Implementing the quantum random walk

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Recently, several groups have investigated quantum analogues of random walk algorithms, both on a line and on a circle. It has been found that the quantum versions have markedly different features to the classical versions. Namely, the variance on the line, and the mixing time on the circle increase quadratically faster in the quantum versions as compared to the classical versions. Here, we propose a scheme to implement the quantum random walk on a line and on a circle in an ion trap quantum computer. With current ion trap technology, the number of steps that could be experimentally implemented will be relatively small. However, we show how the enhanced features of these walks could be observed experimentally. In the limit of strong decoherence, the quantum random walk tends to the classical random walk. By measuring the degree to which the walk remains ''quantum,'' this algorithm could serve as an important benchmarking protocol for ion trap quantum computers.

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I. INTRODUCTION

The idea that a computational device based on the laws of quantum mechanics might be more powerful than a computational device based on classical mechanics has been around for about two decades [1]. The study of computational devices based upon quantum mechanics is known as quantum computation. For an introduction to the field, see for example Nielsen and Chuang [2]. Active research in this field has exploded since the discovery by Shor [3] that a quantum computer could, in theory, factor large semiprimes exponentially faster than can currently be done on a classical computer. Since Shor's algorithm, Grover has devised an algorithm which can, in principle, search an unsorted database quadratically faster than any classical algorithm [4]. However, new quantum algorithms which out perform their classical counterparts are proving difficult to find. One path which is being followed to find quantum algorithms involves looking at effective classical algorithmic techniques, and trying to adapt them to quantum computation. Classically, the random walk has found applications in many fields including astronomy, solid-state physics, polymer chemistry, and biology. For a review of the theory and applications for random walks, see for example Barber and Ninham [5]. The hope is that a quantum version of the random walk might lead to applications unavailable classically. Quantum random walks have been investigated by a number of groups [6–12]. In this paper, we propose a scheme to implement the discrete quantum random walk on a line [8] and on a circle [9], using an ion trap quantum computer. For a review of ion trap quantum computation see Wineland *et al.* [13]. With current ion trap technologies, it will not be possible to implement a large number of steps in the walk, however it should be possible to implement enough steps to experimentally highlight the differences between the classical and quantum random walks, providing an important proof of principle.

The structure of this paper is as follows. In Sec. II we review the simple models of random walks on both a line and a circle, highlighting the differences between the classical and quantum versions in both cases. The classical walks are Gaussian, and therefore can be described by their standard deviations. The quantum walks are highly non-Gaussian, however we analyze the standard deviations of these walks also, in order to make a fair comparison with the classical walks. In Sec. III we discuss how we shall be representing the algorithms in an ion trap quantum computer. We then discuss the pulses required to evolve the system, first for the walk on the line, and then for the walk on a circle. Finally, in Sec. IV, we discuss a relatively simple measurement procedure which can be used to highlight the difference between the classical and quantum random walks.

II. CLASSICAL VERSUS QUANTUM RANDOM WALKS

Classical random walks can take many different forms, starting from the simple discrete random walk on a line, to random walks on graphs, to continuous-time random walks, such as brownian motion. In this paper, we are only considering discrete time, discrete space, random walks on a line and on a circle.

A. Classical walk on a line

Imagine a person standing at the origin of a line with a coin in their hand. They flip the coin, and if it comes up heads, they take a step to the right, if it is tails, they take a step to the left. They then repeat this procedure, flipping the coin, and taking a step based on the result. The probability $P_N(d)$ of being in a position d after N steps is

$$P_N(d)=\frac{1}{2^N}\left(\begin{array}{c} N \\ \frac{d+N}{2} \end{array}\right). \tag{2.1}$$

Table I contains the probabilities for the first few values of N. The nonzero elements of the distribution are simply terms from Pascal's triangle, divided by the appropriate factor of two. There are two features of this random walk that we would like to compare to the quantum analog. First, the

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TABLE I. The probability of being found at position d after N steps of the classical random walk on the line.

