Classical analog of quantum search

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Quantum search is a quantum-mechanical technique for searching N possibilities in only $\sqrt{N}$ steps. A similar algorithm applies in a purely classical setting when there are N oscillators, one of which is of a different resonant frequency. We could identify which one this is by measuring the oscillation frequency of each oscillator, a procedure that would take about N cycles. We show, how by coupling the oscillators together in a very simple way, it is possible to identify the different one in only $\sqrt{N}$ cycles. In case there are multiple oscillators of a different frequency, we can estimate the number of these in a time which is the square root of that required by the sampling method. The analog also leads to some energy routing algorithms for mechanical systems.

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I. INTRODUCTION

Quantum computing algorithms, such as quantum search, make use of the fact that a quantum system is simultaneously in multiple states to carry out certain computations in parallel in the same hardware. To implement the actual quantum-search algorithm one needs a quantum-mechanical system where one can carry out certain elementary quantum-mechanical operations in a controlled way, it is *not* possible to implement the algorithm on classical hardware. Yet, in this paper we show that a very similar algorithm works in a classical system. The difference is that in a classical system the hardware is proportional to N; whereas in the quantum system, the hardware is only proportional to $\log_2 N$.

The algorithm of this paper is of interest, both in its own right as a classical algorithm and also for the insight it provides into quantum computing. For example, it is well established that the quantum-search algorithm, which can search N possibilities in only $\sqrt{N}$ steps, is the best possible algorithm for exhaustive searching. Yet there is no simple argument as to why this is the best algorithm or why the algorithm should need $\sqrt{N}$ steps. This paper gives an elementary argument as to why it needs $\sqrt{N}$ cycles to identify the different oscillator.

II. BACKGROUND

Any quantum-mechanical transformation is a rotation of the state vector in N-dimensional complex Hilbert space, where N is the number of states. Therefore, any quantum-mechanical algorithm too is a rotation of the state vector in N-dimensional complex Hilbert space. The quantum-search algorithm [1] is a special case since it is a rotation in a carefully defined two-dimensional vector space. This was first noticed by Farhi and Gutmann who used it to develop the following variant of the search algorithm [2].

Consider an N-state system, whose Hamiltonian is known to be $|w\rangle\langle w|$. w is known to be a basis state, the problem is to find out which one this is. We are allowed to add on any additional term to the Hamiltonian (provided this does not depend on w) and let the system evolve in any way we choose. The question is as to how rapidly can we identify w?

Any obvious technique will need $O(N)$ time. For example, if we examine each state separately by coupling it to an auxiliary state, it will take $O(1)$ time to examine each state and thus $O(N)$ time in all. However, by using an analogy with the quantum-search algorithm, it is possible to devise a scheme to identify w that requires only $O(\sqrt{N})$ time.

The idea is to first add an additional term of $(1/N)(|1\rangle+\cdots+|N\rangle)(\langle 1|+\cdots+\langle N|)$ to the given Hamiltonian. Then start the system from the superposition $(1/\sqrt{N})(|1\rangle+\cdots+|N\rangle)$, let it evolve for a time $O(\sqrt{N})$ and finally carry out an observation—with a high probability the state observed after this will be $|w\rangle$. This technique is similar to the search algorithm in that it consists of a rotation of the state vector in a two-dimensional vector space defined by $|w\rangle$ and $(1/\sqrt{N})(|1\rangle+\cdots+|N\rangle)$.

To simplify notation, assume that w is the first of the N states, i.e., $w=1$. The total Hamiltonian then becomes

$$H\approx\frac{1}{N}(|1\rangle+\cdots+|N\rangle)(\langle 1|+\cdots+\langle N|)+|1\rangle\langle 1|. \quad (1)$$

Writing this in the subspace spanned by $|1\rangle$ and $|B\rangle \equiv(1/\sqrt{N})\Sigma_{j=2}^{N}|j\rangle$, and leaving out terms of order $1/N$, the above Hamiltonian becomes

$$H\approx(|1\rangle\langle 1|+|B\rangle\langle \mathrm{B}|)+\frac{1}{\sqrt{N}}(|1\rangle\langle B|)+|B\rangle\langle 1|). \quad (2)$$

