Phase-space structure of the Penning trap with octupole perturbation

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A Lie transformation is developed to study the structure of classical phase space for a perturbed Penning trap. In general, perturbations may result from imperfections or may be deliberately introduced into the system by the application of fields. We study the lowest-order nontrivial perturbation in the trap, which is octupolar, using classical perturbation methods. The original three-degree-of-freedom problem is reduced to a single degree of freedom by (i) symmetry arguments, (ii) generation of apt action-angle variables, and (iii) computation of the classical *normal form*. The phase-space structure of the resulting normalized Hamiltonian, in the 1:1 resonance, is then analyzed. In the process we discover a saddle-node bifurcation. This approach provides for a global view of the reduced phase space, and, thereby, allows for a systematic study of the impact of several simultaneously applied perturbations.

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I. INTRODUCTION

The Penning trap [1] is an exceptionally stable technique for trapping charged particles. Consequently, it has become one of the most versatile experimental devices in atomic and molecular physics [2–5]. Applications of the Penning trap are broad and have included tests of fundamental theories, e.g., of QED, and the determination of fundamental physical constants such as the fine-structure constant [6–8]. As well as being useful for the study of isolated charged particles, as in the classic geonium experiment [9], modified Penning traps have been used with advantage to study single-component plasmas. Recently Penning traps have been shown to allow the trapping of large numbers of ions at very low temperatures: remarkably, ion crystals containing approximately 10^6 ions have been observed by Bragg diffraction. van Eijkelenborg *et al.* [10] have trapped the molecular ions HCO^+ and N_2H^+, which, through their interactions with Mg^+ ions, were sympathetically cooled to 4 K. Characteristic and mass-dependent breathing-mode frequencies were then used to detect the molecules. Thus, the Penning trap served essentially as a mass spectrometer. More recently the Penning trap has been used as a quantum computing device by taking advantage of hysteresis and bistability of a parametrically driven electron that served as a 1-bit memory [11,12]. There have also been a number of classical and quantum studies of the Penning trap together with perturbed variants thereof [13–16].

Given the variety of perturbations that are of interest, and the possibility of uncovering new effects by simultaneously applying several perturbations of various strengths, it is clearly desirable to develop a method for studying and conceptualizing the dynamics. For example, several unexpected phenomena have been discovered in perturbed Rydberg atoms using similar approaches [17]. Like the unperturbed hydrogen atom, the classical and quantum theories of the Penning trap itself are self-contained and exact. Indeed, the stability of the Penning trap derives largely from the simple nature of the confining potential: the system displays (i) axial symmetry, thus leading to angular momentum conservation and (ii) an essentially harmonic form of confining potential. This provides a natural launching point for investigating perturbations that destroy one or other, or both, of these symmetries. For example, Squires *et al.* [18] have studied the breakdown of axial symmetry in a so-called combined Penning-Ioffe trap and identified stable orbits that adiabatically continue from the axially symmetric case. These investigators were also able to identify particular frequencies and resonances that give rise to instability. In this spirit, the current paper studies the effect of octupolar perturbations on the phase-space structure of the Penning trap with the objective of developing a way of systematizing the search through phase space.

As indicated, while perturbations may arise from mechanical imperfections they are often deliberately introduced in a controlled fashion as a way to perturb the motions of the confined particles, e.g., in a mass spectrometric experiment. In the ideal trap, confinement along the z axis is realized by applying a three-electrode structure, two of them being, in the ideal case, hyperboloids of revolution, located along the z axis, and the third one, a ring electrode similar to the form of the inner surface of a toroid. Taken together these generate the electrostatic quadrupole potential that acts as a trap along the z direction. Along the x- and y-axes the motion, however, is unstable. To effect stable motion in the radial plane, a magnetic field along the z direction is added that causes charged particles to circulate along cyclotron-type orbits around the center of the trap. For details and description of several ion traps, the reader is referred to Ref. [2].