N/d	-4	-3	-2	-1	0	1	2	3	4
0					1				
1				$\frac{1}{2}$	0	$\frac{1}{2}$			
2			$\frac{1}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$		
3		$\frac{1}{8}$	0	$\frac{3}{8}$	0	$\frac{3}{8}$	0	$\frac{1}{8}$	
4	$\frac{1}{16}$	0	$\frac{4}{16}$	0	$\frac{6}{16}$	0	$\frac{4}{16}$	0	$\frac{1}{16}$

mean of the walk is zero. This is intuitively obvious, we are using a fair coin, so we are as likely to step left as we are to step right. The other property of the distribution that we are interested in is the standard deviation. It is not hard to calculate that the standard deviation of this distribution, σ_c, is given by

$$\sigma_c = \sqrt{N}. \tag{2.2}$$

B. Quantum walk on a line

Now let us consider a quantum version of the walk on a line. The first modification we can make is to replace the coin with a qubit. In this paper, we shall be representing the two levels of the qubit with the states $|\downarrow\rangle$ and $|\uparrow\rangle$ rather than $|0\rangle$ and $|1\rangle$. If we start with the qubit in the down state, and apply a Hadamard operation, we get an equal superposition of up and down,

$$\hat{H}|\downarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle, \quad \hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.3}$$

If we were to measure the qubit, and step left or right depending upon the result, we would obtain exactly the classical walk described above. Now, rather than a person holding a coin, suppose we have a particle, whose motion is confined to one dimension. We can now treat the particle as a quantum system, and perform the quantum walk as follows. During each iteration, we apply the Hadamard operation, followed by the operation which steps right if the qubit is down, and steps left if the qubit is up. That is, we apply the operator,

$$\hat{U} = e^{i\hat{p}\hat{\sigma}_z}\hat{H}, \tag{2.4}$$

where $\hat{p}$ is the momentum operator of the particle confined to one dimension, and $\hat{\sigma}_z$ is the Pauli-z operator acting on the qubit. Therefore, the state of the system after N steps is

$$|\Psi_N\rangle = (e^{i\hat{p}\hat{\sigma}_z})^N|\Psi_0\rangle, \tag{2.5}$$

where $|\Psi_0\rangle$ is the initial state of the system. The mean of the distribution produced by this quantum random walk is not necessarily zero. It is dependent upon the initial state of the qubit. For example, choosing the initial state of the qubit to be down gives a nonzero mean after the second step. For the remainder of this paper, we shall only be considering the distribution created with the initial qubit state $1/\sqrt{2}|\downarrow\rangle + i/\sqrt{2}|\uparrow\rangle$ which has a mean of zero for all values of N,

TABLE II. The probability of being found at position d after N steps of the quantum random walk on the line, with the initial qubit state $1/\sqrt{2}|\downarrow\rangle + i/\sqrt{2}|\uparrow\rangle$.

N/d	-4	-3	-2	-1	0	1	2	3	4
0					1				
1				$\frac{1}{2}$	0	$\frac{1}{2}$			
2			$\frac{1}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$		
3		$\frac{1}{8}$	0	$\frac{3}{8}$	0	$\frac{3}{8}$	0	$\frac{1}{8}$	
4	$\frac{1}{16}$	0	$\frac{6}{16}$	0	$\frac{2}{16}$	0	$\frac{6}{16}$	0	$\frac{1}{16}$

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|\downarrow\rangle + i|\uparrow\rangle). \tag{2.6}$$

Table II contains the probability distribution associated with the first few states $|\Psi_N\rangle$. The nonzero elements of the distribution are no longer simply terms from Pascal's triangle, which arose in the classical case. The deviations from the classical distribution are caused by quantum interference effects. Now it is no longer simple to calculate the standard deviation of the distribution. However, numerical simulations reveal that the standard deviation, σ_q, is almost independent of the initial state of the qubit, and is approximately linear in N,

$$\sigma_q \approx \frac{3}{5}N. \tag{2.7}$$

The standard deviation is plotted in Fig. 1 up to $N=40$ for both the classical and quantum walk distributions.