Thus the quantum dynamics of the system is essentially that of two *degenerate* levels with mixing amplitude of $O(1/\sqrt{N})$. The initial state $1/\sqrt{N}\Sigma_{j=1}^{N}|j\rangle\approx|B\rangle$ ''rotates'' to $|1\rangle$ in a time inversely related to the mixing matrix element. Since this element is $O(1/\sqrt{N})$, the time taken by this search algorithm is $O(\sqrt{N})$.

The discrete quantum-search algorithm is very similar. The main difference is that instead of having the Hamiltonian be constant throughout, it is adjusted so that the item-specific portion acts separately from the mixing portion, i.e.,

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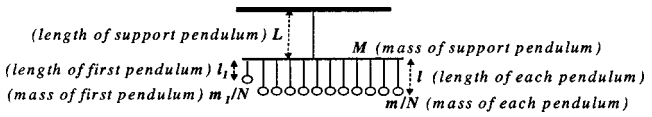

FIG. 1. N pendulums are suspended from a single pendulum.

there are alternate steps of $|w\rangle\langle w|$ and $(1/N)(|1\rangle+\cdots+|N\rangle)(\langle 1|+\cdots+\langle N|)$. This perspective is described in [3]. Thus, at the heart of the search algorithm is a resonance phenomenon. In the following sections we discuss a classical analog of the same phenomenon involving coupled oscillators. Variants of the quantum-search algorithm have previously been proposed with classical waves [4–6]. Ours is quite different in that it explicitly incorporates the coupling between various modes.

III. CLASSICAL ANALOGY

The analysis and results of the following two sections hold for any system of classical oscillators, either mechanical or electrical. For concreteness we consider the oscillators to be pendulums.

The following is the problem. There are N pendulums—one of which is slightly shorter than the rest. The problem is to identify which one this is. By carefully coupling them together and letting them oscillate for $O(\sqrt{N})$ cycles, a substantial portion of the energy can be transferred to the shorter pendulum whose amplitude becomes very high. This is accomplished by a resonance phenomenon very similar to that in quantum search. Using this, it is possible to identify the different pendulum as described in Sec. V.

IV. N COUPLED PENDULUMS

We show that by suspending the N pendulums from a bigger pendulum (Fig. 1) and adjusting the masses and lengths of the bigger pendulum appropriately, it is possible to achieve a coupling similar to that of the N states in the quantum-search algorithm as described in Sec. II. As in Sec. II we make the first pendulum special while the rest of the $(N-1)$ of them are identical.

The Lagrangian of the system of Fig. 1 is given by

$$L=\frac{1}{2}\left\{M\dot{X}^2-KX^2+\frac{1}{N}[m_1\dot{x}_1{}^2-k_1(x_1-X)^2]+\frac{1}{N}\sum_{j=2}^{N}[m\dot{x}_j{}^2-k(x_j-X)^2]\right\};$$

$$K\equiv\left(M+\frac{m}{N}\right)\frac{g}{L},\quad k\equiv m\frac{g}{l},\quad k_1\equiv m_1\frac{g}{l_1},\tag{3}$$

where X is the displacement of the support pendulum, x_j is the displacement of the jth pendulum hanging from the support; M,L are respectively, the mass and the length of the support pendulum, m_1/N, l_1 are respectively, the mass and length of the first pendulum, and m/N, l are respectively, the mass and the length of each of the other pendulums (g is the acceleration due to gravity). It was probably simpler to keep the Lagrangian of Eq. (3) in terms of the m's, l's, and g. However, as mentioned before, the framework of this paper applies to any system of oscillators, electrical or mechanical. In order to be able to quickly translate the results to other applications, we express the Lagrangian in Eq. (3) in a more general notation in terms of the stiffnesses (k's).

