $m=0$ modes are resonantly coupled. These resonances are at $\lambda=0.68,1.95$ with $|m_{12}|=0.038,0.028$, respectively. The characteristic coupling time is given by Eq. (30), which can be written as follows:

$$T=\frac{15\sqrt{2}\hbar\sqrt{\xi}}{8\pi\mu m_{12}b_1(0)}\propto(N^{-3/5}\overline{\omega}^{-13/10}). \tag{51}$$

We can calculate the coupling rate for the parameters of our second-harmonic generation experiment [1], which were: $\lambda=1.95$, $N_0=1.5\times10^4$, $\omega_r\simeq120$ Hz. Inserting the relevant quantities into Eq. (51), we obtain a transfer time of 5.7 ms and 3.7 ms for small initial excited populations of $|b_1(0)|^2=0.02$ and $|b_1(0)|^2=0.05$, respectively. For stronger initial excitation, the transfer times would be even smaller. One can see from Eq. (51) that for larger atom number and stiffer traps the coupling times become smaller as well. This shows that the process of coupling the two quadrupole modes happens on a timescale of a few milliseconds. It is consistent with our experimental observation [1], where the second-harmonic was observed as soon as the driving of the fundamental finished (excitation time of about 30 ms).

D. Coupling between a scissors mode and another quadrupolar excitation

Now we will discuss coupling between the off-diagonal quadrupolar excitations. We already showed in Sec. III that these modes have odd parity and are characterized by $W \propto y_i y_j, i \neq j$. Thus their overlap with the condensate ground state is zero and the only contribution to the coupling matrix element comes from $M_{12}^{(1)}$ given in Eq. (48). But this integral is zero because the product $W_1^2 W_2$ is odd. So the total matrix element M_{12} equals zero and there is no direct coupling via a second-harmonic process between two scissors modes. Our recent experimental observation of a down-conversion process between two scissors modes must, therefore, have a more sophisticated interpretation than the conversion of one quantum of the higher mode into two quanta of the lower scissors mode.

Similarly, there is no coupling between a higher-lying scissors mode and a lower-lying diagonal quadrupole mode. In this case, $M_{12}^{(1)}$ of Eq. (48) is zero beacuse W_2 is odd. $M_{12}^{(2)}$ of Eq. (46) is also zero because $c_2 = 0$ and $f_2^{\pm}$ are odd. Thus the total matrix element is zero. However, there is second-harmonic coupling from a lower scissors mode to a higher-lying diagonal quadrupolar mode. For resonant coupling two quanta of the scissors mode are converted into one quantum of the higher-lying even-parity mode and one finds that the matrix element is again given by expression (49).

VI. CONCLUSION

Starting from the NLSE, we have derived a simple model describing the nonlinear coupling of quasiparticle amplitudes of two modes. The model can be used to study squeezing effects that are directly related to the matrix element governing the coupling process. We have demonstrated how to calculate this matrix element analytically. We then focused on the quadrupole excitations and found that all resonant ($\omega_2 = 2\omega_1$) direct-coupling processes between the six quadrupole modes are described by the expression in Eq. (49) unless they are forbidden ($M_{12} = 0$). All second-harmonic processes involving odd-parity (scissors) modes are forbidden except for up-conversion from a scissors mode to a higher-lying even-parity mode. It is possible to show that there are other allowed nonlinear processes involving all three of the scissors modes. This gives rise to nondegenerate parametric amplification and multimode squeezing. Full details and the derivation of an effective Hamiltonian describing all allowed nonlinear processes between the quadrupole modes (not just the second harmonic generation described here) will be given in a future paper.

ACKNOWLEDGMENTS

We would like to thank H. Nilsen, H. Ritsch, C. Lechner, J. Dunningham, S. Choi, and K. Burnett for fruitful discussions. We acknowledge support from the EPSRC, St. John's College, Oxford (G.H.), Trinity College, Oxford (S.A.M.) and the EC (O.M.M.).

APPENDIX: MATRIX ELEMENTS FOR THE DIAGONAL QUADRUPOLAR MODES

We want to calculate the coupling matrix element for any two diagonal quadrupole modes in a triaxial trap for which the function $W(y_1, y_2, y_3)$ is represented by a polynomial as given in Eq. (17). It is useful to derive a number of relations for the polynomial coefficients that allow us to simplify the expressions for normalization constants, overlap coefficients, and the coupling matrix element.

The lowest mode of Eq. (8) is the so-called Goldstone mode with $\omega_0 = 0, u_0(\mathbf{r}) = \Psi_g(\mathbf{r})$ and $v_0(\mathbf{r}) = -\Psi_g^*(\mathbf{r})$, which arises from the $U(1)$ symmetry breaking. For this particular mode the orthogonality and symmetry relations of Eq. (6) take the form

$$\int d^3\mathbf{r}(\Psi_g^* u_i + \Psi_g v_i) = 0. \tag{A1}$$

Equation (A1) implies that $\int d^3\mathbf{r}\psi_g f^+ = 0$. Substituting Eq. (47) into this equation and integrating over the angles and radial coordinate gives

$$\sum_j \overline{b_j} = -5. \tag{A2}$$

The characteristic polynomials for the quadrupole modes are real and thus u_i, v_i can be taken as real, which allows us to derive from the orthogonality relations (6),

$$\int d^3\mathbf{x} f_i^+ f_j^- = \delta_{ij}. \tag{A3}$$

If we now insert two different polynomials corresponding to two different solutions for f_i^+, f_j^- into Eq. (A3), we obtain the relation $\int d^3\mathbf{y} W_1 W_2 = 0$ and from that

$$\sum_i \overline{b_i}\overline{d_i} = 5, \tag{A4}$$

$$\overline{b_1}\overline{d_2} + \overline{b_2}\overline{d_1} + \overline{b_1}\overline{d_3} + \overline{b_3}\overline{d_1} + \overline{b_2}\overline{d_3} + \overline{b_3}\overline{d_2} = 20, \tag{A5}$$

where the $\overline{b_i}$'s denote the polynomial coefficients of mode 1 and the $\overline{d_i}$'s the polynomial coefficients of mode 2. These coefficients are found from Eqs. (18), (19) in Sec. III. We also have to calculate the overlap coefficients c_i between the condensate and the quasiparticle wave functions,

$$c_i = \int d^3\mathbf{r}\psi_g^* u_i = \frac{1}{2}\int d^3\mathbf{r}\psi_g^*(f_i^+ + f_i^-) = \frac{1}{2}\int d^3\mathbf{r} f_i^-. \tag{A6}$$