The highly nonlinear nature of the dynamics makes perturbed ion traps ideal candidates for studies of integrability and chaotic behavior in classical and quantum mechanics [15]. Bifurcations, unstable equilibria, and separatrices are the seeds of chaotic behavior [19] and govern the chaos-order-chaos transitions whose onset may be sudden and unexpected [13–15,17,20,21] and so possibly overlooked in experiments. In particular, bifurcations in *phase space* will have ramifications for spectroscopy as they lead to the emergence of new families of orbits. We emphasize that, in this paper, we are not necessarily looking for bifurcations that may be implied from a study of the potential-energy, or *effective* potential-energy, surface. It is certainly true that much of the richness of the dynamic of trapped ions can be understood based on the emergence or disappearance of equilibria in the (effective) potential. In the Paul trap a double well emerges for particular values of the parameters that allows the stable trapping of two ions in a dumbbell configuration [15]. Here our emphasis is, instead, on phase-space bifurcations that may not be related to any observable bifurcations in the potential-energy surface. An example of the importance of this type of bifurcation can be found in studies of the vibrational dynamics of small molecules; there, classical studies have proved remarkably effective in pinpointing transitions between librational and rotational modes [22].

To achieve our objective we apply a technique that has mainly been used in celestial mechanics, with, however, recent applications in atomic and chemical physics; normal form theory (NFT). To obtain the normal form one must first identify so-called ''good'' action-angle variables that are appropriate to the dynamics at hand. The determination of action-angle variables reflecting the actual dynamics of a system is a significant issue in treating resonant (degenerate) classical systems and one which has preoccupied celestial mechanicians and astronomers for more than a century (see, [23,24] and references herein). By means of the normalization, in the language of modern-day nonlinear classical dynamics, the actual chaotic (nonintegrable) effective two-degree-of-freedom Hamiltonian is replaced by a nonchaotic (integrable) approximation that is designed to provide good agreement with the real dynamics [21,25]. This paper is the first in a series that will use normal form methods to understand the various modes that can be induced in perturbed Penning traps containing one or more ions through the introduction of perturbations. By exciting the relevant modes of various ion species the ion cloud absorbs energy and this allows the detection of species of spectroscopic interest, e.g., in chemistry and astrophysics. In this paper we present the basic theory for studying the phase-space flow of a single charged particle in a perturbed Penning trap. Subsequently, the separate and combined effects of applying several perturbations, introducing other ions and breaking the overall axial symmetry, will be investigated using this approach and its extensions.

In addition to approximating the local dynamics close to a resonance, normalization, in the sense of NFT, also has a second interpretation: it is a *reduction* [26]. In the process of normalization, the number of degrees of freedom falls by one unit. Upon analyzing this fact from a geometric standpoint, one will recognize that ignoring one coordinate, let us say ψ_1, and holding its conjugate moment Ψ_1 as a parameter amounts to partitioning the phase space into leaves consisting of all states for which the parameter (the moment Ψ_1) has a given value, collecting into classes all points within that leaf that are images of one another by the canonical transformations generated by the integral Ψ_1, and then ''reducing'' the phase space on each leaf $\Psi_1=$const. by handling each class as an individual phase space. Thus the key that allows us to systematize the search through phase space as the strength and number of perturbations applied are varied.

The paper is organized as follows; Sec. II is devoted to the description of the problem of the motion in an unperturbed Penning trap, and obtaining the Hamiltonian in action-angle variables. In Sec. III we present the perturbing potential function in a realistic trap, focussing our interest on the axially symmetric case. For the octupole perturbation the problem (Sec. IV) can be reduced to only two degrees of freedom by making use of this symmetry. Then, we describe a set of action-angle variables that are particularly apt for the application of a Lie transformation. Of special interest is the analysis of the resonant case 1:1. In the presence of this resonance, we are able, first, to normalize the Hamiltonian, and second, to find the equilibria. In the process we discover a saddle-node bifurcation in the normalized problem. Our analysis provides a global picture of the various bifurcations and allows one to understand the effects of varying the perturbing influences on the dynamics. Conclusions are in Sec. V.

II. THE HAMILTONIAN

In an ideal Penning trap, the electrode surfaces are infinite hyperboloids of revolution whose equations are

$$\frac{r^2}{r_0^2}-\frac{z^2}{z_0^2}=\pm 1,$$

where r_0 is the inner radius of the ring electrode and z_0 is half the distance between the two end caps. A positive voltage U is applied to the end-cap electrodes with respect to the ring electrode. Besides this electric field, a time-independent spatially homogeneous magnetic field B is present along the z direction.