Now we change variables so that we consider the center-of-mass mode $\vec{x}$ of pendulums 2,...,N, and other modes of excitation of the same pendulums orthogonal to the center-of-mass mode, which we denote by y_l, $l=1,..,(N-2)$. In terms of these variables, the Lagrangian may be written as

$$L=\frac{1}{2}\left\{M\dot{X}^2-KX^2+\frac{1}{N}[m_1\dot{x}_1{}^2-k_1(x_1-X)^2]+\left(1-\frac{1}{N}\right)\times[m\dot{\bar{x}}^2-k(\bar{x}-X)^2]+\frac{1}{N}\sum_{l=1}^{N-2}(m\dot{y}_l{}^2-ky_l^2)\right\}.\tag{4}$$

Note that the y's decouple from the rest of the variables. If we consider an initial condition where each y is zero, they will stay zero. Hence we can omit these variables and concentrate on the three crucial ones: X, x_1, $\bar{x}$. Defining $\xi\equiv(1/\sqrt{N})x_1$, and ignoring some irrelevant $O(1/N)$ terms, the reduced Lagrangian (without the y's) may be written as

$$L_{\rm red}\approx\frac{1}{2}\left[M\dot{X}^2-KX^2+m_1\dot{\xi}^2-k_1\left(\xi-\frac{1}{\sqrt{N}}X\right)^2+m\dot{\bar{x}}^2-k(\bar{x}-X)^2\right].\tag{5}$$

The Lagrangian $L_{\rm red}$ represents two strongly coupled degrees of freedom, X and $\bar{x}$, and a variable ξ that is weakly coupled to others. We first solve the X, $\bar{x}$ system. This gives rise to two modes with frequencies, which we denote by ω_a and ω_b. The natural frequency of the ξ degree of freedom, corresponding to the special pendulum, is $\omega_1=\sqrt{k_1/m_1}$ (ignoring the $O(1/\sqrt{N})$ coupling ξ has with the other modes). If ω_1 is arranged to be very close to either ω_a or ω_b, we get a resonant transfer of energy between the two weakly coupled systems. The number of cycles required for significant transfer of energy to the special pendulum varies inversely with the coupling and will be $O(\sqrt{N})$.[1]

We next analyze the three-mode system defined by the reduced Lagrangian (5) by writing its equations of motion, in matrix form these are

[1]Clearly, when the deviation of the length of the pendulum approaches zero, there should be no energy transfer to this pendulum. Yet the previous analysis seems to suggest that the time will be $O(\sqrt{N})$ cycles irrespective of the deviation. The reason for this becomes clear by examining the frequency diagram of Fig. 2 when the deviation between ω_1 and $\bar{\omega}$ becomes zero. Then whatever value we choose for ω_c, it will result in an order one difference between ω_1 and ω_a, i.e., we will never be able to satisfy the resonance condition.

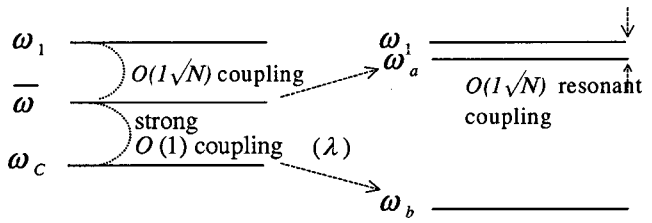

FIG. 2. The center-of-mass mode $\overline{\omega}$ and the coupling mode ω_c interact to produce two new modes (ω_a and ω_b). One of these, ω_a, is resonantly coupled to the oscillation mode of the different pendulum ω_1 with an $O(1/\sqrt{N})$ coupling.

$$\hat{M}\ddot{\vec{X}}(t) = -\hat{K}\vec{X}(t), \tag{6}$$

where the displacement vector $\vec{X}$, the mass matrix $\hat{M}$, and the stiffness matrix $\hat{K}$, are defined as follows:

$$\vec{X}(t) \equiv \begin{pmatrix} X(t) \\ \vec{x}(t) \\ \xi(t) \end{pmatrix}, \quad \hat{M} \equiv \begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m_1 \end{pmatrix},$$

$$\hat{K} \equiv \begin{pmatrix} K+k+\dfrac{k_1}{N} & -k & -\dfrac{k_1}{\sqrt{N}} \\ -k & k & 0 \\ -\dfrac{k_1}{\sqrt{N}} & 0 & k_1 \end{pmatrix}.$$

