Clauser-Horne inequality for three-state systems

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We show a new Bell-Clauser-Horne inequality for two entangled three-dimensional quantum systems (so-called qutrits). This inequality is not violated by a maximally entangled state of two qutrits observed through a symmetric three-input- and three-output-port beam splitter only if the amount of noise in the system is greater than $(11-6\sqrt{3})/2\approx 0.308$. This result is in a perfect agreement with the previous numerical calculations presented in Kaszlikowski *et al.* [Phys. Rev. Lett. **85**, 4418 (2000)]. Moreover, we prove that for noiseless case, the necessary and sufficient condition for the threshold quantum efficiency of detectors below which there is no violation of local realism for the optimal choice of observables is equal to $6(15-4\sqrt{3})/59\approx 0.821$. This efficiency result again agrees with the numerical predictions.

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I. INTRODUCTION

It is well known that the sufficient and necessary condition for the lack of existence of a local realistic description of two entangled qubits in an experiment in which Alice and Bob measure two dichotomic observables is the violation of at least one out of four Clauser-Horne (CH) [1–3] inequalities. However, for a system of two entangled three-dimensional quantum objects (so-called qutrits), there has hitherto been no such inequality, i.e., the inequality giving *necessary and sufficient* conditions for the existence of local realism. Numerical calculations based on the method of linear programing, which give necessary and sufficient conditions [4,5] (as well as the analytical proof presented in Ref. [6] for the choice of observables giving the same violation as the numerical calculations) clearly demonstrates that the violation of local realism for qutrits is stronger than for qubits.

In general, the numerical calculations [4,5] as well as the analytical proof confirming their validity [6] still do not allow us to appreciate fully the nature of quantum correlations and the possibility of their local and realistic description. In this paper, we present the set of Bell inequalities that seem to be a straightforward extension of the CH inequalities to a bipartite system consisting of two qutrits. We do not have a proof that this set of inequalities gives us sufficient and necessary condition for local realism (we prove only the necessary condition). However, it correctly reproduces numerical results for $N=3$ obtained in Refs. [4,5] concerning the threshold noise admixture above which there is a local realistic description as well as the critical efficiency of detectors below which quantum correlations can be simulated classically (i.e., by local hidden variables). This strongly suggests that we also have sufficiency condition.

The paper is organized as follows. In Sec. I, we give the set of nine inequalities for bipartite system of qutrits, the proof for which is given in Appendix A. In Sec. II, we show that for the experiment with the maximally entangled qutrits observed via symmetric six-port beam splitters, the inequality is violated by quantum mechanics. Using the inequality, we calculate the threshold noise admixture and critical quantum efficiency of detectors that confirm the numerical results presented in Refs. [4,5]. We give final conclusions in Sec. III. Finally, in Appendix B, we prove that the calculated critical quantum efficiency of detectors is also a sufficient condition for the existence of local realism for the considered choice of observables.

II. INEQUALITY

In a Bell-type experiment with two qutrits Alice and Bob measure one of the two trichotomic (three possible outcomes) observables: A_1 or A_2 for Alice and B_1 for B_2 for Bob. The outcomes of the measurement of observable A_k $(k=1,2)$ at Alice's side is denoted by a_k $(a_k=1,2,3)$ whereas the outcomes of the measurement of observable B_l $(l=1,2)$ at Bob's side is denoted by b_l $(b_l=1,2,3)$. For each pair of observables A_k,B_l $(k,l=1,2)$ we calculate the joint quantum probabilities $P_{QM}^{kl}(a_k;b_l)$, i.e., the probabilities of obtaining by Alice and Bob simultaneously the results a_k and b_l (coincidence "clicks" of detectors) and single quantum probabilities $P_{QM}^{k}(a_k),Q_{QM}^{l}(b_l)$, i.e., the probabilities of obtaining the result a_k by Alice irrespective of Bob's outcome and result b_k by Bob irrespective of Alice's result.

