Nature of the stochastic processes defined by Bohm's momentum and quantum force

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The nature of the stochastic process defined by Bohm's momentum is elucidated for stationary energy eigenstates and nonstationary states in classically nonchaotic and chaotic Hamiltonian systems. In addition, the nature of the stochastic process defined by Bohm's quantum force is elucidated for stationary energy eigenstates in nonchaotic and chaotic systems, and for nonstationary states in nonchaotic systems. From these results, the following can be concluded. For stationary energy eigenstates, the process defined by the momentum is generically a stationary, Dirac-delta process in both nonchaotic and chaotic systems; in contrast, the process defined by the quantum force is nongeneric. For nonstationary states, the processes defined by the momentum and quantum force are both nongeneric in nonchaotic systems. Furthermore, for nonstationary states, the process defined by the momentum is, with a high level of confidence, a stationary, stable, brown (f^{-2} power spectrum) process in the chaotic kicked pendulum. It is conjectured that this is also true for other chaotic systems. The preceding conclusion and conjecture complement those in [Phys. Rev. A **63**, 042105 (2001)] for the process defined by the quantum force for nonstationary states in chaotic systems.

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I. INTRODUCTION

According to Bohm's [1,2] causal or ontological interpretation of quantum theory, which reproduces precisely all the predictions of the Copenhagen interpretation, matter has a well-defined trajectory independent of observers. The motion of a particle is [1,2] governed by a first-order ordinary differential equation for the position $\mathbf{x}(t)$,

$$\frac{d\mathbf{x}(t)}{dt}=\frac{1}{m}\nabla S(\mathbf{x},t)\Big|_{\mathbf{x}=\mathbf{x}(t)}, \tag{1}$$

where m is the particle mass and $S(\mathbf{x},t)$ is the phase of the quantum wave function, or equivalently, by a coupled first-order ordinary differential equations for the position $\mathbf{x}(t)$ and momentum $\mathbf{p}(t)$ of the particle that has the form of Newton's second law of motion,

$$\frac{d\mathbf{x}(t)}{dt}=\frac{1}{m}\mathbf{p}(t), \tag{2a}$$

$$\frac{d\mathbf{p}(t)}{dt}=[-\nabla V(\mathbf{x},t)-\nabla Q(\mathbf{x},t)]|_{\mathbf{x}=\mathbf{x}(t)}, \tag{2b}$$

provided that the initial momentum is subject to the constraint $\mathbf{p}(0)=\nabla S(\mathbf{x},0)|_{\mathbf{x}=\mathbf{x}(0)}$. Bohm [1] viewed the second term on the right-hand side of Eq. (2b), i.e., $-\nabla Q(\mathbf{x},t)$, as a physical force, which he called "quantum" force, that acts on the particle at each position in addition to the external classical force $-\nabla V(\mathbf{x},t)$. The quantum force is derived from the "quantum" potential [1,2]

$$Q(\mathbf{x},t)=-\frac{\hbar^2}{2m}\frac{\nabla^2 R(\mathbf{x},t)}{R(\mathbf{x},t)}, \tag{3}$$

which is determined by the amplitude $R(\mathbf{x},t)$ of the quantum wave function.

Furthermore, according to Bohm [1], because we are ignorant of the precise actual initial position of the particle, we are forced to use a statistical ensemble of particles, each of which evolves deterministically according to either Eq. (1) or Eqs. (2a) and (2b). The time evolution of the ensemble requires the specification of an initial wave function for all members of the ensemble and, for each member of the ensemble (labeled by index j), an initial *random* position $\mathbf{x}^{(j)}(0)$ where $\mathbf{x}^{(j)}(0)$ is distributed according to a chosen probability density. If the position probability density of the ensemble equals $|\psi(\mathbf{x},0)|^2$ initially (this is assumed throughout this paper), it will equal $|\psi(\mathbf{x},t)|^2$ for all times, where ψ is the time-dependent wave function [1]. For each member of the ensemble (labeled by index j), its Bohmian momentum

$$\mathbf{p}^{(j)}(t)=\nabla S(\mathbf{x},t)|_{\mathbf{x}=\mathbf{x}^{(j)}(t)} \tag{4}$$

and also the quantum force it experiences at the Bohmian position

$$\mathbf{F}^{(j)}(t)=-\nabla Q(\mathbf{x},t)|_{\mathbf{x}=\mathbf{x}^{(j)}(t)} \tag{5}$$

will generally depend on its initial random position $\mathbf{x}^{(j)}(0)$. Hence, the set of time histories from the ensemble for each momentum component (component is labeled by subscript k), $\{p_k^{(j)}(t)\}$, defines a stochastic process, and the set of time histories from the ensemble for each quantum-force component (component is labeled by subscript k), $\{F_k^{(j)}(t)\}$, also defines a stochastic process. This has not been appreciated until recently [3].

