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How much state assignments can differ

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We derive necessary and sufficient conditions for a group of density matrices to characterize what different people may know about one and the same physical system.

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I. INTRODUCTION

According to Peierls [1],

> [T]he most fundamental statement of quantum mechanics is that the wavefunction, or, more generally the density matrix, represents our *knowledge* of the system we are trying to describe.

In answer to the question ''whose knowledge?'' Peierls goes on to say:

> [Density matrices] may differ, as the nature and amount of knowledge may differ. People may have observed the system by different methods, with more or less accuracy; they may have seen part of the results of another physicist. However, there are limitations to the extent to which their knowledge may differ. This is imposed by the uncertainty principle. For example, if one observer has knowledge of S_z of our Stern-Gerlach atom, another may not know S_x, since measurement of S_x would have destroyed the other person's knowledge of S_z, and vice versa. This limitation can be compactly and conveniently expressed by the condition that the density matrices used by the two observers must commute with each other.

In another essay to which he refers the reader of Ref. [1], Peierls adds a corollary to his first condition [2]:

> At the same time, the two observers should not contradict each other. This means the product of the two density matrices should not be zero.

In discussing the extent to which density matrices assigned by observers with differing knowledge may differ, it is useful to define a set of density matrices to be *compatible* when there could be circumstances under which they would represent the knowledge different people have of one and the same physical system.

Fuchs [3] has pointed out a simple counterexample to Peierls' first compatibility condition, that density matrices used by two different observers must necessarily commute, but one of us has argued [4] that Peierls' second compatibility condition remains a necessary constraint. The critique of Peierls' conditions in Ref. [4] leaves open, however, the question of what might constitute necessary and sufficient conditions for several density matrices to be mutually compatible. Here, we provide an answer for systems described by a finite-dimensional Hilbert space.

In Sec. II, we show that if each of several different density matrices has a (not necessarily unique) expansion (in not necessarily orthogonal states) of the form

$$\rho=\sum_i p_i|\phi_i\rangle\langle\phi_i|, \quad p_i>0, \tag{1}$$

and there is at least one state common to every one of the expansions, then there are circumstances under which those density matrices can represent the different knowledge available to different people about one and the same physical system.

In Sec. III, we argue that if different density matrices do represent the knowledge available to different people about one and the same physical system, then the supports of all of those density matrices must have at least one state in common. [The *support* $S(\rho)$ of a density matrix ρ is the subspace spanned by all its eigenvectors with nonzero eigenvalues.]

In Sec. IV, we show that the supports of several different density matrices have at least one state in common if and (for a finite-dimensional Hilbert space) only if each has an expansion of the form (1) with at least one state common to all the expansions. Together with the results of Secs. II and III, this shows that several density matrices are mutually compatible if and only if the supports of all them have at least one state in common, or equivalently, if and only if all of them have expansions of the form (1) with at least one state common to all expansions.

In Sec. V, we comment on these results.

II. A SUFFICIENT CONDITION FOR COMPATIBILITY

Let ρ_a and ρ_b be two [5] different assignments of density matrix to one and the same system *S*. Let them have expansions (1) that share a common state $|\phi\rangle$,

$$\rho_a=p_a|\phi\rangle\langle\phi|+\sum_{i\geqslant 1} p_{ai}|\phi_{ai}\rangle\langle\phi_{ai}|,$$

$$\rho_b=p_b|\phi\rangle\langle\phi|+\sum_{i\geqslant 1} p_{bi}|\phi_{bi}\rangle\langle\phi_{bi}|, \tag{2}$$

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with all p non-negative and p_a and p_b both greater than zero. Then there are conditions under which the knowledge Alice and Bob can each have of S can be encapsulated in density matrices ρ_a and ρ_b. Here is one way this can come about.

Let there be two more systems S_a and S_b. Alice and Bob both know that the composite system S_a+S_b+S is in the pure state $|\Psi\rangle$ given (to within an overall normalization constant) by

$$|\Psi\rangle=|a_0\rangle|b_0\rangle|\phi\rangle+\sum_{i\geqslant 1}\sqrt{p_{ai}/p_a}\,|a_0\rangle|b_i\rangle|\phi_{ai}\rangle+\sum_{i\geqslant 1}\sqrt{p_{bi}/p_b}|a_i\rangle|b_0\rangle|\phi_{bi}\rangle, \tag{3}$$

where the $|a_n\rangle$ and $|b_n\rangle$ are orthonormal sets of states for S_a and S_b. Alice has access only to S_a and Bob only to S_b. None of them have access to S, which neither they nor anybody else act upon. Both of them know all of this.

Alice measures a nondegenerate observable A on S_a whose eigenstates are $|a_0\rangle,|a_1\rangle,|a_2\rangle,\ldots$ and finds the result associated with $|a_0\rangle$, while Bob measures an observable B on S_b, finding the result associated with $|b_0\rangle$. This is a possible set of joint outcomes, since the amplitude of $|a_0\rangle|b_0\rangle|\phi\rangle$ is nonzero. Neither knows what, if anything, may have happened to the subsystems accessible to the other.