Solving Eq. (6) by assuming a solution with time dependence $e^{i\rho t}$, it follows after some straightforward analysis that ρ^2 is given by the eigenvalues of the matrix Λ as

$$\Lambda \equiv \hat{M}^{-1/2}\hat{K}\hat{M}^{-1/2} = \begin{pmatrix} \omega_c^2 & -\lambda & -\dfrac{k_1}{\sqrt{NMm_1}} \\ -\lambda & \overline{\omega}^2 & 0 \\ -\dfrac{k_1}{\sqrt{NMm_1}} & 0 & \omega_1^2 \end{pmatrix}. \tag{7}$$

Here $\omega_c^2 \equiv (1/M)[K+k+(k_1/N)]$ (ω_c corresponds to the frequency of the *coupling* degree of freedom, i.e., the frequency of the large pendulum), $\overline{\omega}^2 \equiv k/m$ ($\overline{\omega}$ is the frequency of the center-of-mass mode), $\omega_1^2 \equiv k_1/m_1$ (ω_1 corresponds to the frequency of the different pendulum), $\lambda \equiv (k/\sqrt{Mm})$ (λ is the coupling between the large pendulum and the center-of-mass mode).

Inspecting the matrix Λ makes it clear that the first two modes are strongly coupled, whereas the first mode is only weakly coupled to the third mode by a term of order $1/\sqrt{N}$. We can thus change basis so that the (1, 2) block is diagonalized. The corresponding frequencies are given by the eigenvalues of the (1, 2) block

$$\omega_{a,b}^2 = \tfrac{1}{2}[\omega_c^2 + \overline{\omega}^2 \pm \sqrt{(\omega_c^2 - \overline{\omega}^2)^2 + 4\lambda^2}]. \tag{8}$$

In the rotated basis, each of the first two modes will have $O(1/\sqrt{N})$ coupling with the third mode; the matrix Λ gets transformed into a matrix $\tilde{\Lambda}$ of the following form:

$$\tilde{\Lambda} \equiv \begin{pmatrix} \omega_a^2 & 0 & -\dfrac{\alpha}{\sqrt{N}} \\ 0 & \omega_b^2 & -\dfrac{\beta}{\sqrt{N}} \\ -\dfrac{\alpha}{\sqrt{N}} & -\dfrac{\beta}{\sqrt{N}} & \omega^2 \end{pmatrix};$$

$$(\alpha, \beta \text{ are of order } 1). \tag{9}$$

We start this system by giving a push to the large support pendulum, delivering order-1 energy. This energy will initially be in the (1,2) subsystem. However, under the condition of resonance, in $O(\sqrt{N})$ cycles, the special pendulum will swing with an amplitude of order 1. All the other $(N-1)$ identical pendulums would move in lock step; their total energy would be of order 1, but individual pendulums will have energy of $O(1/N)$, and their amplitudes would be $O(1/\sqrt{N})$.

It must be noted that precise information about the different oscillator is required in order to satisfy the resonance condition—we would have to know precisely how much longer or shorter this pendulum was as compared with the remaining pendulums. This would determine the value of M and L (the mass and length of the support pendulum from which the rest of the pendulums are suspended).

V. THE ALGORITHM

As described above, we have a means for transferring a large portion of the energy from the support pendulum into an aberrant pendulum, assuming we have precise information about the length of this pendulum but do not know which this is. This procedure can be used to identify which pendulum this is (as in the quantum-search algorithm). In order to better define the problem, it is important to list some of the associated constraints.

A. Rules of the game

(1) The system is started by giving a single push to the support pendulum.

(2) We can redesign parameters and observe the motion of a constant number of pendulums.

(3) Observations can only be resolved with a finite precision that is independent of N.