Since in a Penning trap both a constant electric and a constant magnetic field are present, the potential is taken to be $(\phi,\frac{1}{2}\boldsymbol{B}\times\boldsymbol{x})$. For a particle of mass m moving in an ideal Penning trap, the Lagrangian is

$$\mathcal{L}=\frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{eB}{2c}(x\dot{y}-y\dot{x})-e\phi, \tag{1}$$

where the quadrupole electric potential is

$$\phi=\frac{U}{2\,z_0^2+r_0^2}(2z^2-x^2-y^2).$$

The Hamiltonian will be found as the Legendre transformation of the Lagrangian function

$$\mathcal{H}(\boldsymbol{X},\boldsymbol{x})=\boldsymbol{X}\cdot\dot{\boldsymbol{x}}-\mathcal{L},$$

where the canonically conjugate moments $\boldsymbol{X}$ are the partial derivatives

$$\boldsymbol{X}=\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}}=\boldsymbol{\nabla}_{\dot{x}}\mathcal{L}.$$

Thus, it results that the Hamiltonian is

$$\mathcal{H}=\frac{1}{2m}(X^2+Y^2+Z^2)-\frac{\omega_c}{2}(xY-yX)+\frac{m}{2}\frac{\omega_m^2}{4}(x^2+y^2)+\frac{m}{2}\omega_z^2z^2, \tag{2}$$

where

$$\omega_c=\frac{eB}{mc}$$

is the *cyclotron frequency*,

$$\omega_z=\sqrt{\frac{4eU}{m(2\,z_0^2+r_0^2)}}$$

is the *axial frequency*, and

$$\omega_m=\sqrt{\omega_c^2-2\,\omega_z^2}$$

is a frequency related to the magnetron frequency (see [2] p. 51 for details).

From here on, we take the unit of mass to be $m=1$.

This Hamiltonian is integrable and may be put as a linear combination of three harmonic oscillators. Indeed, the z variable is uncoupled with x and y, hence, the classical Poincaré transformation

$$Z=\sqrt{2\,\omega_z\Phi_3}\cos\varphi_3,\quad z=\sqrt{\frac{2\,\Phi_3}{\omega_z}}\sin\varphi_3 \tag{3}$$

converts the term

$$\mathcal{H}_z=\tfrac{1}{2}Z^2+\tfrac{1}{2}\omega_z^2z^2$$

into $\mathcal{H}_z=\omega_3\Phi_3$, where we put $\omega_z=\omega_3$.

On the other hand, the Lissajous transformation [27] defined by

$$\begin{aligned}X&=-\omega_1[s\sin(\varphi_1+\varphi_2)+d\sin(\varphi_2-\varphi_1)],\\ Y&=\omega_1[s\cos(\varphi_1+\varphi_2)+d\cos(\varphi_2-\varphi_1)],\\ x&=s\cos(\varphi_1+\varphi_2)-d\cos(\varphi_2-\varphi_1),\\ y&=s\sin(\varphi_1+\varphi_2)-d\sin(\varphi_2-\varphi_1),\end{aligned} \tag{4}$$

with s and d given by

$$\omega_1s^2=(\Phi_1+\Phi_2)/2,\quad \omega_1d^2=(\Phi_1-\Phi_2)/2,$$

renders the expression

$$\mathcal{H}_{xy}=\frac{1}{2}(X^2+Y^2)+\frac{1}{8}\omega_m^2(x^2+y^2)-\frac{\omega_c}{2}(xY-yX)$$

into

$$\mathcal{H}_{xy}=\omega_1\Phi_1-\omega_2\Phi_2,$$

where

$$\omega_1=\omega_m/2,\quad \omega_2=\omega_c/2.$$

Thus, the canonical transformation made of the composition of the two canonical ones, Lissajous (4) and Poincaré (3), converts the Hamiltonian (2) corresponding to the ideal Penning trap into

$$\mathcal{H}_0=\omega_1\Phi_1-\omega_2\Phi_2+\omega_3\Phi_3. \tag{5}$$

of immediate integration [3].

III. PERTURBED PENNING TRAP

A real Penning trap differs from the ideal trap in several aspects. It has a finite size, the geometrical shape of the electrodes may differ from that of an ideal trap, there may be misalignment in the electrodes or the magnetic field, and other perturbations may be added intentionally. Thus, the scalar potential Φ and the vector potential that currently describe the electromagnetic field on the trap routinely differ from the one corresponding to the ideal hyperbolic Penning trap.

