Classical analog of entanglement

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We show that quantum entanglement has a very close classical analog, namely, secret classical correlations. The fundamental analogy stems from the behavior of quantum entanglement under local operations and classical communication and the behavior of secret correlations under local operations and public communication. A large number of derived analogies follow. In particular, teleportation is analogous to the one time pad, the concept of ''pure state'' exists in the classical domain, entanglement concentration and dilution are essentially classical secrecy protocols, and single-copy-entanglement manipulations have such a close classical analog that the majorization results are reproduced in the classical setting. This analogy allows one to import questions from the quantum domain into the classical one, and vice versa, helping to get a better understanding of both. Also, by identifying classical aspects of quantum entanglement, it allows one to identify those aspects of entanglement that are uniquely quantum mechanical.

I. INTRODUCTION

In his pioneering paper of 1964 [1], Bell pointed out the nonlocal character of quantum-mechanical long-distance correlations. Since then, many novel aspects of nonlocality have been uncovered, such as teleportation [2], superdense coding [3], and the capability to reduce the number of required bits of classical communication for implementing certain communication tasks (in the so called "communication-complexity scenario") [4]. Furthermore, entanglement and nonlocality are at the core of quantum computation [5] and its capability of performing computations faster than any classical computer. An enormous effort has been dedicated during the last few years to understand the qualitative and quantitative properties of nonlocality. In effect, quantum nonlocality has become to be considered one of, if not the most representative aspect of quantum mechanics. Quite surprisingly we found, as we describe in the present paper, that there exists a quite close classical analog of quantum entanglement, namely, *secret classical correlations*.

Our motivation in looking for a classical analog of quantum entanglement is twofold. First, such an analogy allows us to identify aspects of quantum entanglement that were hitherto considered to be purely quantum but which are, in fact, not quantum at all. Indeed, all those aspects of entanglement that are common with the classical analog, are not of a quantum nature. As a corollary, we also get a better understanding of what are the true quantum features of quantum entanglement. Second, this analogy allows one to transfer questions from quantum entanglement to the classical domain (classical information cryptography) and vice versa and thus lead to a better understanding of both subjects. In fact, the inspiration for our paper stems from the work of Gisin and Wolf [6], which asked if there is a classical analog of bound entanglement.

The analogy we suggest is summarized in the following table:

Quantum entanglement	Secret classical correlations
Quantum communication	Secret classical communication
Classical communication	Public classical communication
Local actions	Local actions

Thus, we suggest that a classical analog of a pair of entangled particles is that of one sample of two secret, correlated, random variables (one at each remote party). Here, by secret communication we mean communication through a channel to which an eavesdropper has no access. By public communication we mean communication through a channel to which an eavesdropper has full access (can hear everything), but neither alter the messages sent, nor introduce new messages. Finally, in the quantum context, by local actions we mean subjecting the qubits to unitary evolutions as well as to measurements and other nonunitary evolutions. The classical analog of unitary transformations is that of replacing the value of the original random variable by some new value related to the old one by a one-to-one function, while the analog of the case of quantum nonunitary evolutions is that of transformation by nonbijective functions.[1]

The main idea of this analogy is that, similarly to quantum entanglement, secret classical correlations act as a (fungible) *resource* and obey a "second law of thermodynamics"

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[1]Note that when we replace the original value of the random variable by another via a nonbijective function, we consider that we actually erase the original information, so information is lost. This is completely analogous to what happens in the quantum case. Of course, one may argue that in neither case information is lost. For example, in the noncollapse interpretations of the quantum case all we have is an entanglement of the measured system with the measuring device; this entanglement, however, involves so many degrees of freedom that it cannot be reversed. Similarly, erasing, say pencil markings from a paper, still preserves the original information in some subtle arrangement of the graphite granules mixed with bits of paper and erasing gum, but this involves so many degrees of freedom that the original information cannot be recovered.

principle—the amount of secrecy doesn't increase under LOPC (local actions and public communication).

The modern paradigm is that of quantum nonlocality as a *resource* as we describe below.

(i) Nonlocal correlations between two or more remote parties can be created by quantum communication, i.e., by sending quantum particles (qubits) from a common source to the parties, or from one party to another.

(ii) Second law of thermodynamics. The amount of nonlocality between the remote parties cannot be increased by local actions and/or classical communication (LOCC). Indeed, one can view this statement as the very *definition* of what nonlocality is. The above version of the second law can be further extended to allow for quantum communication, catalysis, etc. For example [7], "by local actions, classical communication, and exchange of n qubits, the amount of nonlocality between remote parties cannot be increased by more that n ebits."

(iii) The remote parties can, by local actions and classical communication, transform nonlocality from one form into another.

For example, suppose two parties, Alice and Bob, have a large number of pairs of particles, each pair in some pure, nonmaximally entangled state. By appropriate actions, they can end up with a smaller number of pairs in maximally entangled states [8,7]. In effect, at least in the case of bipartite pure states, nonlocality is absolutely fungible—any form can be transformed into any other, and the transformation is reversible. Thus, it doesn't really matter in which form the parties are supplied with nonlocality, they can always convert it into the form that is required for implementing the specific task (for example teleportation) they want to do.

(iv) Nonlocality is consumed for producing useful tasks (teleportation, superdense coding, remote implementation of joint unitary transformations [9], etc.). As with quantum nonlocal correlations, secret correlations are also a resource.

(v) Secret correlations can be established between remote parties by secret communication.

(vi) Second law of thermodynamics. The amount of secret correlations cannot be increased by local actions and/or public communication (LOPC).[2] In fact, as in the case of nonlocality, we can take this law to be the very definition of the amount of secret correlations, i.e., the amount of secret correlations between remote parties is that part of their correlations that cannot be increased by local actions and public classical communication.

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[2] In everyday practice, secret messages are exchanged by public communication by so called "public-key-distribution" protocols. We do not consider here this case since these are only pseudosecret messages—their secrecy is based on encoding that is difficult to decode due to computational complexity; in principle, however, an eavesdropper could decode the message.

change of n secret bits, the amount of secret correlations between remote parties cannot be increased by more that n secret correlation bits."

(vii) The remote parties can, by local actions and public communication, transform secret correlations from one form into another.

(viii) Analogous to entanglement, secret correlations are a fungible resource—they can be stored, transformed from one form into another, and can be consumed to perform useful tasks, such as secret communication via the one-time pad [10].

The possibility of transforming secret correlations from one form into another enables us, similarly to the case of quantum correlations, to obtain a *quantitative* description of secrecy.