These constraints are meant to reflect realistic limitations on the system. Also, these constraints are what make the problem interesting. For example, if we could observe the system with arbitrary precision, then we could deduce the presence of the short pendulum just by observing the motion of any pendulum in only a constant number of cycles, even without any resonance. However, this demands a precision of $O(1/N)$.

B. Algorithm

The following procedure ascertains whether or not there is a special pendulum in the set that is connected to the support pendulum. Once we have a procedure for identifying the presence (or absence) of a desired item in a specified set, it is possible to identify precisely which one this is by $\log_2 N$ repetitions of the identification procedure in a binary search fashion.

Select any one of the pendulums and shorten its length so that it is of the same length as the short pendulum (assuming it is not already a short pendulum). It is assumed that we know the length of the short pendulums. Set the system in motion by giving a push to the support. Observe the cyclic variation in the amplitude of the shortened pendulum for $O(\sqrt{N})$ cycles.

In case the set of pendulums connected to the support originally had a short pendulum then, including the one we had shortened, it will have two short pendulums. If it did not originally have a short pendulum, then it will have just one short pendulum. An analysis similar to the preceding section shows that the resonant coupling transfers a large fraction of the energy to and from the short pendulums with a periodicity of $O(\sqrt{N/\tau})$ cycles, where τ is the number of short pendulums. Thus there will be a difference of a factor of $\sqrt{2}$ in the periodicity, depending on whether there are one or two short pendulums. This periodicity is inferred from the cyclic variations in the amplitude of the shortened pendulum.

VI. WHY DOES IT TAKE $O(\sqrt{N})$ CYCLES?

The quantum-search algorithm has been rigorously proved to be the best possible algorithm for exhaustive search, i.e., no other algorithm can carry out an exhaustive search of N items in fewer than $O(\sqrt{N})$ steps. The proof for this is complicated and based on subtle properties of unitary transformations [7]. Fortunately, in the classical analog, there is a simple argument as to why it needs $O(\sqrt{N})$ cycles to transfer the energy to the desired pendulum.

As described in Sec. IV, the oscillation mode of the single pendulum is resonantly coupled to one of the two modes arising out of the interaction of the center-of-mass mode [which has a mass $O(N)$ times that of the single pendulum] with the mode of the coupling pendulum [which too has a mass $O(N)$ times that of the single pendulum]. Therefore, the modes that arise out of this interaction also behave as oscillators with a mass $O(N)$ times that of the single pendulum.

The question is as to how rapidly can we transfer energy from a pendulum of mass $O(N)$ to that of a pendulum with a mass of order 1 through a resonant coupling. Assume both pendulums to have an energy of order 1. Then the amplitude of the larger pendulum is $O(1/\sqrt{N})$ times that of the smaller pendulum. Since they have the same frequencies, the peak velocity of the larger pendulum is also $O(1/\sqrt{N})$ times that of the smaller pendulum.

Consider an elastic collision between a sphere of mass of N, traveling with a velocity of $O(1/\sqrt{N})$, with another sphere of unit mass traveling with a velocity less than 1. As shown in Fig. 3, in the center-of-mass frame, the larger sphere is almost stationary and the smaller sphere bounces off the larger sphere. The speed of the smaller sphere stays unaltered and the velocity changes sign (in order to conserve kinetic energy). Translating back to the original frame, we see that the magnitude of the velocity of the smaller sphere has increased by $2/\sqrt{N}$. Therefore, it will take $O(\sqrt{N})$ such interactions for the velocity of the smaller sphere to be able to rise from 0 to 1 (or equivalently to transfer an energy of order 1).