A local realistic description is equivalent to the existence of a joint probability distribution $P(a_1,a_2;b_1,b_2)$, in which the so-called marginals

$$P^{kl}(a_k;b_l)=\sum_{a_{k+1}=1}^{3}\sum_{b_{l+1}=1}^{3}P(a_1,a_2;b_1,b_2),$$

$$P^{k}(a_k)=\sum_{a_{k+1}=1}^{3}\sum_{b_1=1}^{3}\sum_{b_2=1}^{3}P(a_1,a_2;b_1,b_2), \qquad (1)$$

$$Q^l(b_l)=\sum_{a_1=1}^{3}\sum_{a_2=1}^{3}\sum_{b_{l+1}=1}^{3}P(a_1,a_2;b_1,b_2),$$

where $k+1$ and $l+1$ are modulo 2, recover quantum probabilities, i.e., $P^{kl}_{QM}(a_k;b_l)=P^{kl}(a_k;b_l)$, $P^k_{QM}(a_k)=P^k(a_k)$ and $Q^l_{QM}(b_l)=Q^l(b_l)$. With these formulas [Eq. (1)] one can prove that the following set of 36 inequalities (of which at most 32 are obviously independent) is valid:

$$\begin{aligned}&P^{1+\alpha\,1+\beta}(2+x;1+y)+P^{1+\alpha\,2+\beta}(2+x;1+y)\\&-P^{2+\alpha\,1+\beta}(2+x;1+y)+P^{2+\alpha\,2+\beta}(2+x;1+y)\\&+P^{1+\alpha\,1+\beta}(1+x;2+y)+P^{1+\alpha\,2+\beta}(1+x;2+y)\\&-P^{2+\alpha\,1+\beta}(1+x;2+y)+P^{2+\alpha\,2+\beta}(1+x;2+y)\\&+P^{1+\alpha\,1+\beta}(2+x;2+y)+P^{1+\alpha\,2+\beta}(1+x;1+y)\\&-P^{2+\alpha\,1+\beta}(2+x;2+y)+P^{2+\alpha\,2+\beta}(2+x;2+y)-P^{1+\alpha}(1\\&+x)-P^{1+\alpha}(2+x)-Q^{2+\beta}(1+y)\\&-Q^{2+\beta}(2+y)\leqslant 0,\end{aligned}\qquad(2)$$

where $x,y=0,1,2;$ $\alpha,\beta=0,1$ and where the addition is modulo 3 for x,y and modulo 2 for α,β.[1] The proof is straightforward but laborious and it is given in the Appendix A. The above inequality is the necessary condition for the existence of a local and realistic description of the considered experiment. It is interesting to notice that each of the above inequalities is the sum of two CH inequalities and some additional term. For instance, for $x=y=0$, we have the following two CH inequalities: $P^{11}(2;1)+P^{12}(2;1)-P^{21}(2;1)+P^{22}(2;1)-P^1(2)-Q^2(1)$, $P^{11}(1;2)+P^{12}(1;2)-P^{21}(1;2)+P^{22}(1;2)-P^1(1)-Q^2(2)$ and the term $P^{11}(2;2)+P^{12}(1;1)-P^{21}(2;2)+P^{22}(2;2)$, which bears a resemblance to an incomplete CH inequality, i.e., the CH inequality without single detection probabilities and with the term $P^{12}(2;2)$ replaced by $P^{12}(1;1)$.

[1]It is possible to obtain additional inequalities from the existing ones using the freedom in the labeling of the results as well as the local settings of the measuring apparatus. However, this freedom in the labeling is of little importance here. First, if one searches for the maximal violation of the inequality, this is equivalent to testing all possible unitary transformations of the local observation bases (associated with alternative local observations). Such unitary transformations also include detector (outcome, eigenfunctions) relabelings. Furthermore, in order to obtain a global maximum, one should scan all possible local settings (which in turn can be used to parametrize the local unitary transformations of the bases of the local observables). This effectively includes the permutation on the settings of the two alternative local observations in the inequalities. Second, at present, there is no proof that the full set of inequalities, which can be generated from Eq. (2) by all such relabelings, forms a sufficient condition for a local realistic description of the probabilities. Therefore, there is no practical need whatsoever to find a minimal subset of the inequalities in our discussion that would in any sense form a complete set (e.g., a full set of linearly independent conditions).