In general, a stochastic process $\{y^{(j)}(t)\}$ is either stationary or nonstationary. Let

$$W_n(y_1,t_1;y_2,t_2;\ldots;y_n,t_n) \tag{6}$$

denote the nth-order joint probability density that the random variable $y(t_1)$ has value y_1, and the random variable $y(t_2)$ has value $y_2,\ldots,$ and the random variable $y(t_n)$ has value y_n. The nth-order correlation is defined as [4]

$$\langle y(t_1)y(t_2)\cdots y(t_n)\rangle = \int y_1 y_2 \cdots y_n \times W_n(y_1,t_1;y_2,t_2;\ldots;y_n,t_n) \times dy_1 dy_2 \cdots dy_n . \tag{7}$$

A stochastic process is stationary if [4]

$$W_n(y_1,t_1+\tau;y_2,t_2+\tau;\ldots;y_n,t_n+\tau) = W_n(y_1,t_1;y_2,t_2;\ldots;y_n,t_n) \tag{8}$$

for all n and τ. In particular [4], the first-order probability density is independent of time, i.e., $W_1(y_1,t_1)=W_1(y_1)$, and the second-order probability density can only depend on the time difference, i.e., $W_2(y_1,t_1;y_2,t_2)=W_2(y_1,y_2,t_2-t_1)$.

In a previous paper [3], the nature of the stochastic process defined by Bohm's quantum force was elucidated for nonstationary quantum states in classically chaotic Hamiltonian systems. In particular, the numerical results [3] suggest that the stochastic process is a stationary, non-Gaussian stable [5], white (flat power spectrum) process. In this paper, the nature of the stochastic process defined by Bohm's momentum is studied for stationary energy eigenstates and nonstationary states in both nonchaotic and chaotic Hamiltonian systems. Results are presented in Sec. II. In addition, extending previous work [3], the nature of the stochastic process defined by Bohm's quantum force is studied for stationary energy eigenstates in nonchaotic and chaotic systems, and for nonstationary states in nonchaotic systems. Results are presented in Sec. III. In Sec. IV, conclusions are drawn from the results of Secs. II and III, and a conjecture on the nature of the stochastic process defined by Bohm's momentum for nonstationary states in chaotic systems is given.

II. MOMENTUM

A. Stationary energy eigenstates

Consider the set of *real*, bound-state energy eigenfunctions $\phi_n(\mathbf{x})$ with corresponding energies E_n of a Hamiltonian system with a time-independent potential $V(\mathbf{x})$. Classically, such systems must have at least two degrees of freedom, i.e., four-dimensional phase space, for the possibility of chaos. If the initial wave function is an energy eigenfunction $\phi_n(\mathbf{x})$, then the wave function at a later time t is

$$\psi(\mathbf{x},t)=\phi_n(\mathbf{x})\exp(-iE_nt/\hbar). \tag{9}$$

For all stationary energy eigenstates above, the Bohmian momentum of each member of the ensemble given by Eq. (4) is

$$\mathbf{p}^{(j)}(t)=\mathbf{0} \tag{10}$$

because the phase of the wave function is independent of $\mathbf{x}$. For each momentum component k, the stochastic process $\{p_k^{(j)}(t)\}$ is thus a stationary process with the following nth-order joint probability density,

$$W_n(p_1,t_1;p_2,t_2;\ldots;p_n,t_n)=\delta(p_1)\,\delta(p_2)\cdots\delta(p_n), \tag{11}$$

regardless of whether the system is classically chaotic or not.

B. Nonstationary states

1. Classically nonchaotic systems

Two nonchaotic Hamiltonian systems are studied here: harmonic oscillator and free particle.