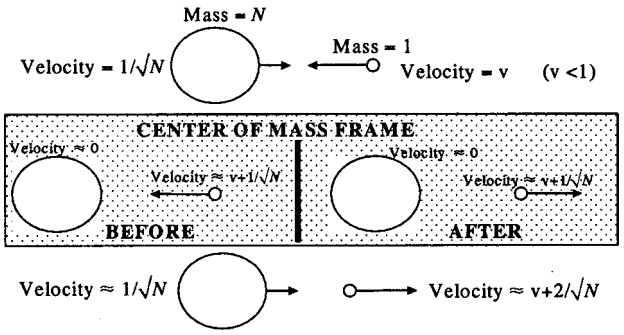

FIG. 3. When a sphere of unit mass moving with unit velocity collides with a larger sphere of mass equal to N that is moving with a velocity of $1/\sqrt{N}$, the magnitude of the velocity of the smaller sphere can change by at most $2/\sqrt{N}$.

VII. APPLICATIONS AND EXTENSIONS

A. Counting

Estimating the number of occurrences is an important problem in statistics and computer science. The first extension of the original quantum-search algorithm was to the problem of counting where it gave a square-root advantage over the best possible classical algorithm [8]. Our classical analog too gives a square-root advantage over the standard estimation technique.

We are given N pendulums; a small fraction of them (say ϵ) are shorter than the rest. The problem is to estimate ϵ. The standard sampling technique is to pick a certain number of pendulums at random and measure their oscillation frequency. Since the probability of getting a shorter pendulum in each sample is ϵ, it will take about $1/\epsilon$ samples before we get a single occurrence of a shorter pendulum. Since it takes $O(1)$ cycles to estimate the oscillation frequency of a pendulum, it will take $O(1/\epsilon)$ cycles to be able to derive any reasonable estimate of ϵ. On the other hand, by extending the technique of the preceding section, it is possible to estimate ϵ in only $O(1/\sqrt{\epsilon})$ cycles.

The approach is to suspend all N pendulums from a single pendulum as in Sec. IV thus coupling them. Now, as before, a resonant coupling is designed between the shorter pendulums and the rest of the system. The strength of this coupling is $O(\sqrt{\epsilon})$. This causes energy to flow back and forth from the shorter pendulums with a periodicity of $O(1/\sqrt{\epsilon})$ cycles. As in Sec. V we design the first pendulum to be a short pendulum. By following its amplitude for $O(1/\sqrt{\epsilon})$ cycles,

we will observe a cyclic variation. The length of this cycle will immediately identify ϵ.

B. Mechanical applications

Consider an application where we need to transfer energy to one of N subsystems. This can be accomplished by coupling the subsystems as described in this paper and making a slight perturbation to the subsystem into which we want the energy to flow into. After $O(\sqrt{N})$ cycles, a large fraction of the energy will flow into the selected subsystem. Alternatively, if we want to transfer energy from one subsystem to another, this can be similarly accomplished by a two-step process. First, make a perturbation to the subsystem from which the energy is coming. If the system is now allowed to oscillate for $O(\sqrt{N})$ cycles, the energy transfers into the support structure. Now, if the perturbation is removed from the source subsystem and made in the destination subsystem, the energy will flow from the support into the destination subsystem. By proper design it is possible to accomplish a lossless transfer of energy from one to another subsystem. This type of scheme would be especially useful in an application where we need the flexibility of transferring energy to any one of N components with minimal changes in hardware: a mechanical router.

C. Quantum-mechanical applications

In quantum-mechanical settings there are several applications where various modes of oscillation are coupled through the *center-of-mass mode*. For example, consider N atoms coupled resonantly to a photon mode in an optical cavity [9]. The atoms are trapped in the cavity by some kind of electromagnetic fields. The photon mode plays the role of the support pendulum through which the particles are coupled. Consider the basis state $|i\rangle$ to be the state where the photon excitation is localized on the ith atom. Due to the coupling there is a certain amplitude for the excitation to transfer to another atom. Since the atoms are close together in the cavity, this amplitude is the same between any two atoms. Therefore, the Hamiltonian is of the form $a\Sigma_i|i\rangle\langle i| + b\Sigma_{i,j}|i\rangle\langle j|$. This is exactly the kind of Hamiltonian that motivated our analysis in Sec. II. A similar analysis applies in the case of an ion trap [10] or in the case of Josephson junctions [11] coupled through a mutual inductance.

ACKNOWLEDGMENTS

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