Clearly, the standard deviation is significantly different for the quantum and classical random walks on a line. Now let us consider the random walks which arise when periodic boundary conditions are applied to the random walks.

C. Classical walk on a circle

In the paper by Aharonov *et al.* [9], they consider random walks on the circle, where the step size is an irrational multiple of π. Here, we shall only be considering the simple

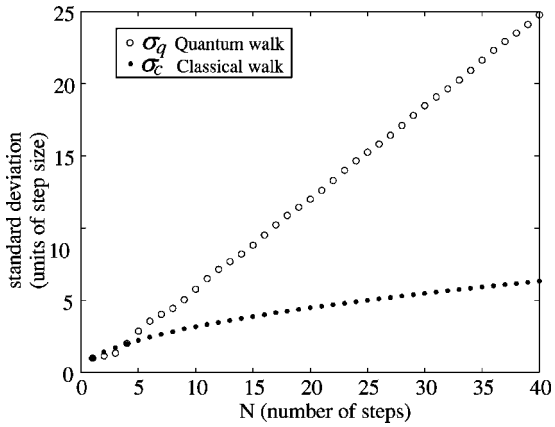

FIG. 1. Standard deviation for both the quantum and classical random walks up to $N=40$.

TABLE III. The probability of being found at position θ after N steps of the quantum random walk on the circle.

N/d	0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$
0	1	0	0	0
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
2	$\frac{1}{2}$	0	$\frac{1}{2}$	0
3	0	1	0	0
4	0	0	1	0
5	0	$\frac{1}{2}$	0	$\frac{1}{2}$
6	$\frac{1}{2}$	0	$\frac{1}{2}$	0
7	0	0	0	1

distribution which arises when the step size is taken to be $\pi/2$. Let us assume that the particle is initially found, with probability one, at some point on a circle denoted by $\theta=0$,

$$P_0(\theta=0)=1. \tag{2.8}$$

After one step of the algorithm, the classical distribution is given by

$$P_1(\theta)=\begin{cases} 0, & \theta=0,\pi, \\ \dfrac{1}{2}, & \theta=\pm\dfrac{\pi}{2}, \end{cases} \tag{2.9}$$

and after the second step,

$$P_2(\theta)=\begin{cases} \dfrac{1}{2}, & \theta=0,\pi, \\ 0, & \theta=\pm\dfrac{\pi}{2}. \end{cases} \tag{2.10}$$

It is not difficult to see that the probability distribution for all subsequent odd number of steps will be given by Eq. (2.9), and the distribution for all subsequent even number of steps will be given by Eq. (2.10).

D. Quantum walk on a circle

Let us consider the quantum random walk on the circle. Once again, we start with the particle at some point on a circle denoted by $\theta=0$, thus the initial probability distribution is given by Eq. (2.8). The probability distributions after one and two steps are also given by Eqs. (2.9) and (2.10), respectively, however after the third step, interference effects results in the distribution

$$P_3\left(\theta=\frac{\pi}{2}\right)=1. \tag{2.11}$$

Calculation of the states after subsequent steps reveals that the quantum random walk around the circle, with a step size of $\pi/2$ is periodic with a period of eight. The eight probability distributions which arise are given in Table III.