In the bi-partite case, the analogy is now obvious, as follows:

Shared, undirected resources

e-bit$_{AB}$	shared secret bit$_{AB}$

Directed resources

qubit$_{A \to B}$	secret bit$_{A \to B}$
classical bit$_{A \to B}$	public classical bit$_{A \to B}$

The situation of multi-partite secret correlations is more complicated, as is the situation of multi-partite entanglement. It is now clear that there are many different, irreducible, types of multi-partite entanglement [11,12]; this is also the case for secret correlations.

At this point it is legitimate to ask what is the role of secrecy. That is, why do we consider *secret* classical correlations to be the analog of entanglement and not simply *any* classical correlations. There are two main reasons. The first reason is that while such an analogy is certainly possible, it would be rather uninteresting. Indeed, one of the main aspects of manipulating entanglement is that there is a way in which the different parties may communicate (classical communication) which doesn't increase the amount of entanglement. Similarly in the case of secret classical correlations, public communication doesn't increase the amount of secrecy. In the case of arbitrary classical correlations however there is no way in which the remote parties could communicate and not increase the correlations. So when trying to build an LOCC ("local operations and classical communications") analog in the case of arbitrary classical correlations, we have no choice but to completely eliminate the communication, which leads to a very uninteresting situation.

The second reason is far more profound. Consider, for example, two parties, Alice and Bob who share a maximally entangled state $|\Psi\rangle = (1/\sqrt{2})(|0\rangle|0\rangle + |1\rangle|1\rangle)$. Suppose now that Alice and Bob "degrade" the state by "erasing" the entanglement. They can do this *in a minimal way* by Alice randomizing the phase of her basis state vectors $\{|0\rangle, |1\rangle\}$. Then Alice and Bob will be left with a mixture of $(1/\sqrt{2})(|0\rangle|0\rangle + |1\rangle|1\rangle)$ and $(1/\sqrt{2})(|0\rangle|0\rangle - |1\rangle|1\rangle)$ with equal probabilities. This mixture contains no entanglement (it is equivalent to an equal mixture of $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$) but contains secret correlations between Alice and Bob. Thus secret correlations are, in fact, very closely related to entanglement.

The analogies described above are the "fundamental" analogies. From them follow an entire set of derived analo-

gies. We would like to emphasize, however, that it is only the fundamental analogies (such as the behavior under LOCC/LOPC) that have truly deep significance and that one shouldn't expect the derived analogies to be very close (though many of them are). Derived analogies are summarized in the following table:

Quantum	Classical
Teleportation	One time pad
Entanglement concentration	Secret-correlation concentration
Entanglement dilution	Secret-correlation dilution
Entanglement purification	Classical privacy amplification
Single-copy transformations	Single-copy transformations
Probabilistic single copy transformations	Probabilistic single-copy transformations
Catalytic transformations	Catalytic transformations
Bound entanglement	Bound information ?

II. QUANTUM STATES AND CLASSICAL ANALOGS

In the previous section, we suggested that classical secret correlations are a good analog for quantum entanglement. Again, the basis of the analogy is the similar behavior of secret correlation and quantum entanglement under LOPC/LOCC. To make the analogy more detailed and to obtain the "derived" analogies mentioned above we need to define more precisely the analogy between quantum states and secret correlations.

Consider two remote parties, Alice and Bob. A general quantum state is described by a density matrix ρ_{AB} or, equivalently, by a pure state Ψ_{ABE} in which A and B are entangled with a third party, the "environment." The classical equivalent of the general quantum state is a probability distribution $P(X_A, X_B, X_E)$ where X_A, X_B, and X_E are random variables known to Alice, Bob, and Eve (the eavesdropper), respectively. One copy of a quantum state Ψ_{ABE} corresponds to one sample of the probability distribution $P(X_A, X_B, X_E)$.

A quantum bipartite pure state can always be written in the Schmidt basis [13] as

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |i\rangle_A |i\rangle_B. \qquad (1)$$

If Alice and Bob measure their particles in the Schmidt basis then they get correlated random variables, X_A and X_B, which come according to the distribution $p(X_A=i, X_B=j) = \delta_{ij} p_i$. In other words, they both get the same sample from a random variable $X \sim \{p_i\}$. Furthermore, the values of X_A and X_B are secret—there is no third party E who knows them. We propose classical distributions of this form as the classical "pure" state. That is, a bipartite classical pure state is a distribution

$$p(X_A=i, X_B=j, X_E=k) = \delta_{ij} p_i \widetilde{P}(E_k), \qquad (2)$$

where $\widetilde{P}(E_k)$ is the distribution of the eavesdropper's variable X_E and it is completely irrelevant, except for the fact that it is completely uncorrected to the distribution of X_A and X_B.[3] Strictly speaking, we propose Eq. (2) as the classical analog of the pure-state Schmidt decomposition, and any classical state that is locally equivalent, i.e., can be transformed into the above form by local, one-to-one mappings (the equivalent of local unitaries) we consider to be a pure state.

Another interesting case is that of distributions of the form $p(X_A=i, X_B=j, X_E=k) = P(X_A=i, X_B=j)\widetilde{P}(E_k)$ in which E is completely uncorrelated with A and B, but A and B are not completely correlated with each other. Such a distribution is obtained when Alice and/or Bob measure a quantum pure state in some other basis than the Schmidt one. Such a distribution has some characteristics of a pure state and some characteristics of a mixed state. We will discuss this case in more detail in Sec. XI.

For more than two parties the analog of a density matrix $\rho_{ABC}\ldots$ is a probability distribution $P(X_A, X_B, X_C, \ldots)$. It is not yet clear to us what the general analog of a multipartite pure state is. This is due, in part, to the fact that for multipartite states the analog of the Schmidt decomposition is far more complicated. We shall give some multipartite results in Sec. XII.

III. TELEPORTATION AND THE ONE TIME PAD

The first "derived" analogy is probably the most striking of all. The fundamental quantum-communication protocol, that is, teleportation turns out to be analogous to the fundamental secret communication protocol, the one time pad [14].

Alice begins with the qubit (secret bit) to be sent, which may be entangled (secretly-correlated) with any number of other particles (bits). She does a Bell measurement (addition modulo 2) on the qubit (secret bit) to be sent and the qubit (bit) of resource she holds. She then sends the outcome (result) of this operation as a classical bit (public bit) to Bob. He then does a conditional unitary (bit flip) upon his part of the ebit (shared secret bit). Bob now holds the qubit (secret bit) Alice was sending him.