For the one-dimensional harmonic oscillator, if the initial wave function is a minimum-uncertainty Gaussian wave packet with position standard deviation $\sqrt{\hbar/2m\omega}$, then the wave packet remains Gaussian without spreading in position and momentum,

$$\psi(x,t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}(x-a\cos\omega t)^2\right] \times\exp\left\{-\frac{i}{2}\left[\omega t+\frac{m\omega}{\hbar} \times(2ax\sin\omega t-\tfrac{1}{2}a^2\sin 2\omega t)\right]\right\}. \tag{12}$$

For this nonstationary state that is initially centered at a in position, the Bohmian momentum of each member of the ensemble given by Eq. (4) is [6]

$$p^{(j)}(t)=-m\omega a\sin\omega t\big|_{x=x^{(j)}(t)}=-m\omega a\sin\omega t. \tag{13}$$

The stochastic process $\{p^{(j)}(t)\}$ is thus a nonstationary process with the following nth-order joint probability density,

$$W_n(p_1,t_1;p_2,t_2;\ldots;p_n,t_n)=\delta(p_1-(-m\omega a\sin\omega t_1))\,\delta(p_2-(-m\omega a\sin\omega t_2))\cdots\delta(p_n-(-m\omega a\sin\omega t_n)). \tag{14}$$

For the one-dimensional free particle, if the initial wave function is a minimum-uncertainty Gaussian wave packet, then the wave packet remains Gaussian that spreads in position but not in momentum,

$$\psi(x,t)=\left(\frac{1}{2\pi\sigma_t^2}\right)^{1/4}\exp\left(\frac{-x^2}{4\sigma_t^2}\right)\exp\left\{i\left[\frac{f(t)x^2}{4\sigma_t^2}-\tfrac{1}{2}\tan^{-1}[f(t)]\right]\right\}, \tag{15}$$

where $f(t)=\hbar t/2m\sigma_0^2$, and $\sigma_t^2=\sigma_0^2[1+f^2(t)]$ is the position variance at time t. For this nonstationary state, which remains centered at 0 both in position and momentum, the Bohmian position of each member of the ensemble is given by

$$x^{(j)}(t)=x^{(j)}(0)[1+f^2(t)]^{1/2}. \tag{16}$$

So the Bohmian momentum of each member of the ensemble given by Eq. (4) is

$$p^{(j)}(t)=\frac{\hbar f(t)}{2\sigma_t^2}x\bigg|_{x=x^{(j)}(t)}=a(t)x^{(j)}(0), \qquad (17)$$

where

$$a(t)=\frac{\hbar f(t)[1+f^2(t)]^{1/2}}{2\sigma_t^2}. \qquad (18)$$

The stochastic process $\{p^{(j)}(t)\}$ is thus nonstationary. The first-order probability density $W_1(p_1,t_1)$ is just the probability density of $a(t_1)x^{(j)}(0)$. Given that the probability density of the random position $x^{(j)}(0)$ is $|\psi(x^{(j)}(0),0)|^2$, i.e., a Gaussian centered at 0 with variance σ_0^2 [see Eq. (15)], it is easy to show using the transformation of variable technique that $W_1(p_1,t_1)$ is also a Gaussian centered at 0 but with variance $a^2(t_1)\sigma_0^2$,

$$W_1(p_1,t_1)=\left[\frac{1}{2\pi a^2(t_1)\sigma_0^2}\right]^{1/2}\exp\left[\frac{-p_1^2}{2a^2(t_1)\sigma_0^2}\right]. \qquad (19)$$

Higher-order joint probability densities $(n\geqslant 2)$ are given by

$$\begin{aligned}W_n(p_1,t_1;p_2,t_2;\ldots;p_n,t_n)\\ =W_1(p_1,t_1)\,\delta\left(p_2-\frac{a(t_2)}{a(t_1)}p_1\right)\\ \times\delta\left(p_3-\frac{a(t_3)}{a(t_1)}p_1\right)\cdots\delta\left(p_n-\frac{a(t_n)}{a(t_1)}p_1\right).\end{aligned} \qquad (20)$$

For the one-dimensional free particle, if the initial wave function is

$$\psi(x,0)=\mathrm{Ai}(Bx/\hbar^{2/3}), \qquad (21)$$

where B is a constant and Ai is the Airy Function, then [7]

$$\begin{aligned}\psi(x,t)=\mathrm{Ai}((B/\hbar^{2/3})(x-B^3t^2/4m^2))\exp[i(B^3t/2m\hbar)\\ \times(x-B^3t^2/6m^2)].\end{aligned} \qquad (22)$$

For this stationary state that propagates without spreading in position, the Bohmian momentum of each member of the ensemble given by Eq. (4) is

$$p^{(j)}(t)=\frac{B^3t}{2m}\bigg|_{x=x^{(j)}(t)}=\frac{B^3t}{2m}. \qquad (23)$$

The stochastic process $\{p^{(j)}(t)\}$ is, therefore, a nonstationary process with the following nth-order joint probability density