Anybody informed of the results of both measurements would conclude that the state of S was now $|\phi\rangle$. But since Alice does not know the results of Bob's measurement, or even whether he undertook to perform any measurements, she can only reason as follows: immediately after her measurement she assigns to S_b+S the state (to within normalization)

$$|\Psi_a\rangle=|b_0\rangle|\phi\rangle+\sum_{i\geqslant 1}\sqrt{p_{ai}/p_a}|b_i\rangle|\phi_{ai}\rangle. \tag{4}$$

Not knowing what, if anything, Bob may have done to his own subsystem, Alice assigns to S the reduced density matrix obtained from $|\Psi_a\rangle\langle\Psi_a|$ by taking the partial trace over S_b. This is precisely ρ_a.

In the same way, knowing only the results of his own measurement, Bob will describe S with the reduced density matrix ρ_b.

This construction generalizes to any number of observers. For Alice, Bob, and Carol, for example, one would have additional subsystems S_a, S_b, and S_c, the joint state (3) would become

$$|\Psi\rangle=|a_0\rangle|b_0\rangle|c_0\rangle|\phi\rangle+\sum_{i\geqslant 1}\sqrt{p_{ai}/p_a}|a_0\rangle|b_i\rangle|c_i\rangle|\phi_{ai}\rangle+\sum_{i\geqslant 1}\sqrt{p_{bi}/p_b}|a_i\rangle|b_0\rangle|c_i\rangle|\phi_{bi}\rangle+\sum_{i\geqslant 1}\sqrt{p_{ci}/p_c}|a_i\rangle|b_i\rangle|c_0\rangle|\phi_{ci}\rangle, \tag{5}$$

and the state (4) that Alice assigns to the other subsystems after she measures her own to be in the state $|a_0\rangle$ would become

$$|\Psi_a\rangle=|b_0\rangle|c_0\rangle|\phi\rangle+\sum_{i\geqslant 1}\sqrt{p_{ai}/p_a}|b_i\rangle|c_i\rangle|\phi_{ai}\rangle, \tag{6}$$

which again leads her, in the absence of any knowledge of what Bob and Carol may or may not have done, to assign the reduced density matrix ρ_a to the system S.

III. A NECESSARY CONDITION FOR COMPATIBILITY

Suppose Alice, Bob, Carol, ... describe a system with density matrices $\rho_a,\rho_b,\rho_c,\ldots$. Each of their different density-matrix assignments incorporates some subset of a valid body of currently relevant information about the system, all of which could, in principle, be known by a particularly well-informed Zeno [6].

Let us say that a system is *found to be in* a particular pure state if the projection operator on that state is measured and the result is 1. We then refer to that state as the *outcome* of the measurement [7]. The set of all states that a density matrix ρ forbids to be outcomes of a measurement is called the *null space* of ρ. It is the subspace of all eigenvectors of ρ with zero eigenvalue, and the orthogonal complement of the support $S(\rho)$.

A necessary condition for the compatibility of these differing density-matrix assignments follows from these considerations:

(i) If anybody describes a system with a density matrix ρ, then nobody can find it to be in a pure state in the null space of ρ. For although anyone can get a measurement outcome that everyone has assigned nonzero probabilities, nobody can get an outcome that anybody knows to be impossible.

(ii) A system that cannot be found to be in either of two distinct states cannot be found to be in any superposition of those states. For any density matrix whatever that incorporates the information that both outcomes are impossible, must also assign zero probability to any superposition of those outcomes [8].

(iii) There must be some states in which a system can be found.

It follows from (i) and (ii) that when different people assign different density matrices to one and the same physical system, the union of all their different null spaces must span a subspace S_0 of states in which the system cannot be found. According to (iii) there must then be states $|\psi\rangle$ that are not in S_0. The projection of such a $|\psi\rangle$ on the orthogonal complement of S_0 lies in the support of every one of the different density matrices. So for a collection of different state assignments to be compatible, the supports of all the different density matrices must have at least one state in common.

IV. EITHER CONDITION IS NECESSARY AND SUFFICIENT

If the Hilbert space is finite dimensional, then the supports $S(\rho_a),S(\rho_b),\ldots$ of several different density matrices

$\rho_a, \rho_b, \ldots$ can have at least one state in common if and only if each has an expansion of the form (1) with at least one state common to all the expansions. To show this, we must show that a state $|\psi\rangle$ can appear as one of the $|\phi_i\rangle$ in an expansion (1) of a density matrix ρ into not necessarily orthogonal states if and only if $|\psi\rangle$ belongs to the support of ρ [9].

That any $|\psi\rangle$ occurring in (1) must be in the support of ρ follows directly from the fact that every $|\phi_i\rangle$ in (1) must be orthogonal to any vector $|\lambda\rangle$ in the null space of ρ, since

$$\langle\lambda|\rho|\lambda\rangle = 0 \tag{7}$$

and every p_i in (1) is strictly greater than zero.