The necessary and sufficient resources are given by

$$1\,\text{ebit}_{AB} + 2\,\text{classical bits}_{A\to B} \Rightarrow 1\,\text{qubit}_{A\to B}, \qquad (3)$$

$$1\,\text{shared secret bit}_{AB} + 1\,\text{public bit}_{A\to B} \Rightarrow 1\,\text{secret bit}_{A\to B}. \qquad (4)$$

By necessary we mean that, if we were to try to do the teleportation with less than one ebit—by using a less than maximally entangled state, for example—the teleportation will not give a perfect output, and the classical information

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[3]Note that, quantum mechanically, in order to say that the state of Alice and Bob is pure we don't need to specify that the state of Alice, Bob, and the Environment is of the form $|\psi\rangle_{ABE} = |\psi\rangle_{AB}|\widetilde{\psi}\rangle_E$, but it is enough to know the state ρ_{AB} of Alice and Bob alone. On the other hand, the classical correlations of Alice and Bob alone do not allow us to know if Eve is, or is not, correlated with Alice and Bob, therefore, we must always describe the full state of Alice, Bob, and Eve.

will give some information about the qubit we are sending. If we try to use a less than completely correlated shared secret bit to send a secret bit then Eve gets some information about the secret bit. The resources are sufficient since we can achieve the operations using them.

Note that the resources are used up in the process: once we have used an ebit (shared secret bit) to send a qubit (shared secret bit) we cannot reuse it. Quantum mechanically this is obvious, since the original maximally entangled state is destroyed by Alice's measurement. Classically, however, Alice and Bob do not lose their correlated bits—Alice and Bob need not erase or physically modify in any way their original correlated bits but just use them for some mathematical operations. What is lost, however, is the secrecy of these bits—they cannot be reused.

Furthermore, it is obvious to see that the one-time-pad secret communication can be used to implement the analog of teleportation of entangled states and of entanglement swapping.

Finally, let us note an important fact. Quantitatively, the amount of resources in the classical and quantum cases are similar, but not identical: but we need two classical $bits_{A \to B}$ to send one qubit, whereas only one public $bit_{A \to B}$ to send one secret bit.

IV. ENTANGLEMENT AND SECRET-CORRELATION MANIPULATIONS—SINGLE COPY

The ability to manipulate entanglement, i.e., transforming entanglement from one form into another by local actions and classical communications is one of the most important aspects of entanglement. This leads to elevating entanglement to the status of a (fungible) resource: to a large extent it doesn't matter in which form entanglement is supplied, we can transform it into the specific form we need for different applications, very much as say, transforming the chemical energy stored in coal into electrical energy for use in electric engines. Similarly, one can imagine that Alice and Bob are supplied with secret correlations in some given form, i.e., according to some specific probability distribution, and they want to obtain secret correlations obeying a different probability distribution. We find that the quantum and classical scenarios are in very close analogy.

In this section, we treat the case of bipartite pure-state single-copy manipulations. In the quantum context this means that the two parties, Alice and Bob, share a single pair of particles in some pure state $|\Psi\rangle_{AB}$. In the classical context, Alice and Bob share a single sample of a classical pure state (2).

In the case of a single copy, entanglement is not a completely interconvertible resource [as it is in the case of many copies (see Sec. VIII)], but many more restrictions apply.

For bipartite pure quantum states, it is possible to turn one state into another *with certainty* if and only if (iff) a certain set of conditions, collectively known as majorization, holds [15,16]. Here we show that for classical secret pure states, the transformation is possible if and only if an analogous condition holds.

Quantum mechanically, the majorization condition is the

following. Consider two quantum pure states $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$, written in their Schmidt bases

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}|i\rangle_A|i\rangle_B, \tag{5}$$

$$|\phi\rangle_{AB} = \sum_i \sqrt{q_i}|i\rangle_A|i\rangle_B, \tag{6}$$

with the squared Schmidt coefficients p_i and q_i arranged in decreasing order, $p_1 \geq p_2 \geq \cdots$ and $q_1 \geq q_2 \geq \cdots$. The vector $\vec{q} = \{q_i\}$ is said to majorize the vector $\vec{p} = \{p_i\}$ iff

$$\sum_{i=1}^k q_i \geq \sum_{i=1}^k p_i \quad \forall k. \tag{7}$$

$|\phi\rangle_{AB}$ is said to majorize $|\psi\rangle_{AB}$ iff $\vec{q}$ majorizes $\vec{p}$. The transformation $|\psi\rangle_{AB} \mapsto |\phi\rangle_{AB}$ is possible with certainty if and only if $|\phi\rangle_{AB}$ majorizes $|\psi\rangle_{AB}$ [16]. (Note that it is the final state that must majorize the starting one.)

For classical secret correlations, suppose Alice and Bob begin with an arbitrary classical bipartite pure state, which we may write as

$$p(X_A = i, X_B = j, X_E = k) = \delta_{ij} p_i \widetilde{P}(E_k). \tag{8}$$

Their task is to produce some other state,

$$p(Y_A = i, Y_B = j, Y_E = k) = \delta_{ij} q_i \widetilde{P}'(E_k). \tag{9}$$

We shall prove that they can do this iff $\vec{q}$ majorizes $\vec{p}$. However, to understand what is going on, let us first consider a simple example that has all the important features. The quantum version was first considered in Ref. [15].

Suppose Alice and Bob share one sample of the classical pure state X, where

$$p_1 = p_2 = p_3 = \tfrac{1}{3}, \tag{10}$$

and they would like to turn it into a sample of the pure state Y, where

$$q_1 = q_2 = \tfrac{1}{2}. \tag{11}$$

A probabilistic method (analogous to the procrustean method for the quantum case [8]) is for Alice to send message m_1 (which means "OK") if X is 1 or 2, and to send message m_2 (which means "not OK") if X is 3. If message m_1 is sent then Alice and Bob keep their sample, and they now have a shared secret random variable of the form Y. Indeed, in this case Eve only knows that the value of the secret variable is either 1 or 2 but she doesn't know which one—Alice and Bob's data is, therefore, still perfectly secret, and it is now either 1 or 2 with probability 1/2. If message m_2 is sent then the procedure failed and Alice and Bob have to throw away their sample. The reason is that Eve, who monitors the public communication, learns that Alice and Bob's variable is equal to 3, and there is no more Alice and Bob can do.