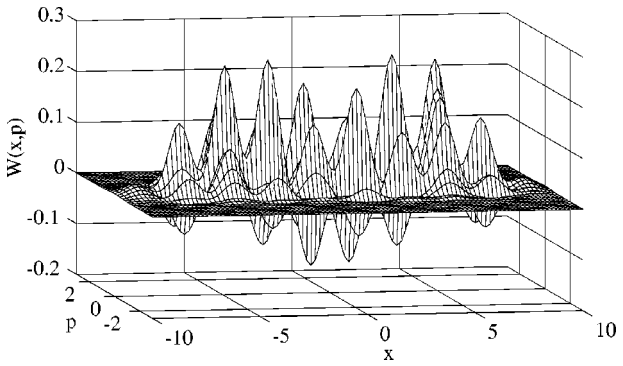

FIG. 2. Wigner function of the particle after five steps of the quantum random walk on the line. (The electronic level of the ion has been traced over.)

III. IMPLEMENTING THE WALKS IN AN ION TRAP

The analysis thus far has assumed that all operations can be applied without error and the particle can exist in position eigenstates. Now we shall relax these assumptions, and describe how the algorithm can be implemented in an ion trap.

The ion trap provides a convenient setting for the quantum random walks we have described, as it contains the required discrete and continuous quantum variables. For the remainder of this paper, we shall be discussing implementations based on a single $^9Be^+$ ion, confined in a coaxial-resonator radio frequency ion trap, as described in [14] and references therein.

The preparation involves laser cooling the ion to the motional and electronic ground state $|0\rangle|\downarrow\rangle$, as described in [15]. A sequence of four Raman beam pulses are then applied [14] to create the state $(|\alpha\rangle|\downarrow\rangle+|-\alpha\rangle|\uparrow\rangle)/\sqrt{2}$, where $|\alpha\rangle$ denotes the coherent state of the oscillator,

$$|\alpha\rangle=\frac{e^{2\alpha_R\alpha_I i}}{\pi^{1/4}}\int dx e^{\sqrt{2}i\alpha_I x}e^{-1/2(x-\sqrt{2}\alpha_R)^2}|x\rangle \tag{3.1}$$

and $\alpha\equiv\alpha_R+i\alpha_I$.

The first pulse is a $\pi/2$ pulse, which creates an equal superposition of $|0\rangle|\downarrow\rangle$ and $|0\rangle|\uparrow\rangle$. A displacement beam is then applied which excites the motion correlated to the $|\uparrow\rangle$ internal state. The third pulse is a π pulse which exchanges the internal states, and finally the displacement beam is applied again. The combined action of the four pulses is to effectively perform the operator $\hat{U}$, defined in Eq. (2.4). The quantum random walk on the line is accomplished by repeating this sequence of pulses N times. Figure 2 contains the Wigner function obtained by tracing over the internal degree of freedom after five steps of the quantum random walk algorithm.

The quantum random walk on the circle can be implemented in an ion trap by ''walking'' the particle around a circle in phase space, rather than a circle in real space. In order to accomplish this task, we need to generate an operator of the form

$$\hat{W}=e^{i\pi\hat{a}^\dagger\hat{a}\hat{\sigma}_z/2}\hat{H}, \tag{3.2}$$

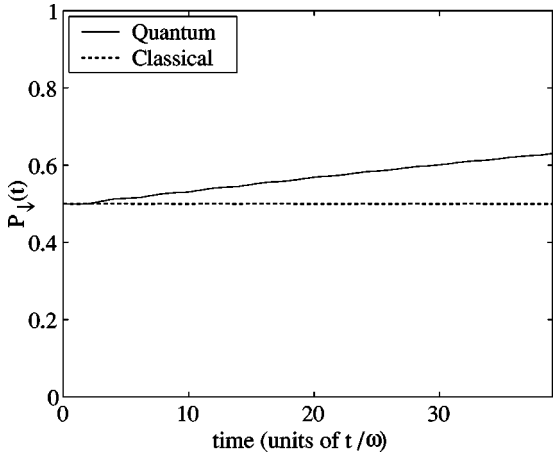

FIG. 3. Probability of measuring the ion in the ground state after applying the random walk for a time t/ω, decoupling the internal and motional states, and applying the measurement operator $\hat{M}$.

where $\hat{a}$ and $\hat{a}^\dagger$ correspond to the annihilation and creation operators of the harmonic oscillator. This operator can be produced in an ion trap by applying far-detuned laser pulses to the ion [16], followed by a $\pi/2$ pulse.