Imperfections of the electrostatic field can be presented in a form of the series of spherical Legendre functions [2,3,28]

$$\delta\Phi(r,\theta,\varphi)=\sum_{0\leqslant l\leqslant m\leqslant\infty}a_{l,m}r^lP_l^m(\cos\theta)\cos(m\varphi), \tag{6}$$

where (r,θ,φ) are the spherical coordinates, and P_l^m the associated Legendre polynomials.

Symmetries of the perturbation reduce the number of terms in the above expression. Indeed, symmetry under space reflections with respect to the origin requires $a_{l,m}=0$ for all odd l. On the other hand, axial symmetry about the z axis requires the vanishing of all coefficients $a_{l,m}$ with $m>0$. Thus, when these two symmetries take place, the perturbation can be written in Cartesian coordinates as follows:

$$P(x,y,z)=\sum_{0\leqslant l\leqslant\infty}a_{2l}r^{2l}P_{2l}(z/r)=\sum_{0\leqslant l}V_{2l}.$$

After some computations, one finds that the quadrupole term is

$$V_2 = \tfrac{1}{2} a_2 (2z^2 - x^2 - y^2).$$

The octupole term is

$$V_4 = \tfrac{1}{8} a_4 [8z^4 - 24z^2(x^2+y^2) + 3(x^2+y^2)^2].$$

The contribution of the term, $l=6$, is

$$V_6 = \frac{a_8}{16}[16z^6 - 120z^4(x^2+y^2) + 90z^2(x^2+y^2)^2 - 5(x^2+y^2)^3].$$

The dominating terms of the nonlinear contribution depend on the actual trap geometry [5]. Thus, for example, in a trap where the ideally hyperbolic electrodes are approximated by electrodes of spherical section, the main nonlinear term is the octupole [29].

Quadrupole perturbation does not contribute to the difficulty of the problem, since the problem $\mathcal{H}_0 + V_2$ still is integrable. Indeed, after rearranging terms, it results

$$\mathcal{H} = \mathcal{H}_0 + V_2 = \frac{1}{2m}(X^2+Y^2+Z^2) - \frac{\omega_c}{2}(xY - yX) + \frac{m}{2}\frac{\tilde{\omega}_m^2}{4}(x^2+y^2) + \frac{m}{2}\tilde{\omega}_z^2 z^2,$$

with $\tilde{\omega}_m^2 = \omega_m^2 - 4a_2$ and $\tilde{\omega}_z^2 = \omega_z^2 + 2a_2$. This Hamiltonian has exactly the same form as the Hamiltonian (2), which, as above proved, is integrable. Thus, only higher terms in the multipole expansion will be considered.

The quadrupole ($l=2$) case was analytically solved by Horvath *et al.* [5]. Our goal here is to study the octupole ($l=4$) contribution, to obtain information about stability regions and to show the behavior of the phase flow.

IV. OCTUPOLE PERTURBATION WITH AXIAL SYMMETRY

Let us consider that the perturbation is given by the octupole term with axial symmetry, which is to say, $l=4$. In such a case, the Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0 + V_4.$$

This Hamiltonian is invariant by rotations about the z axis, thus, by virtue of Noether's theorem, the projection along this axis of the angular momentum vector is an integral, therefore, the term $xY - yX$ is a constant that may be dropped from the Hamiltonian that is reduced to

$$\mathcal{H} = \tfrac{1}{2}(X^2+Y^2+Z^2) + \tfrac{1}{2}\omega_1(x^2+y^2) + \tfrac{1}{2}\omega_3^2 z^2 + \tfrac{1}{8}a_4[8z^4 - 24z^2(x^2+y^2) + 3(x^2+y^2)^2]. \tag{7}$$