The above method works with probability $\frac{2}{3}$. Can Alice and Bob do better? The second distribution majorizes the first, since $\frac{1}{2} \geqslant \frac{1}{3}$, $\frac{1}{2} + \frac{1}{2} \geqslant \frac{1}{3} + \frac{1}{3}$, and $\frac{1}{2} + \frac{1}{2} + 0 \geqslant \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$. Thus, according to the majorization theorem we shall shortly prove, there exists a method that works with certainty. The protocol for achieving this goes as follows. Alice reads the value of X. If it is 1, she flips an unbiased coin that tells her to send message m_1 or m_2 with equal probability. If $X=2$ she flips an unbiased coin to send m_2 or m_3, and if $X=3$ she flips an unbiased coin to send m_1 or m_3. She then publicly sends the message, so that everyone can read it. If m_1 is sent, Eve knows that X is 1 or 3 with equal probability. If m_2 is sent, Eve knows that X is either 1 or 2, with equal probability. And if m_3 is sent, Eve knows that X is 2 or 3 with equal probability. Now Alice and Bob just have to do a simple relabelling of X to produce Y. If m_1 is sent, they both do $1 \mapsto 1, 3 \mapsto 2$. If m_2 was sent they do $1 \mapsto 1, 2 \mapsto 2$. If m_3 is sent they do $2 \mapsto 1, 3 \mapsto 2$. Whatever message was sent, Y is now a shared random variable that is (as far as Eve is concerned) a shared secret bit of the form (11).

Now we shall look at the general case. For which pure states X and Y is it possible to turn a single sample of X into a single sample of Y? Consider the most general possible protocol. We assume that Alice, Bob, and Eve all know the protocol.[4] Alice and Bob start by having a single sample of the pure state X. They each have also access to some local source of secret randomness—they may each throw dice. Of course, Alice knows only the outcomes of her dice and Bob of his. During the protocol Alice and Bob may publicly communicate, perhaps in many rounds, with each message determined by X, the public messages already sent, and by the results of the local dice. At the end of the protocol there will be some total public message that consists of all the messages that were exchanged by Alice and Bob. All three parties, Alice, Bob, and Eve know this total message. In addition, Alice and Bob know the value of X (which is common to both of them since the state is pure), and each of them knows the outcomes of his/her own dice. Based on all this knowledge Alice and Bob must decide on the values of Y_A and Y_B. Formally, we can write

$$Y_A = f_A(X_A, m, d_A), \qquad (12)$$

$$Y_B = f_B(X_B, m, d_B), \qquad (13)$$

where by m we denote the total message, and by d_A and d_B we denote the outcome of all Alice's and Bob's dice.

The above procedure can be simplified. Since we begin with a pure state, $X_A = X_B = X$. Furthermore, since we want to end with a pure state, we require $Y_A = Y_B$. This requirement implies that Y_A and Y_B cannot depend on the outcome

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[4]if Alice and Bob had a secret protocol, this would be like having an additional shared random variable, whose different outcomes told them which protocol to use. Thus, they would have an additional resource. Here we insist they have only one shared resource, X.

of the dice d_A and d_B. Also given the initial value X and the message m, Alice and Bob must perform the same function f. Thus we get

$$Y_A = Y_B = f(X, m). \qquad (14)$$

Furthermore, since Bob's actions may not depend on the outcomes of his dice but only on X and m, for every procedure that involves many rounds of communication between Alice and Bob, we can formulate an equivalent procedure in which the total message is entirely generated by Alice—she could simply throw all dice herself—and then the message is communicated in a single transmission to Bob. Let us now formalize this procedure for turning X into Y.

Alice looks at $X = x_i$, which occurs with probability p_i. She then throws a biased dice which tells her to send message m_j with some probability $p(m_j | x_i)$ which depends upon x_i. She then publicly announces m_j. Alice and Bob now follow the instructions in the message, which say to do $x_i \mapsto y_k(x_i, m_j)$. Forgetting what X is (i.e., summing over x_i) this gives them some joint distribution for y_k and m_j, $p(y_k, m_j)$. Since Alice and Bob want y_k to be secret from Eve, who knows only the protocol and the message, this distribution must factorize: $p(y_k, m_j) = p(y_k)p(m_j)$. $p(y_k)$ is the final distribution, and so we want $p(y_k) = q_k$ (the distribution of Y).

This secrecy procedure can be thought of as a single-party problem, which goes as follows. We begin with a sample from X, which occurs with probability p_i. We may look at the sample, and then roll some dice, which gives outcome m_j with probability $p(m_j | x_i)$. We then perform the map $x_i \mapsto y_k(x_i, m_j)$. We then forget what X is, which gives some joint distribution for y_k and m_j, $p(y_k, m_j)$. We desire this distribution to factorize, $p(y_k, m_j) = p(y_k)p(m_j)$, and that $p(y_k) = q_k$. Note that this single-party procedure is not a secrecy procedure, however, it is possible iff the above secrecy transformation is.

To find for which p_i and q_k this single-party problem is possible, and thus to find for which p_i and q_j the secrecy transformation is possible, we shall look at the time-reversed problem. This goes as follows. We start with a sample from Y, which occurs with probability q_k. We then roll dice, which give outcome m_j with probability $p(m_j)$, independent of the outcome of Y. This gives a joint distribution $p(y_k, m_j) = q_k p(m_j)$. Now we must do the inverse of the map $x_i \mapsto y_k(x_i, m_j)$ to turn our Y into an X. If the map is one to one, and hence invertible, this will give us a distribution $p(x_i, m_j)$. Like any joint distribution, this can be written as $p(x_i, m_j) = p(x_i)p(m_j | x_i)$. If we now forget the value of Y and of m_k, we get a new distribution for X, $p(x_i)$. We desire $p(x_i) = p_i$. If the map is many to one, then we can give it a probabilistic inverse, which is a "one-to-many" map where the probabilities of getting various x's, given any particular y_k, are given by the relative frequencies of the x_i's when y_k is produced in the forward-time protocol. This probabilistic one-to-many map can be replaced by a probabilistic choice of several one-to-one maps, which will have the same effect upon the protocol since we forget which map we did at the end. Thus in the reversed-time single-party problem, we need

only consider maps that are one-to-one. This also applies to the forward-time single-party problem, and to the forward-time secrecy protocol: we only need consider maps that are one-to-one, i.e., permutations.