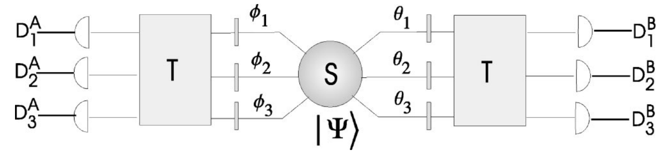

FIG. 1. Experiment with two six-port beam splitters. Two spatially separated six-port devices are fed with a two-photon-beam entangled state (3). Each local experimenter has three phases in the beams that define the observable that is measured. Also, at each site, there are three photon detectors D^A_k and D^B_l that register photons in the output ports of the device.

III. VIOLATION OF THE INEQUALITY

We now show that the above inequality is violated by quantum mechanics. To this end, let us consider the following Bell experiment. The source produces maximally entangled state $|\psi\rangle$ of two qutrits

$$|\psi\rangle=\frac{1}{\sqrt{3}}(|1\rangle_A|1\rangle_B+|2\rangle_A|2\rangle_B+|3\rangle_A|3\rangle_B),\qquad(3)$$

where $|k\rangle_A$ and $|k\rangle_B$ describe kth basis state of the qutrit A and B, respectively. Such a state can be prepared with pairs of photons using parametric down-conversion (see Ref. [9]), in which case kets $|k\rangle_A$ and $|k\rangle_B$ denotes photons propagating to Alice and Bob in mode k. Starting with this state, Alice and Bob measure one of two trichotomic observables defined by a six-port beam splitter (three input and three output ports). The extended theory of such devices can be found in Ref. [9]. A brief description is provided below.

The *unbiased* six-port beam splitter [9] is a device with the following property: if a photon enters any single input port (out of the three), there is equal probability that it leaves one of the three output ports. In fact, one can always construct a special six-port beam splitter with the distinguishing trait that the elements of its unitary transition matrix, $\hat{T}$, are *solely* powers of the complex number, $\alpha=\exp(i2\pi/3)$, namely, $T_{kl}=(1/\sqrt{3})\alpha^{(k-1)(l-1)}$. It has been shown in Ref. [9] that *any* six-port beam splitter can be constructed from the above-mentioned one by adding appropriate phase shifters at its exit and input ports (and by a trivial relabeling of the output ports). If the output beams of the beam splitter are directly fed into detectors, as it will be in the case under consideration, the exit phase shifts can be, of course, neglected. The phase shifters in front of the input ports of the beam splitter can be tunable and used to change the phase of the incoming photon (Fig. 1). The full set of the three phase shifts in front of the input ports of a beam splitter, which we denote for convenience as a ''vector'' of phase shifts $\vec{\phi}=(\phi_1,\phi_2,\phi_3)$, can be treated as the set macroscopic local parameters that can be arbitrary controlled by the observer, and which define the observable that is measured. Therefore, a six-port beam splitter, together with the three phase shift devices, performs the unitary transformation $\hat{U}(\vec{\phi})$ with the entries $U_{kl}=T_{kl}\exp(i\phi_l)$.

We calculate quantum probabilities in a standard way, i.e.,

$$\begin{aligned}P^{kl}_{QM}(a_k;b_l)&=\mathrm{Tr}\,\Pi_{a_k}\otimes\Pi_{b_l}\hat{U}(\vec{\phi}_k)\otimes\hat{U}(\vec{\theta}_l|\psi\rangle\langle\psi|\hat{U}^{\dagger}(\vec{\phi}_k)\\&\quad\otimes\hat{U}^{\dagger}(\vec{\theta}_l)),P^{k}_{QM}(a_k)\\&=\mathrm{Tr}(\Pi_{a_k}\otimes I|\psi\rangle\langle\psi|),Q^{l}_{QM}(b_l)=\mathrm{Tr}(I\otimes\Pi_{b_l}|\psi\rangle\langle\psi|)\end{aligned}$$