After substituting Eqs. (47) into Eq. (A6), we obtain

$$c_i = \frac{1}{2} A_i l_c^3 \sqrt{\frac{n_0}{N_0}} \frac{8\pi}{15}\left(1 + \frac{1}{7}\sum_j \overline{b_j}\right) = \frac{1}{7} A_i \sqrt{\frac{N_0}{n_0}}, \tag{A7}$$

where we used Eq. (A2) and $N_0 = n_0 l_c^3 8\pi/15$ [Thomas-Fermi relation for $\mu(N_0)$] for the last step. We also have to calculate the normalization amplitudes for these modes and obtain from Eq. A3),

$$A_i = \left(\epsilon_i \xi l_c^3 \frac{4\pi}{105} \times \left[3\sum_j \overline{b}_j^2 + 2(\overline{b}_1\overline{b}_2 + \overline{b}_1\overline{b}_3 + \overline{b}_2\overline{b}_3) - 35 \right] \right)^{-1/2}. \tag{A8}$$

We can now calculate the coupling matrix element from Eqs. (46), (48). We shall introduce a constant R_{12} defined as follows:

$$R_{12} = \left(\sqrt{\frac{n_0}{N_0}} A_1^2 A_2 \xi l_c^3/2 \right) \frac{4\pi}{105}. \tag{A9}$$

The first part of the matrix element $M_{12}^{(1)}$ is then

$$M_{12}^{(1)} = -R_{12}\int_0^1 dy \Big\{ y^6 \Big[15\sum_i \overline{b}_i^2\overline{d}_i + 3(2\overline{b}_1\overline{b}_2\overline{d}_1 + \overline{b}_1^2\overline{d}_2 + 2\overline{b}_1\overline{b}_3\overline{d}_1 + \overline{b}_1^2\overline{d}_3 + 2\overline{b}_1\overline{b}_2\overline{d}_2 + \overline{b}_2^2\overline{d}_1 + 2\overline{b}_1\overline{b}_3\overline{d}_3 + \overline{b}_3^2\overline{d}_1 + 2\overline{b}_2\overline{b}_3\overline{d}_2 + \overline{b}_2^2\overline{d}_3 + 2\overline{b}_2\overline{b}_3\overline{d}_3 + \overline{b}_3^2\overline{d}_2) + 2\overline{b}_1\overline{b}_2\overline{d}_3 + 2\overline{b}_1\overline{b}_3\overline{d}_2 + 2\overline{b}_2\overline{b}_3\overline{d}_1 \Big] + y^4\Big[490 + 21\sum_i \overline{b}_i^2 + 7(2\overline{b}_1\overline{b}_2 + 2\overline{b}_2\overline{b}_3 + 2\overline{b}_3\overline{b}_1) \Big] - 525y^2 + 105 \Big\} y^2 \Delta\epsilon_{12}(1-y^2)dy. \tag{A10}$$

The second part of the matrix element is given in Eq. (A11). Note that this part arises due to the finite overlap between the condensate ground state and the untilded Bogoliubov wave functions u_i, v_i.

$$M_{12}^{(2)} = 4R_{12}\int_0^1 \Delta\epsilon_{12}\left\{ 35y^4 - \frac{325}{7}y^2 + \frac{90}{7} \right\} \times y^2(1-y^2)dy - 2\sqrt{\frac{n_0}{N_0}} A_2 \int_0^1 y^6(1-y^2)dy. \tag{A11}$$

We note that for resonant interaction $\Delta\epsilon_{12} = \epsilon_2 - 2\epsilon_1 = 0$, so that $M_{12}^{(1)} = 0$ and the only term surviving in $M_{12}^{(2)}$ is $-2\sqrt{n_0/N_0}A_2\int_0^1 y^6(1-y^2)dy$.

[1] G. Hechenblaikner, O.M. Marago, E. Hodby, J. Arlt, S. Hopkins, and C.J. Foot, Phys. Rev. Lett. **85**, 692 (2000).

[2] E. Hodby, O.M. Maragò, G. Hechenblaikner, and C.J. Foot, Phys. Rev. Lett. **86**, 2196 (2001).

[3] P. Öhberg, E.L. Surkov, I. Tittonen, S. Stenholm, M. Wilkens, and G.V. Shlyapnikov, Phys. Rev. A **56**, R3346 (1997)

[4] S.A. Morgan, S. Choi, and K. Burnett, Phys. Rev. A **57**, 3818 (1998)

[5] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).

[6] O.M. Maragò, S.A. Hopkins, J. Arlt, E. Hodby, G. Hechenblaikner, and C.J. Foot, Phys. Rev. Lett. **84**, 2056 (2000).

[7] M.O. da C. Pires and E.J.V. de Passos, J. Phys. B **33**, 3929 (2000).

[8] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[9] D.F. Walls and G.J. Milburn, *Quantum Optics* (Springer, New York, 1995).

[10] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).