As explained above, if we find the conditions for which the reversed-time single-party problem is possible, we will have the conditions for which the forward time-secrecy transformation is possible. Physically, this time-reversed single-party problem goes as follows. We begin with a ball in some box according to the distribution q_i. We do not know which box the ball is in, and are not allowed to look to see where it is. We then apply some shuffle (one-to-one relabelling) to the boxes, choosing which shuffle to make according to a distribution, $p(m_j)$, which we may choose. We then forget which shuffle we did, and look at the new distribution of the balls, p_i. The question is for which q_i and p_i is this possible? Clearly p_i should be more random than q_i. This is a well-known problem, and is the context in which majorization appears in classical physics. The answer is that it is possible iff $\vec{p}$ majorizes $\vec{q}$. Intuitively, this is easy to see, and the proof can be found, for example, in Ref. [17].

Above, we have proved the majorization result in the classical context by using arguments referring solely to the classical context. We could have used, however, the known results for quantum-entanglement manipulation to prove the classical ones. The reason is as follows. On one hand, it was found out that transforming pure quantum states (with certainty) from one into another involves only actions and measurements in the Schmidt decomposition basis. These actions do not involve phases, but are simply classical actions upon the basis, which are performed coherently to make a quantum evolution. One could, however, imagine starting by measuring the quantum state in the Schmidt basis, and then performing the corresponding classical actions and measurements upon the state. This transforms one classical state into another, and will not give Eve any knowledge about the state since the quantum procedure did not entangle the quantum state with the environment. Thus, if we can transform with certainty a quantum pure state $|\Psi\rangle$ (5) into a quantum pure state $|\Phi\rangle$ (6), we can also transform with certainty X (8), the classical pure state equivalent of $|\Psi\rangle$, into Y (9), the classical pure state equivalent of $|\Phi\rangle$.

To prove the reverse, that is, that X can be transform with certainty into Y only if the quantum analogs can be transformed from one into the other, we note that we can turn any classical transformation of pure states into a quantum one, simply by applying the classical operations coherently, and performing the quantum actions in the Schmidt basis. Thus, there cannot be any classical procedure that does better than the optimal quantum one. So the classical transformation is possible iff the quantum one is.[5]

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[5]Note, however, that although we can use the quantum result to prove the classical one, we cannot use the classical result to prove the quantum result. The reason is that although we can turn any classical transformation into a quantum one, we cannot generate this way all possible quantum protocols—indeed, they may involve phases outside the Schmidt basis.

V. PROBABILISTIC SINGLE-COPY MANIPULATIONS

It may not be possible to transform a single copy of a resource from one form into another with certainty, but it may be possible to do it with some probability. What is the largest probability with which this can be done? For quantum states, the problem was considered in Refs. [15,18], and the general answer is given [18] in the simple form,

$$\min_k \frac{1 - \Sigma_{i=1}^k p_i}{1 - \Sigma_{k=1}^k q_i}. \tag{15}$$

We shall now show that for classical secret states, the answer is the same.

As for the nonprobabilistic transformations, we may simplify the most general protocol, which then goes as follows. Alice first looks at her sample that comes according to the distribution $p(x_i)$. She then chooses a message m_j according to $p(m_j|x_i)$. Most of the possible messages will be ones for which the transformation succeeds: these must say to do a one-to-one map[6] $X \mapsto Y$. The other messages say "fail": for these it does not matter what transformation we do, and it does not help to send more than one "fail" message. So we may assume we have only one "fail" message, m_{fail}, which says to do $x_i \mapsto y_1$. Alice and Bob then do $x_i \mapsto y_k(x_i, m_j)$ according to the message. This gives them a distribution $p(y_k, m_j)$. In the case they succeed, this distribution must factorize,

$$p(y_k, m_j) = \begin{cases} p(y_k)p(m_j) & \text{for } j \neq \text{fail} \\ \delta(y_k = 1)p(m_{\text{fail}}) & \text{for } j = \text{fail}. \end{cases} \tag{16}$$

By defining $p(\text{success}) = \lambda$, so that $p(m_j) = \lambda p(m_j|\text{success})$ for $j \neq$ ''*fail*'' and $p(m_{\text{fail}}) = 1 - \lambda$ and by requiring $p(y_k) = q_k$ (so that the protocol succeeds) we obtain

$$p(y_k, m_j) = \begin{cases} \lambda q_k p(m_j|\text{success}) & \text{for } j \neq \text{fail}, \\ (1-\lambda)\delta(y_k = 1) & \text{for } j = \text{fail}. \end{cases} \tag{17}$$

The time-reversed, single-party version of this problem is to start by flipping a coin (H/T) with probabilities ($\lambda, 1-\lambda$). We look at the result, and if it is T we start with $y_k = 1$, send a message m_{fail}, and are allowed to do anything (including probabilistic things) to transform $Y \mapsto X$. If the coin is H we get a sample y_k according to $p(y_k) = q_k$, but do not know which sample we get. We then pick some message according to $p(m_j)$, and do the corresponding shuffle $y_k \mapsto x_i$. This gives some distribution $p(x_i, m_j)$. Finally, we forget whether the coin was H or T, and also which message was sent. This then gives us $p(x_i)$, which we would like to be p_i. Our aim is, for a given q_k and p_i, to find the maximal λ for which this is possible. This problem is closely related to the one where majorization first appeared in classical physics, and the maximal value of λ is as given at the start of

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[6]There is no loss in generality in forgetting about the many-to-one maps, for the same reasons as in the nonprobabilistic manipulations.

this section. Once again the quantum and classical pure-state manipulations are possible under the same conditions.

VI. CATALYSIS OF SINGLE-COPY TRANSFORMATIONS

There is an interesting entanglement transformation called catalysis [19] that transfers easily to the classical case. Suppose we begin with some pure state