,

where $\vec{\phi}_k$ denotes the set of phase shifts at Alice's side when she measures the observable A_k, $\vec{\theta}_l$ denotes the set of phase shifts at Bob's side when he measures the observable B_l and Π_{a_k},Π_{b_l} are projectors on the states $|a_k\rangle,|b_l\rangle$, respectively. One has

$$\begin{aligned}P^{kl}_{QM}(a_k;b_l)&=\frac{1}{3}\left|\sum_{m=1}^{3}\exp[i(\phi^m_A+\phi^m_B)]U_{mk}U_{ml}\right|^2\\&=\frac{1}{27}\left[3+2\sum_{m>n}^{3}\cos(\mathbf{\Phi}^m_{kl}-\mathbf{\Phi}^n_{kl})\right],\end{aligned}\qquad(4)$$

where $\mathbf{\Phi}^m_{kl}\equiv\phi^m_A+\phi^m_B+[m(a_k+b_l-2)](2\pi/3)$. These probabilities have the property that if the sum a_k+b_l is the same modulo 3 then they are equal, i.e., $P^{kl}_{QM}(1;2)=P^{kl}_{QM}(2;1)=P^{kl}_{QM}(3;3)$, etc. Thus, they can be divided into three equivalence classes and it is enough to provide only a representative member of each class, for instance in the example above, $P^{kl}_{QM}(1;2)$.

Following Ref. [4], we define the amount of violation of local realism as the minimal noise admixture F_{thr} to the state (3) below which the measured correlations cannot be described by local realism for the given observables. Therefore, we assume that Alice and Bob perform their measurements on the following mixed state ρ_F:

$$\rho_F=(1-F)|\psi\rangle\langle\psi|+F\rho_{noise},\qquad(5)$$

where $0\leqslant F\leqslant 1$ and where ρ_{noise} is a diagonal matrix with entries equal to 1/9. This matrix is a totally chaotic mixture (noise). For $F=0$ (pure maximally entangled state), a local realistic description does not exist whereas for $F=1$ (pure noise) it does. Therefore, there exists some threshold value of F, which we denote by F_{thr}, such that for every $F\leqslant F_{thr}$ a local realistic description is not valid. The bigger the value of F_{thr}, the stronger the violation of local realism is according to the measure defined here.

The quantum probabilities calculated (denoted with tilde) on Eq. (5) acquire the form

$$\begin{aligned}\tilde{P}^{kl}_{QM}(a_k;b_l)&=(1-F)P^{kl}_{QM}(a_k,b_l)+\frac{F}{9},\\\tilde{P}^{k}_{QM}(a_k)&=P^{k}_{QM}(a_k),\\\tilde{Q}^{l}_{QM}(b_l)&=Q^{l}_{QM}(b_l).\end{aligned}\qquad(6)$$

Let us now assume that Alice measures two observables defined by the following sets of phase shifts $\vec{\phi}_1=[0,(2\pi/3),-(4\pi/3)]$ and $\vec{\phi}_2=(0,\pi,\pi)$, whereas Bob measures two observables defined by the sets of phase shifts $\vec{\theta}_1=[0,(5\pi/6),(7\pi/6)]$ and $\vec{\theta}_2=[0,(\pi/2),(3\pi/2)]$. It can be verified numerically (analytical verification is too difficult because one has to find the maximum of a twelve-variable function defined on some bounded twelve-dimensional region) that for these phase shifts we get the strongest violation of the inequality (2). This is independent confirmation of the results presented in Ref. [5]. Straightforward calculations give the following values of the probabilities for each experiment (note that we give only the probabilities for $F=0$):

$$\begin{aligned}&P^{11}_{QM}(1;1)=P^{12}_{QM}(2;2)=P^{21}_{QM}(3;3)=P^{22}_{QM}(1;1)=\frac{1}{27},\\&P^{11}_{QM}(3;3)=P^{12}_{QM}(3;3)=P^{21}_{QM}(1;1)=P^{22}_{QM}(3;3)\\&\qquad=\frac{4+2\sqrt{3}}{27},\\&P^{11}_{QM}(2;2)=P^{12}_{QM}(1;1)=P^{21}_{QM}(2;2)=P^{22}_{QM}(2;2)\\&\qquad=\frac{4-2\sqrt{3}}{27}.\end{aligned}\qquad(7)$$