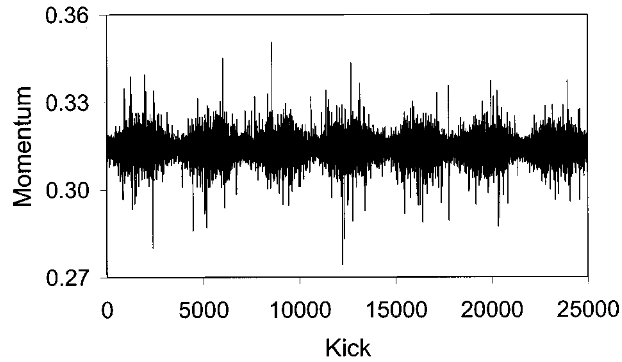

FIG. 1. A realization of the stochastic process defined by Bohm's angular momentum (in arbitrary unit) in a prototypical classically chaotic Hamiltonian system: the periodically delta-kicked pendulum.

$$\begin{aligned}W_n(p_1,t_1;p_2,t_2;\ldots;p_n,t_n)=\delta\left(p_1-\frac{B^3t_1}{2m}\right)\delta\left(p_2\right.\\ \left.-\frac{B^3t_2}{2m}\right)\cdots\delta\left(p_n-\frac{B^3t_n}{2m}\right).\end{aligned} \qquad (24)$$

2. Classically chaotic system

Here a well-known prototypical classically chaotic Hamiltonian system: the periodically delta-kicked plane pendulum [8] is studied.

Figure 1 shows a realization of the stochastic process $\{p^{(j)}(t)\}$ (p here is angular momentum) where the momentum just before each instantaneous gravitational kick is plotted. For this sample function, the system parameters are $a\equiv mLgT=0.005$ and $b\equiv T/mL^2=50$, where the kicking period $T=1$ (m and L are, respectively, the mass and length of the pendulum; g is the acceleration due to gravity). The initial wave function is a superposition of free-rotor energy eigenstates

$$\psi(\theta,0)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}A_k(0)e^{ik\theta}, \qquad (25)$$

where the expansion coefficient $A_k(0)$ is Gaussian centered at k_0,

$$A_k(0)=\left(\frac{2\sigma_0^2}{\pi}\right)^{1/4}\exp[-\sigma_0^2(k-k_0)^2]\exp(-ik\theta_0) \qquad (26)$$

with $\sigma_0=0.01$, $\theta_0=\pi$, and $k_0=3142$. The initial Bohmian angle is π. And we chose $\hbar=0.0001$ for ease of wave-function propagation. Details of the numerical integration of the time-dependent Schrödinger's equation and Bohm's equation of motion (1), in order to yield the momentum time series, are given in [3]. All quantities stated above with dimensions are in arbitrary units.

Figure 2 shows the estimated univariate probability density of the univariate data set consisting of all the 25 001 momentum values in Fig. 1, together with a univariate stable

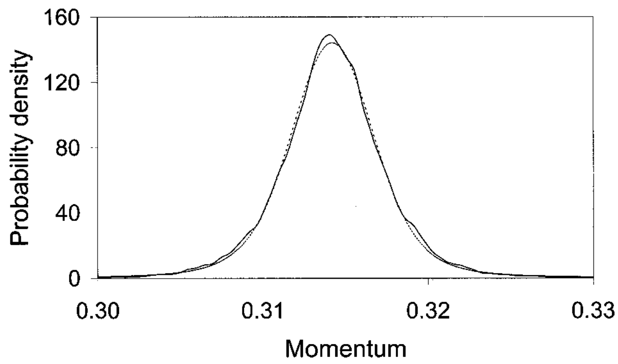

FIG. 2. Univariate probability density of the univariate data set obtained from the whole sample function in Fig. 1: smoothed data density (solid line) and fitted stable density (dotted line). The fitted stable density is essentially symmetric ($\beta = -0.0122$) with a non-Gaussian characteristic exponent $\alpha = 1.628$.