The existence of this integral would amount to reduce the degrees of freedom by one. Indeed, by the canonical transformation (4), the term (x^2+y^2) depends only on the angle φ_1, while the transformation (3) makes the term z^2 depending on the angle φ_3, and consequently, the angle φ_2 is cyclic in the Hamiltonian (7)

$$\begin{aligned}\mathcal{H} = \mathcal{H}_0 + a_4\mathcal{H}_1 = \omega_1\Phi_1 + \omega_3\Phi_3 + \frac{a_4}{16\omega_1^2\omega_3^2} &\times [3(8\omega_1^2\Phi_3^2 - 16\omega_1\omega_3\Phi_1\Phi_3 - \omega_3^2\Phi_2^2 + 3\omega_3^2\Phi_1^2) \\ &- 12\omega_3\sqrt{\Phi_1^2-\Phi_2^2}(\omega_3\Phi_1 - 4\omega_1\Phi_3)\cos 2\varphi_1 \\ &+ 16\omega_1\Phi_3(3\omega_3\Phi_1 - 2\omega_1\Phi_3)\cos 2\varphi_3 \\ &+ 3\omega_3^2(\Phi_1^2-\Phi_2^2)\cos 4\varphi_1 + 8\omega_1^2\Phi_3^2\cos 4\varphi_3 \\ &- 24\Phi_3\sqrt{\Phi_1^2-\Phi_2^2}\,\omega_1\omega_3\cos 2(\varphi_1+\varphi_3) \\ &- 24\Phi_3\sqrt{\Phi_1^2-\Phi_2^2}\,\omega_1\omega_3\cos 2(\varphi_1-\varphi_3)].\end{aligned} \tag{8}$$

Considering the relatively simple nature of the perturbations, a straightforward Lie transformation is sufficient to carry out a reduction. The Lie transformation itself is easy to build; symbolic processors currently available simplify the task as long as it is not carried out beyond excessively high orders. In the present case, there is no need for carrying out the reduction beyond order two.

Normalization of a Hamiltonian of type

$$\mathcal{H}(\boldsymbol{p},\boldsymbol{P},\epsilon) = \sum_{n\geqslant 0}\epsilon^n\mathcal{H}_n(\boldsymbol{p},\boldsymbol{P}),$$

we recall [30,26,31], is a one-parameter family of canonical transformations

$$\nu:(\boldsymbol{p}',\boldsymbol{P}',\epsilon)\rightarrow(\boldsymbol{p},\boldsymbol{P})$$

that changes $\mathcal{H}$ into a function

$$\nu\mathcal{H}(\boldsymbol{p}',\boldsymbol{P}',\epsilon) = \mathcal{H}(\boldsymbol{p}(\boldsymbol{p}',\boldsymbol{P}',\epsilon),\boldsymbol{P}(\boldsymbol{p}',\boldsymbol{P}',\epsilon),\epsilon)$$

in the kernel of the Lie derivative L_0.

The Lie derivative associated with $\mathcal{H}_0$ is the partial differential operator

$$L_0: F \rightarrow (F,\mathcal{H}_0)$$

mapping F onto its Poisson bracket to the right with $\mathcal{H}_0$. The *kernel* of L_0 is the set of functions F such that $L_0(F)=0$; the *image* of L_0, the set of functions F of the form $F = L_0(G)$.

In our case, the Lie derivative of the Hamiltonian $\mathcal{H}_0$ in Eq. (8) is

$$L_0 = \omega_1\frac{\partial}{\partial\varphi_1} + \omega_3\frac{\partial}{\partial\varphi_3},$$

thus, when the two frequencies ω_1 and ω_3 are not commensurable, the term

$$\mathcal{H}_1=\frac{3a_4}{16\omega_1^2\omega_3^2}(8\omega_1^2\Phi_3^2-16\omega_1\omega_3\Phi_1\Phi_3-\omega_3^2\Phi_2^2+3\omega_3^2\Phi_1^2)$$

belongs to the kernel of the Lie derivative, and will be chosen as the normalized perturbed Hamiltonian (we omit the primes for the sake of simplifying the notation). The reduced Hamiltonian depends only on the moments and it is directly integrable.

Experiments with devices that modify the axial frequency by superimposing a quadratic magnetic-field component on the homogeneous magnetic field of the Penning trap are reported in Ref. [6].