All the single probabilities are equal to $\frac{1}{3}$. Putting Eq. (7) into the inequality (2), we find that it is not violated if $F\leqslant(11-6\sqrt{3}/2)=F_{thr}$ [7]. This result is consistent with numerical result presented in Refs. [4,5] as well as the analytical proof presented in Refs. [6,8].

We can also use the CH inequality (2) obtained here to calculate the threshold value of quantum efficiency of detectors above which there does not exist local and realistic description of the experiment for which $F=0$ (no noise). To this end, we replace the probabilities (7) by probabilities denoted by bar $\overline{P}^{kl}_{QM}(a_k;b_l)=\eta^2P^{kl}_{QM}(a_k;b_l)$; $\overline{P}^{k}_{QM}(a_k)=\eta P^{k}_{QM}(a_k)$, $\overline{Q}^{k}_{QM}(b_l)=\eta Q^{k}_{QM}(b_l)$, where η $(0\leqslant\eta\leqslant 1)$ denotes the quantum efficiency of detectors (for simplicity, we assume that the efficiency is the same for all detectors). Putting these probabilities into Eq. (2), we get $\eta_{cr}=6(15-4\sqrt{3})/59$. Furthermore, in Appendix B we show that there exists a local realistic model reproducing quantum-mechanical probabilities for the experiment, provided that the quantum efficiency of detectors is not greater than that calculated here, η_{cr}. In this way, we prove that $\eta_{cr}=6(15$

$-4\sqrt{3})/59$ is the necessary and sufficient condition for the existence of a local hidden variables [10]. This may suggest that the CH inequality for qutrits presented here is also a sufficient condition for the existence of local realistic description of quantum correlations.

IV. CONCLUSIONS

We have derived 36 (of which 32 are independent) inequalities as a necessary condition for the probabilities of correlations observed in two three-dimensional physical systems to be describable in terms of local realism. We have shown that these inequalities are violated by maximally entangled state of two qutrits and the strength of violation defined as the minimal noise admixture F_{thr} hiding nonclassical nature of quantum correlations agrees with the numerical results presented in previous papers [4,5]. Using the derived inequalities, we calculated the critical quantum efficiency of detectors η_{cr} below which there exists a local realistic description of the investigated quantum system, which again perfectly agrees with the numerical computations [5].

As the numerical results give necessary and sufficient conditions for violating local realism, the fact that inequalities presented here give correct values of F_{thr} and η_{cr} at the same time is consistent with the hypothesis that they are also sufficient conditions for local realistic description.

We should mention here that after the inequality presented in this paper had been made available [12], in Ref. [8], a series of Bell inequalities reproducing the threshold values of noise admixture for arbitrary dimension N confirming to a very high accuracy the numerical results from Refs. [4,5] was presented. However, it can be checked easily that these inequalities give overestimated values of threshold quantum efficiencies of detectors needed to violate local realism. Thus, these inequalities are only a necessary condition for local realism. Indeed, it can be shown [11] that the inequality for qutrits given in Ref. [8] can be derived from the inequality (2) in this paper but the converse is not true.