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |ii\rangle_{AB}, \qquad (18)$$

and wish to produce, using LOCC, the state

$$|\phi\rangle_{AB} = \sum_j \sqrt{q_j} |jj\rangle_{AB}. \qquad (19)$$

This is possible [16] iff q_j majorizes p_i. There are, however, states where q_j does not majorize p_i, but where catalysis is possible. That is, where Alice and Bob cannot perform

$$|\psi\rangle \mapsto |\phi\rangle, \qquad (20)$$

but if Alice and Bob share an additional pure state,

$$|\chi\rangle_{AB} \sum_k \sqrt{r_k} |kk\rangle_{AB}, \qquad (21)$$

then they are able to perform, with certainty, the transformation

$$|\psi\rangle|\chi\rangle \mapsto |\phi\rangle|\chi\rangle. \qquad (22)$$

This is, quite simply, because for the tensor-product system, the majorization condition holds. $|\chi\rangle$ acts as a catalyst. It enables the transformation of $|\psi\rangle$ into $|\phi\rangle$, but is not consumed in the process. One example of such a catalysis is transforming the quantum state whose squared Schmidt coefficients are

$$p_1 = 0.4; \quad p_2 = 0.4; \quad p_3 = 0.1; \quad p_4 = 0.1, \qquad (23)$$

into the quantum state

$$q_1 = 0.5; \quad q_2 = 0.25; \quad q_3 = 0.25, \qquad (24)$$

using the catalyst

$$r_1 = 0.6; \quad r_2 = 0.4. \qquad (25)$$

The classical analog of this process follows immediately. That is, Alice and Bob may wish to turn the classical pure state defined by p_i into the classical pure state defined by q_j, using LOPC. This is only possible, as we showed in Sec. IV, when q_j majorises p_i. However, there are cases when this is not possible, but if they also have a sample of the classical pure state r_k, then they can achieve the transformation

$$P \otimes R \mapsto Q \otimes R, \qquad (26)$$

with certainty. The sample R is not revealed or altered by this process, and can be subsequently used independently else-

where. As far as we know, this classical secret correlation catalysis has not been previously considered.

VII. SHUFFLING WITH CATALYSIS

Another classical catalysis problem that has not (to our knowledge) been considered before is the single-party, time-reversed version[7] of the classical pure-state catalysis discussed in the previous section. We call this "shuffling catalysis." We emphasize that this shuffling catalysis has, in itself, nothing to do with secrecy or secret correlations. However, it is possible to perform this shuffling catalysis iff the classical pure-state catalysis is possible. Recalling (from Sec. IV) that the majorization conditions are easier to prove in the shuffling scenario than in the classical secret-correlation scenario, studying shuffling catalysis may help in finding exactly when classical secret correlation (and, by analogy, entanglement) catalysis is possible.

We state the problem of shuffling catalysis to make the idea clear. Suppose we have a sample from a distribution q_j and wish to turn it into a sample from a distribution p_i. We are not allowed to look at the sample to see what it is, we can only throw dice whose probabilities (which we choose) are independent of which sample we have. We then make some permutation (shuffle) upon the outcomes, which suffle decided by the dice, and finally forget which one we did. As mentioned in Sec. IV, this "shuffling" is possible iff q_j majorizes p_i. There are, however, distributions where q_j does not majorize p_i, and so cannot be turned into it directly, but where we can perform catalysis. This means that we can take a sample from a third distribution r_k, such that $q_j \otimes r_k$ majorizes $p_i \otimes r_k$, and then roll an independent dice and permute the possible outcomes of the tensor-product distribution to turn $q_j \otimes r_k$ into $p_i \otimes r_k$. This catalysis is possible iff we can use r_k to turn the shared secret-correlation pure-state p_i into to the pure state q_j. Thus, an example of this shuffling catalysis is the example given in Sec. VI.

VIII. PURE-STATE CONCENTRATION AND DILUTION

For many copies of a bipartite pure state, entanglement is a completely fungible resource. It can be converted from one form to another reversibly. Thus, we can quantify the amount of entanglement by a single number, the entropy of entanglement. We shall show that the same is true for classical pure bipartite states. That is, for such states, secret correlations are a completely fungible resource. They can be converted from one form to another reversibly, and can be quantified by a single number, the entropy of secrecy.

We define the entropy of entanglement for a quantum pure state, $E(|\psi\rangle_{AB})$ as

$$E(|\psi\rangle_{AB}) = -\sum_i p \log_2 p_i, \qquad (27)$$

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[7]See Sec. IV for the meaning of the single-party, time-reversed version of the classical pure-state transformation.

where p_i are the squares of the Schmidt coefficients.

The physical meaning of the entropy of entanglement is the following. When Alice and Bob share a large number N of copies of some arbitrary pure state $|\psi\rangle_{AB}$, they can convert them, in a *reversible way*, using only local operations and classical communication into a number K of copies of the maximally entangled state

$$|\psi_s\rangle_{AB} = \frac{1}{\sqrt{2}}(|11\rangle_{AB} + |22\rangle_{AB}), \qquad (28)$$

where

$$\frac{K}{N} \to E(|\psi\rangle_{AB}), \qquad (29)$$

as $N \to \infty$. That is, the entropy of entanglement represents the yield of singlets per copy of the original state $|\psi\rangle_{AB}$. The operation of converting the states $|\psi\rangle_{AB}$ into maximally entangled states is called entanglement concentration [8] and the reverse operation is called entanglement dilution.

Since entanglement cannot increase under LOCC, the above procedures are optimal, in the sense that concentration and dilution cannot produce more copies: if they could, we would be able to produce entangled states from nothing.[8] We can thus quantify the amount of entanglement in a state by its entropy of entanglement. Any state is worth that many maximally entangled states, since it can be reversibly converted into that many states. We call one of these maximally entangled states an ebit, and shall say that other states have an entanglement of E ebits. Note that this quantity is additive. That is, if we have two states that individually have entanglement E_1 and E_2, together they have entanglement $E_1 + E_2$.

The quantum procedure of entanglement concentration can directly be mapped into an equivalent classical analog. The reason for this is that all the quantum actions used for entanglement concentration take place in the Schmidt decomposition bases, i.e., the unitary actions are all permutations in the Schmidt basis while the measurements are of operators whose eigenstates are direct products in the Schmidt basis. Hence all these actions are essentially classical. Furthermore, the quantum procedure does not require communication, so is completely secure.

The quantum-dilution protocol also has a classical analog. Indeed, the quantum dilution [8] involves only Schumacher compression of quantum information and teleportation. Both these protocols have classical analogs: Schumacher compression maps into Shannon data compression and teleportation is replaced by the one-time-pad secret communication.

Since secret correlations cannot increase under LOPC, these procedures are optimal. They allow us to reversibly convert N copies of the classical pure state $X \sim p_i$ into K copies of the shared secret bit $Y \sim q_j$,

$$P(Y_A = 1, Y_B = 1) = P(Y_A = 2, Y_B = 2) = \tfrac{1}{2}, \qquad (30)$$

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[8]It would be like the Carnot cycle for a perpetual motion machine.

where

$$\frac{K}{N} = -\sum_i p \log_2 p_i. \qquad (31)$$

We can thus quantify the amount of secret correlations by the entropy of secrecy, which is defined as the number of shared secret bits that can be produced per copy of the original state X. We note that this amount is equal to the mutual entropy between X_A and X_B, and is also equal to the local entropy of X_A, and to the local entropy of X_B.