IV. MEASURING THE WALKS

Using current ion trap technologies, wave-packet dispersion is negligible [14], so the main source of decoherence is related to the internal levels of the ion. Decoherence of the electronic levels of the ion during the application of the algorithm has the effect of gradually transforming the quantum random walk to the classical random walk. Rather than considering this to be a negative effect, we can measure the degree to which the ion is acting as a quantum variable rather than a classical variable, and thereby effectively measure the level of decoherence in the ion trap.

The scheme that we envisage for measuring the random walk utilizes similar operators to those employed in the application of the algorithm. After applying the random-walk sequence for some number of steps, the internal state of the ion is decoupled from the motional state by an appropriate Raman pulse. An effective operator such as $\exp(i\hat{p}\hat{\sigma}_y)$ is applied, before finally measuring the internal state of the ion. Thus we are using the internal state of the ion to supply us with information about the motional state.

In the case of the walk on the line, suppose we decouple the internal state from the motional state by measuring whether the ion is in the state $|\uparrow\rangle$ or $|\downarrow\rangle$. We then apply the operator

$$\hat{M}^{\pm} = e^{\pm i\hat{p}\hat{\sigma}_y}. \tag{4.1}$$

The positive Hamiltonian is applied upon obtaining the results $|\uparrow\rangle$, while the negative Hamiltonian is applied otherwise. Finally, we again measure the internal state of the ion. If the quantum random walk has experienced no decoherence, then we measure $|\downarrow\rangle$ with the probabilities given by the solid line in Fig. 3, whereas if the ion suffers complete decoherence we would expect to measure $|\downarrow\rangle$ with probability of one half.

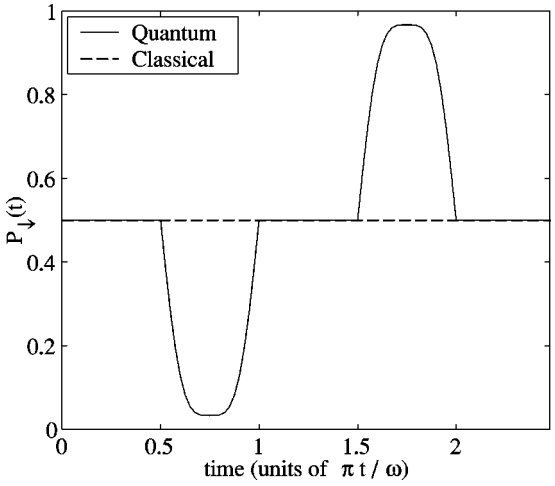

FIG. 4. Probability of measuring the ion in the ground state after applying the random walk on a circle for a time $\pi t/\omega$, decoupling the internal and motional states, and applying the measurement operator $\hat{D}$.

A similar scheme can be used to measure the level of decoherence in the quantum random walk on the circle. Figure 4 again depicts the probability of measuring the ion in the ground state after decoupling the internal and motional states, however, this time we then apply the operator

$$\hat{D} = e^{i\hat{x}\hat{\sigma}_y}. \tag{4.2}$$

In this case, because we have total destructive interference of certain paths during the walk, the deviation of the quantum-to-classical walk is much larger at certain stages of the walk.

V. DISCUSSION

We have described ion trap implementation schemes for quantum random walks, both on the line and on the circle. We have also suggested a measurement process which allows the enhanced features of these walks to be experimentally observed.

At this point, it is unclear whether quantum random walks will have any useful algorithmic applications. However, we believe that they can provide a benchmarking protocol for ion trap quantum computers, and perhaps other implementation schemes which combine continuous and discrete quantum variables.

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