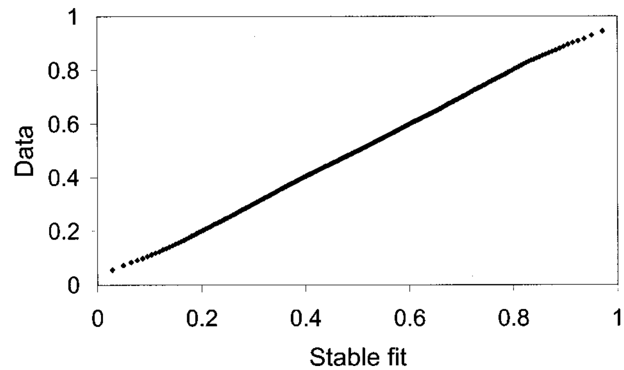

FIG. 3. Variance-stabilized PP (percent-percent) plot for the univariate data set obtained from the whole sample function in Fig. 1.

probability density fit to the data using Nolan's STABLE program [9]. Univariate stable distributions are characterized by four parameters: characteristic exponent $\alpha \in (0,2]$, skewness $\beta \in [-1,1]$, scale $\gamma \in (0,\infty)$, and shift or location $\delta \in (-\infty,\infty)$ [5]. The maximum likelihood fit [10], based on reliable computations of stable densities, produced the following stable parameters in the S^0 parametrization: $\alpha = 1.628$, $\beta = -0.0122$, $\gamma = 1.976 \times 10^{-3}$, and $\delta = 0.3142$. The closeness of the estimated data density and the fitted stable density in Fig. 2 shows that the stable fit is good. Furthermore, the variance-stabilized PP (percent-percent) plot [10,11] in Fig. 3 also indicates a good stable fit since the plotted points are essentially on the diagonal (the closer the points are to the diagonal, the better the fit).

Next, a bivariate data set, consisting, of 25 000 adjacent pairs of values obtained from the whole momentum time series in Fig. 1, was analyzed. First, the scalar product of each bivariate data with a fixed two-dimensional unit vector is formed to yield a set of univariate data. Then, for various angle φ of the unit vector (angle is measured from the positive horizontal axis; counter-clockwise angle is positive): 90°, 67.5°, 45°, 22.5°, 0°, −22.5°, −45°, −67.5°, the corresponding univariate data set is fitted with a univariate stable density using the maximum likelihood method in Nolan's STABLE program. For each of the eight angles, the density plot and variance-stabilized PP plot diagnostics both indicate a good univariate stable fit. The stable parameter functions $\alpha(\varphi)$, $\beta(\varphi)$, $\gamma(\varphi)$, and $\delta(\varphi)$ (in the S^0 parametrization) from the fits are given in Table I. The good univariate stable density fits, and the near constant value of the characteristic exponent $\alpha(\varphi)$ as a function of φ implies (see [10] for proof of the theorem) that the bivariate probability density of the original bivariate data is a stable density. The constant value of the characteristic exponent, and the parameter functions $\beta(\varphi)$, $\gamma(\varphi)$, and $\delta(\varphi)$ completely characterize [10] a bivariate stable density.

Each univariate data set obtained from a different part of the sample function in Fig. 1 is also stable distributed with essentially the same stable parameters as the ones for the data set from the whole sample function. Furthermore, each bivariate data set obtained from a different part of the sample function in Fig. 1 is also stable distributed with essentially the same characteristic exponent and parameter functions $\beta(\varphi)$, $\gamma(\varphi)$, and $\delta(\varphi)$ as the ones for the data set from the whole sample function. This implies [12] that the stochastic process $\{p^{(j)}(t)\}$, of which the sample function in Fig. 1 is a realization, is a stationary process in the wide sense [rather than in the strict sense defined in Eq. (8)]. Furthermore, the fitted univariate and bivariate stable probability densities are essentially the same for different sample functions corre-

TABLE I. Fitted stable parameter functions for a bivariate data set of adjacent momentum values obtained from the whole sample function in Fig. 1.

Angle φ (degrees)	Characteristic exponent $\alpha(\varphi)$	Skewness $\beta(\varphi)$	Scale $\gamma(\varphi)$ (units of 10^{-3})	Shift/location $\delta(\varphi)$
90	1.627	−0.0123	1.976	0.3142
67.5	1.607	−0.0011	1.788	0.4105
45	1.586	0.0182	1.720	0.4443
22.5	1.585	−0.0037	1.768	0.4105
0	1.627	−0.0123	1.976	0.3142
−22.5	1.689	−0.0216	2.246	0.1700
−45	1.702	0.0148	2.356	-1.986×10^{-5}
−67.5	1.675	0.0139	2.238	−0.1700