In the case of commensurability, that is, when the two frequencies ω_1 and ω_3 are equal, the term in Eq. (8) containing $\cos 2(\varphi_1-\varphi_2)$ belongs to the kernel of L_0, thus, it must remain in the new Hamiltonian, which is of two degrees of freedom and, in principle, not integrable. However, it is possible to reduce the degrees of freedom by making a new canonical transformation $(\Phi_1,\Phi_3,\varphi_1,\varphi_3)\mapsto(\Psi_1,\Psi_3,\psi_1,\psi_3)$, similar to the one given in [32,33], as the mapping

$$\Phi_1=\Psi_1+\Psi_3,\quad \varphi_1=\tfrac{1}{2}(\psi_1+\psi_3),$$

$$\Phi_3=\Psi_1-\Psi_3,\quad \varphi_3=\tfrac{1}{2}(\psi_1-\psi_3). \tag{9}$$

In these variables, the Hamiltonian (8) reads

$$\begin{aligned}\mathcal{H}&=\mathcal{H}_0+a_4\mathcal{H}_1\\&=2\omega_1\Psi_1-\frac{a_4}{16\omega_1^2}[3(\Phi_2^2+5\Psi_1^2+10\Psi_1\Psi_3-27\Psi_3^2)\\&\quad+24(\Psi_1-\Psi_3)\sqrt{(\Psi_1+\Psi_3)^2-\Phi_2^2}\cos 2\psi_1+24(\Psi_1\\&\quad-\Psi_3)\sqrt{(\Psi_1+\Psi_3)^2-\Phi_2^2}\cos 2\psi_3\\&\quad-12(3\Psi_1-5\Psi_3)\sqrt{(\Psi_1+\Psi_3)^2-\Phi_2^2}\cos(\psi_1+\psi_3)\\&\quad-16(\Psi_1-\Psi_3)(\Psi_1+5\Psi_3)\cos(\psi_1-\psi_3)\\&\quad-3((\Psi_1+\Psi_3)^2-\Phi_2^2)\cos 2(\psi_1+\psi_3)\\&\quad-8(\Psi_1-\Psi_3)^2\cos 2(\psi_1-\psi_3)].\end{aligned} \tag{10}$$

The perturbation belongs to the algebra of Fourier series in the angle ψ_3; hence, the kernel of L_0 consists of the subalgebra of functions that do not depend on ψ_3, and the normalization that we have in view is a Delaunay normalization that converts $\mathcal{H}$ into its average over the angle ψ_3 [34]. The Lie derivative is very simple; indeed, it is the partial derivative

$$L_0=2\omega_1\frac{\partial}{\partial\psi_1},$$

therefore, the reduced Hamiltonian [those terms of Eq. (10) belonging to the kernel of this Lie derivative] is

$$\begin{aligned}\mathcal{H}&=2\omega_1\Psi_1-\frac{3a_4}{16\omega_1^2}[(\Phi_2^2+5\Psi_1^2+10\Psi_1\Psi_3-27\Psi_3^2)\\&\quad+8(\Psi_1-\Psi_3)\sqrt{(\Psi_1+\Psi_3)^2-\Phi_2^2}\cos 2\psi_3],\end{aligned}$$

which is a Hamiltonian with only one degree of freedom in the variables (ψ_3,Ψ_3), since now Ψ_1 is an integral.

First at all, let us note that the value of Φ_2 is irrelevant, assuming it to be nonzero. Indeed, by defining the dimensionless moments

$$\tilde{\Psi}_1=\Psi_1/\Phi_2,\quad \tilde{\Psi}_3=\Psi_3/\Phi_2,$$

and by the time scaling $\tau=\Phi_2 t$, the Hamiltonian becomes

$$\begin{aligned}\mathcal{H}&=2\omega_1\tilde{\Psi}_1-\frac{3a_4}{16\omega_1^2}[(1+5\tilde{\Psi}_1^2+10\tilde{\Psi}_1\tilde{\Psi}_3-27\tilde{\Psi}_3^2)\\&\quad+8(\tilde{\Psi}_1-\tilde{\Psi}_3)\sqrt{(\tilde{\Psi}_1+\tilde{\Psi}_3)^2-1}\cos 2\psi_3].\end{aligned}$$