IX. ENTANGLEMENT PURIFICATION AND PRIVACY AMPLIFICATION

An important procedure in quantum information is entanglement purification [20], which turns mixed states into pure states, at the many copy level. The number of pure states produced per input mixed state is the yield.

Analogous procedures for turning classical mixed states into classical pure states exist, though are usually subdivided into two stages. The first stage takes the mixed state $P(X_A, X_B, X_E)$ and turns it into a mixed state where Alice and Bob hold the same value, i.e., of the form $P(i, j, k) = \delta_{ij} P(i, i, k)$. The stage is known as information reconciliation [21], because Alice and Bob are agreeing on a common value. The second stage takes the output of the first stage, and factors out Eve, to give a state of the form $\delta_{ij} p_i \widetilde{P}(k)$. In other words, it produces a pure state. This stage is known as privacy amplification [21], because Alice and Bob are increasing the secrecy of their key by reducing (to 0) Eve's knowledge of it.

In general, it is not known what the optimal protocol is, and there may be different optimal protocols for different states. There are a few different schemes for the quantum and classical cases, but we do not wish to discuss the details here, just to draw the analogy. First, any information-reconciliation/privacy-amplification protocol may be used as an entanglement-purification protocol. Second, any entanglement-purification protocol may be used as a information-reconciliation/privacy-amplification protocol. We hope that a detailed study of the two problems together will yield better understanding and new protocols in both the classical and the quantum case.

X. BOUND ENTANGLEMENT

One of our motivations for this work was a paper [6] by Gisin and Wolf, suggesting a classical analog of bound entanglement. A bound entangled state is a bipartite mixed quantum state that cannot be created locally (without any prior entanglement), but from which no maximally entangled states can be distilled, even if there are many copies of the bound entangled state. It is as if the entanglement is "bound" inside the state, and cannot be released. They proposed the classical analog to be a sample from a probability distribution on Alice, Bob, and Eve, $P(X_A, X_B, X_C)$, in which Alice

and Bob have strictly positive intrinsic information,[9] but from which they cannot distill shared secret bits under LOPC, even if they have many samples from the distribution. Though it is not yet known if such a classical state exists, there is strong evidence that, by starting with a bound entangled state ρ_{AB}, taking a natural purification, $|\psi_{ABE}\rangle$, and measuring it in natural bases, we may produce a classical bound state. Here we simply note that bound information fits into our framework as a derived analogy, and is another consequence of the deeper analogy between entanglement and secret classical correlations.

XI. PURE OR MIXED?

We have mentioned in Sec. II that it is not clear whether to classify classical states of the form $P(X_A,X_B)P(X_E)$ where X_A is *not* completely correlated with X_B as pure or as mixed. Such a distribution resembles a pure state because it is not correlated with Eve: this is like a pure state not being entangled with the environment. It also resembles a pure state because we can optimally distill shared secret bits from many copies of such a state at a rate equal to the natural measure of shared correlations, the mutual information [22]; this is the analog of pure-state entanglement concentration. However, it is not known whether such a distillation is reversible. That is, given the shared secret bits, can we produce the original states? If the answer is no, this would be typical behavior of a mixed state. Furthermore, a definite similarity to mixed states is that there is no Schmidt decomposition for such states: in other words, there is no way, using local reversible transformations, to make Alice and Bob have the same values for their samples.

Another similarity to mixed states is that it is not possible, even probabilistically, to use LOPC to produce a pure state from one copy of such a distribution. For example, consider the bipartite, two-dimensional case, where Alice and Bob both receive either a 0 or a 1, with probabilities $p_{00},p_{10},p_{01},p_{11}$. We can assume that at least the first three probabilities are nonzero (otherwise they have a pure state). They wish to use LOPC to make a classical "entangled" pure state, i.e., where $P(00)>0$, $P(11)>0$, $P(01)=P(10)=0$. As discussed in Sec. IV, the most general thing they can do is to first communicate publicly, resulting in some total public message, m_i, where i may depend upon their local dice and upon their samples. They may then change their samples according to some map that is specified by the message. For example, the message could tell Alice to flip her bit, and Bob to leave his alone. Note that the message has to tell them what to do locally: it cannot tell them to look at the other person's bit to decide what they will do. Now, to make a pure state with any probability they need at least one map that is local in the sense described above and which produces

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[9]A classical measure, which, loosely speaking, is designed to test whether or not Alice and Bob share some information that Eve does not have and which they can use. The hope was that, if positive, they would have something useful, and if zero, they would have nothing.

both 00 and 11, and nothing else. We shall show that no such map exists.

Assume that such a map exists. Without loss of generality, we may assume the map does

$$00 \mapsto 00. \qquad (32)$$

Since Bob has to act locally, this means that if he starts with a 0, he has to finish with a 0. Since they must finish with the same thing, this implies

$$10 \mapsto 00. \qquad (33)$$

Since they are symmetric, similar reasoning gives

$$01 \mapsto 00. \qquad (34)$$

Because they have to act locally, we now know that if Alice or Bob sees a 1, they have to finish with a 0. Thus

$$11 \mapsto 00. \qquad (35)$$

And so the map takes everything to 00, which is no good. For classical states in higher dimensions, the same type of reasoning shows that we cannot produce a classical pure state from a single copy of such a state.

So, as we have shown, classical states of the form $P(X_A,X_B)P(X_E)$ have some characteristics in common with pure quantum states, and some in common with mixed quantum states.

XII. MULTIPARTITE RESULTS

It is well known that entanglement is much more complicated for multipartite systems than for bipartite systems [11,12,23]. In particular, already in the case of three parties, it is known that tripartite entanglement is fundamentally different than the bipartite entanglement, even in the many-copy scenario. Furthermore, there might even exist many different inequivalent forms of tripartite entanglement. As more systems are added, the problem becomes vastly more complicated, but we have a few results to guide us, such as the fact that there is genuine entanglement at every level (again, even in the many-copy scenario). Here we show that many of these features have classical analogs.