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APPENDIX A

In this appendix, we sketch the proof of inequality (2) for $\alpha=\beta=0$ and $x=y=0$. For other values of α, β, x, and y, the proof is exactly similar. Let us consider the left-hand side of the inequality. It can be written as a sum of three parts, which we denote by $\mathcal{I}(\mathrm{CH}_1)$, $\mathcal{I}(\mathrm{CH}_2)$ and $\mathcal{I}(\mathrm{G})$ (we use the fact that probabilities appearing in the inequality can be written as marginals of the joint probability distribution)

$$\begin{aligned}\mathcal{I}(\mathrm{CH}_1)=&\sum_{l=1}^{3}\sum_{n=1}^{3}P(2,l;1,n)+\sum_{l=1}^{3}\sum_{m=1}^{3}P(2,l;m,1)\\&-\sum_{k=1}^{3}\sum_{n=1}^{3}P(k,2;1,n)+\sum_{k=1}^{3}\sum_{m=1}^{3}P(k,2;m,1)\\&-\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}P(2,l;m,n)\\&-\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}P(k,l;m,1),\end{aligned}\tag{A1}$$

$$\begin{aligned}\mathcal{I}(\mathrm{CH}_2)=&\sum_{l=1}^{3}\sum_{n=1}^{3}P(1,l;2,n)+\sum_{l=1}^{3}\sum_{m=1}^{3}P(1,l;m,2)\\&-\sum_{k=1}^{3}\sum_{n=1}^{3}P(k,1;2,n)+\sum_{k=1}^{3}\sum_{m=1}^{3}P(k,1;m,2)\\&-\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}P(1,l;m,n)\\&-\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}P(k,l;m,2),\end{aligned}$$

$$\begin{aligned}\mathcal{I}(\mathrm{G})=&\sum_{l=1}^{3}\sum_{n=1}^{3}P(2,l;2,n)+\sum_{l=1}^{3}\sum_{m=1}^{3}P(1,l;m,1)\\&-\sum_{k=1}^{3}\sum_{n=1}^{3}P(k,2;2,n)+\sum_{k=1}^{3}\sum_{m=1}^{3}P(k,2;m,2).\end{aligned}$$

Please notice that CH_1 and CH_2 are Clauser-Horne inequalities for pairs of detectors 1 for Alice 2 for Bob and 2 for Alice and 1 for Bob, respectively. By summing every terms and rearranging if necessary, we get the following expression:

$$\begin{aligned}&-[(P(1,1;1,1)+P(1,1;1,3)+P(1,1;2,1)+P(1,1;2,3)\\&+P(1,1;3,1)+P(1,1;3,3)+P(1,2;1,1)+P(1,2;1,2)\\&+2P(1,2;1,3)+P(1,2;2,3)+P(1,2;2,3)+P(1,2;3,3)\\&+P(1,3;1,1)+P(1,3;1,2)+P(1,3;1,3)+P(1,3;3,1)\\&+P(1,3;3,2)+P(1,3;3,3)+P(2,1;2,1)+P(2,1;2,2)\\&+P(2,1;2,3)+P(2,1;3,1)+P(2,1;3,2)+P(2,1;3,3)\\&+P(2,2;1,2)+P(2,2;1,3)+P(2,2;2,2)+P(2,2;2,3)\\&+P(2,2;3,2)+P(2,2;3,3)+P(2,3;1,2)+P(2,3;2,2)\\&+P(2,3;3,1)+2P(2,3;3,2)+P(2,3;3,3)+P(3,1;1,1)\end{aligned}$$

$$
\begin{aligned}
&+2P(3,1;2,1)+P(3,1;2,2)+P(3,1;2,3)+P(3,1;3,1)\\
&+P(3,2;1,1)+P(3,2;1,2)+P(3,2;1,3)+P(3,2;2,1)\\
&+P(3,2;2,2)+P(3,2;2,3)+P(3,3;1,1)+P(3,3;1,2)\\
&+P(3,3;2,1)+P(3,3;2,2)+P(3,3;3,1)+P(3,3;3,2)],
\end{aligned}
\tag{A2}
$$

which due to the positivity of the joint probability distribution $P(a_1,a_2;b_1,b_2)$ is always negative or identically zero. This completes the proof.