The equations of motion for this Hamiltonian are

$$\frac{d\Psi_3}{d\tau}=-\frac{\partial\mathcal{H}}{\partial\psi_3}=-\frac{3a_4}{\omega_1^2}(\tilde{\Psi}_1-\tilde{\Psi}_3)\sqrt{(\tilde{\Psi}_1+\tilde{\Psi}_3)^2-1}\sin 2\psi_3,$$

$$\begin{aligned}\frac{d\psi_3}{d\tau}=\frac{\partial\mathcal{H}}{\partial\tilde{\Psi}_3}=-\frac{3a_4}{8\omega_1^2}\Bigg[&(5\tilde{\Psi}_1-27\tilde{\Psi}_3)\\&-\frac{4(1-2\tilde{\Psi}_1\tilde{\Psi}_3-2\tilde{\Psi}_3^2)}{\sqrt{(\tilde{\Psi}_1+\tilde{\Psi}_3)^2-1}}\Bigg]\cos 2\psi_3.\end{aligned}$$

Equilibria are the roots of the system made by equating to zero the equations of the motion. Note that not all solutions of this system are valid, since on the one hand, from the definition of the Lissajous variables, $\Phi_1>\Phi_2$, and $\Phi_3\geqslant 0$, and besides, from Eq. (9), it results that $\Psi_1>1/2$, $\Psi_3<\Psi_1$, and $\Psi_1+\Psi_3=\Phi_1>\Phi_2$, therefore, $\tilde{\Psi}_1+\tilde{\Psi}_3>1$. According to this, $\psi_3=\pi/4$ or $\psi_3=3\pi/4$, which cancel out the second equation are not solutions.

The first one of the above equations vanishes for either $\psi_3=0$ or $\psi_3=\pi/2$. For the first case ($\psi_3=0$) the equilibria are the roots of the equation

$$(5\tilde{\Psi}_1-27\tilde{\Psi}_3)-\frac{4(1-2\tilde{\Psi}_1\tilde{\Psi}_3-2\tilde{\Psi}_3^2)}{\sqrt{(\tilde{\Psi}_1+\tilde{\Psi}_3)^2-1}}=0, \tag{11}$$

which always exists for $\tilde{\Psi}_1\geqslant 0.5$. The plot $\tilde{\Psi}_1$ versus $\tilde{\Psi}_3$ of the roots of this equation is represented in Fig. 1 (above).

For the second case ($\psi_3=\pi/2$), the equilibria must obey the equation

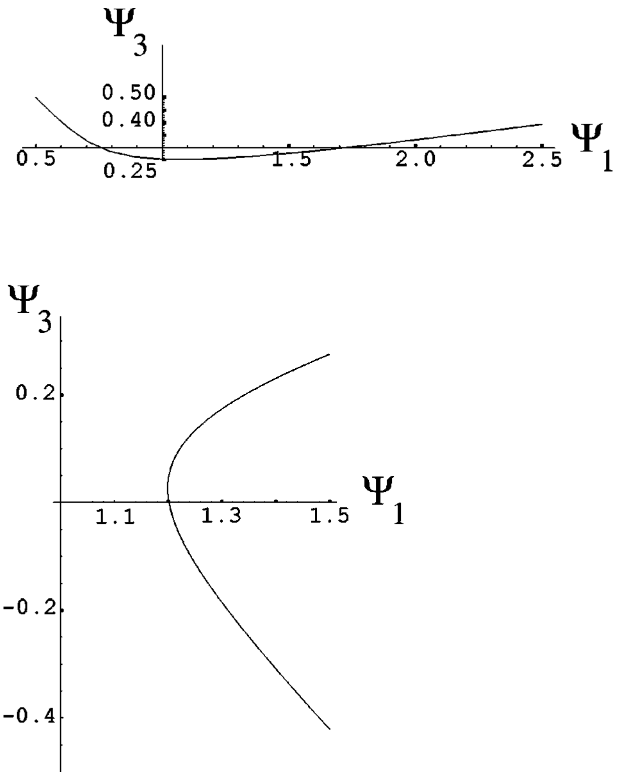

FIG. 1. Geometrical *locus* of the equilibria: above $\psi_3=0$, below $\psi_3=\pi/2$.