First, we shall look at the tripartite case. We propose that the classical equivalent of the Greenberger-Horne-Zeilinger (GHZ) state,

$$|\text{GHZ}\rangle_{ABC} = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle), \qquad (36)$$

is a probability distribution of the form

$$P(X_A,X_B,X_C,X_E) = P(X_A,X_B,X_C)\tilde{P}(X_E), \qquad (37)$$

where $P(X_A,X_B,X_C)$ is given by

$$P(0,0,0) = P(1,1,1) = \tfrac{1}{2}. \qquad (38)$$

We shall call this the C-GHZ (classical GHZ), and the classical singlet (i.e., the bipartite shared secret bit) we shall call

the classical Einstein-Podolsky-Rosen *(C*-EPR) *Is* is easy to see that out of 1 GHZ copy we may generate one *C*-EPR, i.e.,

$$C\text{-GHZ} \mapsto C\text{-EPR}. \tag{39}$$

Clare simply forgets her bit. This may sound unsatisfactory since in the quantum case Alice and Bob end with an EPR, which Clare has no control over, whereas here Clare could always later remember her bit, and so one may argue that we have not really performed the classical transformation. However, since Alice, Bob, and Clare all begin with the same information and communicate only publicly, it is impossible for Alice and Bob to agree upon anything without Clare knowing it. Thus the "stronger" form of the transformation is impossible, and the best we can do is this weak form, with Clare forgetting her bit.

The above transformation is irreversible: i.e., given one *C*-EPR it is impossible to make a *C*-GHZ [11]. This is because the bipartite entropy of secrecy can only decrease under LOPC, and viewing the system as (AB) vs C a $C\text{-EPR}_{AB}$ will have 0 entropy, whereas the $C\text{-GHZ}_{ABC}$ has entropy of 1 (and is symmetric with respect to all the parties). It is possible, however, to do

$$C\text{-EPR}_{AB} + C\text{-EPR}_{BC} \mapsto C\text{-GHZ}. \tag{40}$$

This is done as it would be in the quantum case: Bob makes a joint measurement on his bits (addition modulo 2), and publicly announces the result. Bob now forgets his second bit, and if the public message was 1, Clare flips her bit. They are then done. This procedure can be viewed as Bob using the $C\text{-EPR}_{BC}$ as a one time pad to send Clare the value of the $C\text{-EPR}_{AB}$. It is again clear that we cannot do the reverse transformation: viewing the system as (AC) vs B, the C-GHZ has an entropy of secrecy of 1, whereas the two C-EPR's together have an entropy of 2.

The entropy of secrecy can be used to show that there exists more than just bipartite secrecy, even in the many-copy case. Specifically, the four-party cat state, which has distribution $P(X_A, X_B, X_C, X_D)$ given by

$$P(0,0,0,0) = P(1,1,1,1) = \tfrac{1}{2} \tag{41}$$

(where Eve factors out) cannot be converted reversibly into *C*-EPR pairs. The proof of this is exactly the proof used for the analogous quantum problem [11], and is done by partitioning the four parities into pairs in various ways, and looking at the entropy of entanglement, which must be asymptotically conserved under reversible transformations.

Suppose that we could reversibly convert asymptotically a single four-party cat state into *C*-EPR pairs: n_{AB} between A and B, n_{AC} between A and C, etc. Partitioning the system into (A) vs (BCD) we get the equation

$$n_{AB} + n_{AC} + n_{AD} = 1. \tag{42}$$

Partitioning the system as (B) vs (ACD), (C) vs (ABD), and (D) vs (ABC) gives

$$n_{AB} + n_{BC} + n_{BD} = 1, \tag{43}$$

$$n_{AC} + n_{BC} + n_{CD} = 1, \tag{44}$$

$$n_{AD} + n_{BD} + n_{CD} = 1. \tag{45}$$

On the other hand, partitioning the system as (AB) vs (CD), (AC) vs (BD), and (AD) vs (BC) gives

$$n_{AC} + n_{AD} + n_{BC} + n_{BD} = 1, \tag{46}$$

$$n_{AB} + n_{AD} + n_{BC} + n_{CD} = 1, \tag{47}$$

$$n_{AB} + n_{AC} + n_{BD} + n_{CD} = 1. \tag{48}$$

Summing the first four equations together gives

$$2 \sum_{\text{all pairs}} n_{ij} = 4. \tag{49}$$

while summing together the next three gives

$$2 \sum_{\text{all pairs}} n_{ij} = 3. \tag{50}$$

Thus the transformation is impossible, and the four party classical cat state really is more than just bipartite shared secret correlations.

We thus conclude that there are different types of multipartite secret correlations.

XIII. CONCLUSION

We have described a fundamental analogy between entanglement and secret classical correlations. The analogy is quite simple to state. Both are resources, and the main objects involved in the study of such resources have a one-to-one correspondence, as given in the first table of Sec. I. Due to this basic analogy, many derived analogies follow. In particular, we have shown that teleportation and the one time pad are deeply connected, that the concept of "pure state" exists in the classical domain, that entanglement concentration and dilution are essentially classical secrecy manipulations, and that the single-copy entanglement manipulations have such a close classical analog that the majorization results are reproduced in the classical setting. We have pointed out that entanglement purification is analogous to classical privacy amplification, and hope that the search for better protocols in the two areas can go hand in hand. We finally showed that, as with entanglement, one can look at multipartite shared secret correlations, and gave a flavor of how results in the quantum setting easily transfer into the classical world. Despite all these useful derived analogies, our main point is the fundamental one: entanglement and shared secret correlations are deeply related, and one should never be viewed without the other.

We want to emphasize that by no means do we claim that quantum entanglement is a fundamentally classical effect or that there exists a classical explanation of entanglement. The classical analog of entanglement is nothing more or less than a simple analog, and has a value of its own. On the other hand, all the aspects of quantum entanglement that are com-

mon with the classical analog cannot be considered to be quantum. Thus, many aspects that were hitherto considered to be genuinely quantum lose their status.

The main thrust of this paper was to identify the common aspects of quantum entanglement and classical secret correlations. An even more interesting question is to find those aspects which are *not* common. For example, we have not found any (and believe there is no) analog of superdense coding. It is not the case that by having one secret-correlation bit and by sending one secret bit, Alice can transmit to Bob two public bits. The lack of superdense coding manifests itself, implicitly, also by a difference in the quantitative descriptions of teleportation and one-time-pad secret communication: in the case of teleportation Alice has to send Bob two classical bits while in the one time pad Alice has to send only one public bit. It is only such aspects that are not common to the two settings that are genuinely quantum. We hope that getting rid of those aspects that were believed to be quantum but are not, and identifying the genuine quantum ones will lead to a better understanding of quantum entanglement, and of secret communication.

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