APPENDIX B

To prove that $\eta_{cr}=6(15-4\sqrt{3})/59$ is also a sufficient condition for the existence of local hidden variables, we show that there exists a local hidden variable model reproducing quantum probabilities for η_{cr}. Obviously, such a model must account for the fact that there are probabilities of nondetection events. The full quantum probabilities for each pair of the experiments k,l read $\overline{P}^{kl}_{QM}(a_k;b_l)=\eta^2 P^{kl}_{QM}(a_k;b_l)$ for $a_k,b_l\neq 0$, $\overline{P}^{kl}_{QM}(a_k;0)=\overline{Q}^{kl}_{QM}(0;b_l)=\frac{1}{3}\eta(1-\eta)$ for $a_k,b_l\neq 0$ and $\overline{P}^{kl}_{QM}(0;0)=(1-\eta)^2$, where 0 denotes the lack of detection. In this case, the existence of a local realistic description of the experiment is equivalent to the existence of a joint probability distribution (also denoted by bar to distinguish it from the joint probability distribution for the perfect case) $\overline{P(}a_1,a_2;b_1,b_2)$ with $a_1,a_2,b_1,b_2=0,1,2,3$ that returns quantum probabilities as marginals, i.e.,

$$
\overline{P}^{kl}_{QM}(a_k;b_l)=\sum_{a_{k+1}=0}^{3}\sum_{b_{l+1}=0}^{3}\overline{P(}a_1,a_2;b_1,b_2), \tag{B1}
$$

where $k+1$ and $l+1$ are modulo 2. The model for $\eta_{cr}=6(15-4\sqrt{3})/59$ is given below (probabilities equal to zero are not shown):

$$
\overline{P}(0,0;0,0)=(1-\eta_{cr})^2,
$$

$$
\begin{aligned}
\overline{P}(3,3;3,3)&=\overline{P}(3,3;2,3)=\overline{P}(3,2;3,3)=\overline{P}(3,2;3,1)\\
&=\overline{P}(1,3;2,3)=\overline{P}(1,3;2,2)=\overline{P}(1,1;1,2)\\
&=\overline{P}(1,1;2,2)=\overline{P}(2,1;1,1)=\overline{P}(2,1;1,2)\\
&=\overline{P}(2,2;3,1)=\overline{P}(2,2,;1,1)=\frac{\eta_{cr}^2}{27},
\end{aligned}
$$

$$
\begin{aligned}
\overline{P}(3,3;1,3)&=\overline{P}(3,1;3,3)=\overline{P}(3,1;3,2)=\overline{P}(3,1;1,3)\\
&=\overline{P}(3,1;1,2)=\overline{P}(3,2;3,2)=\overline{P}(1,3;2,1)\\
&=\overline{P}(1,1;3,2)=\overline{P}(1,2;3,1)=\overline{P}(1,2;3,2)\\
&=\overline{P}(1,2;2,1)=\overline{P}(1,2;2,2)=\overline{P}(2,3;1,3)\\
&=\overline{P}(2,3;1,1)=\overline{P}(2,3;2,3)=\overline{P}(2,3;2,1)\\
&=\overline{P}(2,1;1,3)=\overline{P}(2,2;2,1)=\frac{\eta_{cr}^2(4-2\sqrt{3})}{81},
\end{aligned}
$$

$$
\begin{aligned}
\overline{P}(0,3;2,3)&=\overline{P}(0,1;1,2)=\overline{P}(0,2;3,1)=\overline{P}(3,0;3,3)\\
&=\overline{P}(3,3;0,3)=\overline{P}(3,2;3,0)=\overline{P}(1,0;2,2)\\
&=\overline{P}(1,3;2,0)=\overline{P}(1,1;0,2)=\overline{P}(2,0;1,1)\\
&=\overline{P}(2,1;1,0)=\overline{P}(2,2;0,1)=\frac{1}{3}(1-\eta_{cr})\,\eta_{cr}.
\end{aligned}
\tag{B2}
$$

It can be checked directly that the quantum probabilities $\overline{P}^{kl}_{QM}(a_k;b_l)$ are recovered using Eq. (B2).

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