$$(5\tilde{\Psi}_1-27\tilde{\Psi}_3)+\frac{4(1-2\tilde{\Psi}_1\tilde{\Psi}_3-2\tilde{\Psi}_3^2)}{\sqrt{(\tilde{\Psi}_1+\tilde{\Psi}_3)^2-1}}=0, \qquad (12)$$

and it is represented in Fig. 1 (below). Note that this equation has two roots for $\tilde{\Psi}_1 \geqslant 1.199\,009\,336\,219$, and none outside this interval. Thus, there is a bifurcation for this value. Although we do not prove analytically the stability of the equilibria, we can see, by inspection of the phase flow that the equilibria at $\psi_3=0,\pi$ are always stable, whereas of the two equilibria at $\psi_3=\pi/2$, one of the equilibrium points (corresponding to the smaller value of $\tilde{\Psi}_3$) is a local energy maximum and is also stable. The remaining equilibrium is a saddle point and is unstable (see Fig. 2, obtained for the value $\tilde{\Psi}_1=1.5$).

V. CONCLUSIONS

In this short paper we have provided the basis for a systematic study of the structure of phase space for a single ion in a Penning trap perturbed by, in principle, a number of disturbing forces simultaneously applied. For the case at hand we discovered a saddle-node bifurcation. The bifurcation manifests itself by the appearance, at a critical value of $\tilde{\Psi}_1$, of two new equilibria along the line $\psi_3=\pi/2$, one stable and another one unstable. Since an equilibrium in the reduced Hamiltonian accounts for a periodic orbit in the original problem, a closed-form expression for the homoclinic passing by the unstable point would be very helpful; indeed, together with Melnikov's theorem [31], it would serve to decide which perturbations added to the quadrupole potential would induce chaotic behavior in the neighborhood of the unstable periodic solution. Moreover, this approach will be useful for studying several combined perturbations. A nontrivial extension of the method will be to the case of *two* or more interacting ions for which new sets of "good" action-angle variables are needed. Work in all of these directions is underway.

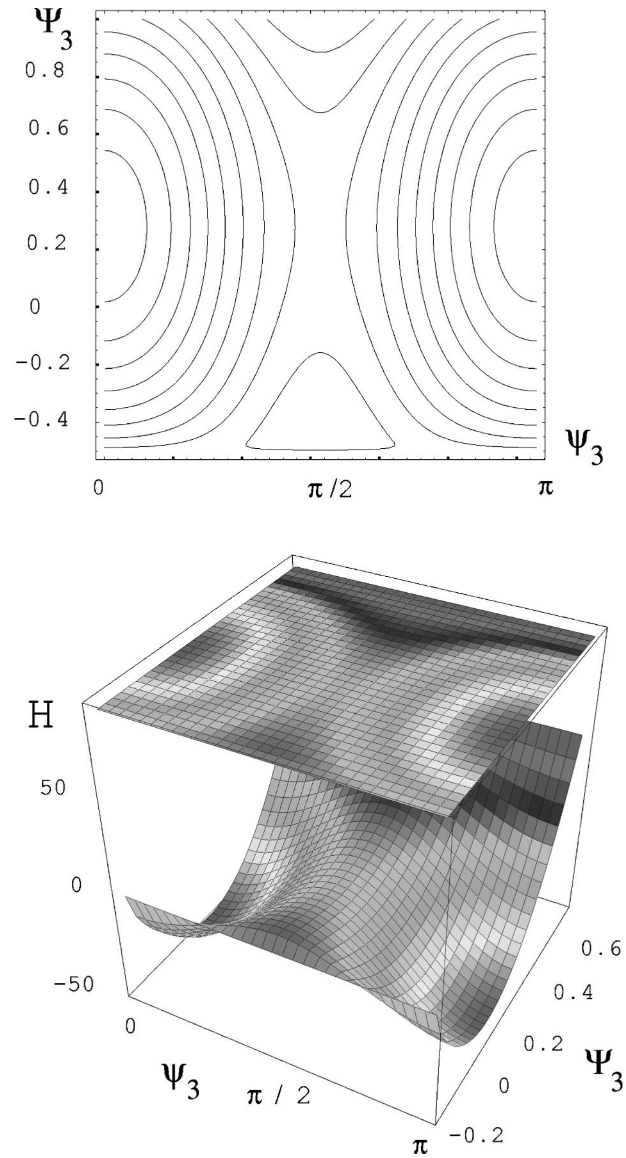

FIG. 2. Energy plot (bottom) and contour levels (top) for $\Psi_1=1.5$. We identify four equilibria, three stable and one unstable.

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