Conversely, to see that any vector in $S(\rho)$ can appear in some expansion of the form (1), consider an expansion of ρ in orthonormal projections onto its eigenvectors,

$$\rho = \sum_i r_i |\psi_i\rangle\langle\psi_i|. \tag{8}$$

The positive r_i in Eq. (8) are all bounded away from zero if the dimension of $S(\rho)$ is finite. (This is the only place where we appeal to the finite dimensionality of the Hilbert space.) Let r_0 be the least value of any of the r_i and define nonnegative s_i by

$$s_i = r_i - r_0. \tag{9}$$

Then

$$\rho = \sum_i s_i |\psi_i\rangle\langle\psi_i| + r_0 P, \tag{10}$$

where the sum is over the nonzero s_i and P is the projection operator onto $S(\rho)$. If $|\psi\rangle$ is any unit vector in $S(\rho)$, then one can find an orthonormal basis $|\eta_i\rangle$ for $S(\rho)$ with $|\eta_0\rangle = |\psi\rangle$. Since

$$P = \sum_i |\eta_i\rangle\langle\eta_i|, \tag{11}$$

Eqs. (10) and (11) do indeed give a decomposition of ρ in which $|\psi\rangle$ appears:

$$\rho = r_0|\psi\rangle\langle\psi| + \sum_{i>0} r_0|\eta_i\rangle\langle\eta_i| + \sum_i s_i|\psi_i\rangle\langle\psi_i|. \tag{12}$$

V. DISCUSSION

(1) The density matrix for a pure state has the one-dimensional space spanned by that state alone as its support. So a special case of our condition is that two pure-state density matrices are compatible if and only if they are identical. This includes the example given by Peierls in the first quotation above, but it provides a rather different explanation. Peierls would say that the reason why Alice cannot know that a spin-1/2 particle is up along **z** while Bob knows at the same time that it is up along **x** is that the process by which one of them acquires his or her knowledge necessarily renders obsolete the knowledge of the other. From our perspective, however, the question of how the knowledge might have been acquired does not enter, provided nothing has been done that renders the knowledge of either invalid. The two state assignments are incompatible, because if Alice knew that nobody could find the particle to be down along **z**, and Bob knew that nobody could find it to be down along **x**, then since $|\downarrow_z\rangle$ and $|\downarrow_x\rangle$ span the whole space, the impossibility of superpositions of impossible outcomes would require *all* outcomes to be impossible.

(2) One could argue that nobody can ever know with certainty that any outcome of any measurement is strictly impossible. The support of any realistic density matrix would then be the entire Hilbert space, and our condition would be vacuous—any set of density matrices would be mutually compatible, though the probability of an outcome leading to such state assignments in the manner of Sec. II could be minuscule. On the other hand, although the quantum theory is famously probabilistic, one should not lose sight of the fact that the *theory* is also capable of deterministic assertions, which strictly prohibit certain measurement outcomes under certain ideal conditions. It is surely a significant feature of the theory that consideration of impossible outcomes and very little else leads, without any invocation of "the uncertainty principle" or "maximal information," to the fact that pure-state assignments must be unique, as well as the more general constraint on mixed-state assignments.

(3) We have limited the density matrices under consideration in Sec. III to those based on a currently relevant subset of a body of information that could, in principle, all have been acquired by a single observer through measurements on the system S or on other systems correlated with S in known ways. If we were to expand the set of density matrices to include guesses, or forms incorporating data based on badly designed measurements or rendered obsolete by subsequent measurements, then, of course, our necessary condition need not apply.

(4) Peierls' second condition that the product of two density matrices be nonzero, is implied by our condition that their supports have at least one state in common, but it is weaker because it takes into account only point (i) of Sec. III but not point (ii). [It too is subject to the reservation in point (3) above.]

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[1] R.E. Peierls, Phys. World **4**, 19 (1991).
[2] Rudolf Peierls, *More Surprises in Theoretical Physics* (Princeton University Press, Princeton, NJ, 1991), p. 11.
[3] Christopher Fuchs (private communication) to N.D. Mermin; e-print quant-ph/0105039. A version of Fuchs' counterexample is given in Ref. [4].
[4] N. David Mermin, in *Quantum (Un)speakables: Essays in Commemoration of John S. Bell*, edited by Reinhold Bertlmann and Anton Zeilinger (Springer-Verlag, Berlin, 2001); e-print quant-ph/0107051.
[5] As noted below, what follows is straightforwardly generalized to any number of assignments.
[6] Section II gives an example of such a body of information and how partial information might be acquired.
[7] This does not, of course, imply anything about the state of the system prior to the measurement.
[8] Such a density matrix need not be any of $\rho_a, \rho_b, \rho_c, \ldots$, but could characterize the knowledge of some Zeno, Yvonne, Xavier, ... who had access to information that included both impossibilities.
[9] This point was first made by Schrödinger, Proc. Cambridge Philos. Soc. **32**, 446 (1936), though he does not explicitly note that the Hilbert space must be finite dimensional.