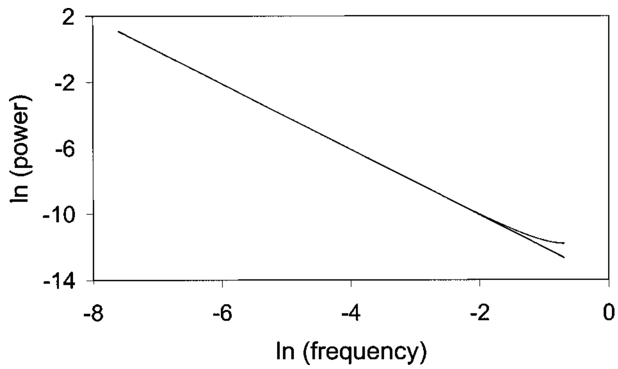

FIG. 4. The natural-log of the power of the whole sample function in Fig. 1 vs the natural-log of the frequency (in arbitrary units) up to the Nyquist frequency of 0.5. The Nyquist frequency is defined [14] as $1/(2\Delta)$ where Δ is the sampling interval, in this case 1 (in arbitrary units). The straight line is the fit.

sponding to different initial Bohman angles, and, therefore [12,13], the process $\{p^{(j)}(t)\}$ is ergodic in the wide sense. Wide ergodicity means [12] that the first-order and second-order probability densities, $W_1(p_1)$ and $W_2(p_1,p_2,t_2-t_1)$, for the process can be obtained, respectively, as the probability densities of univariate and bivariate data sets from a sample function. The second-order correlation function can also be obtained from a sample function: directly as a time average [12], or indirectly as the Fourier transform of the one-sided power spectrum [14].

The natural log of the power of the sample function in Fig. 1 is plotted in Fig. 4 vs the natural log of the frequency. The one-sided power spectrum was calculated using the maximum entropy (all poles) method [14]. The excellent straight-line fit implies that the power spectrum behaves as a power law,

$$P(f)=Cf^{n}, \tag{27}$$

where C and n are constants. The fit yields $C=8.0\times10^{-7}$ and $n=-2.0$.

All the results above were found to be typical for a variety of system parameters and initial wave functions. For the system parameters, different value of $a=mLgT$ and $b=T/mL^2$ was used such that the dimensionless product ab ranges from 10^{-4} to 1. The product ab determines the degree of chaos in classical phase space: the transition from local or weak chaos to global or strong chaos occurs at $ab\approx0.9716$ [8]. For the initial wave function: (i) the parameters σ_0, θ_0, and k_0 of the expansion coefficients in Eq. (26) were varied, (ii) N expansion coefficients of equal amplitude $1/\sqrt{N}$ were used, and (iii) different mth free-rotor energy eigenstate was used as the initial wave function, i.e., $A_k(0)=\delta_{km}$ in Eq. (25). In all of these diverse cases, the stochastic process $\{p^{(j)}(t)\}$ is thus stationary in the wide sense with first-order and second-order probability densities that are, respectively, univariate and bivariate stable densities. The stable parameters, however, generally vary with the system parameters and initial wave function. Moreover, in all cases, the process is a brown [15] colored process because the power spectrum of a sample function obeys a power law [see Eq. (27)] with $n=-2.0$ (the constant C, however, varies with system parameters and initial wave function).

III. QUANTUM FORCE

A. Stationary energy eigenstates

For the stationary state given by Eq. (9), because the phase is independent of $\mathbf{x}$, Eq. (1) implies that the Bohmian position of each member of the ensemble is constant in time

$$\mathbf{x}^{(j)}(t)=\mathbf{x}^{(j)}(0). \tag{28}$$

The quantum potential Q is time independent, and $\nabla Q(\mathbf{x},t)=\nabla Q(\mathbf{x})=-\nabla V(\mathbf{x})$, where V is the time-independent classical potential. Hence Eq. (5) gives a time-independent quantum force experienced by each member of the ensemble at its Bohmian position,

$$\mathbf{F}^{(j)}(t)=\nabla V(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^{(j)}(0)}. \tag{29}$$

For each quantum-force component k, the stochastic process $\{F_k^{(j)}(t)\}$ is thus a stationary process because each realization is time independent. In general,

$$F_k^{(j)}(t)=f_k(\mathbf{x}^{(j)}(0)), \tag{30}$$

where f_k is either a linear or nonlinear real function of the random position $\mathbf{x}^{(j)}(0)$. The first-order probability density $W_1(F_1,t_1)=W_1(F_1)$ is just the probability density of $f_k(\mathbf{x}^{(j)}(0))$, which can be determined, in principle [16], given that the probability density of the random position $\mathbf{x}^{(j)}(0)$ is $\phi_n^2(\mathbf{x}^{(j)}(0))$ [recall that the initial wave function is the real energy eigenfunction $\phi_n(\mathbf{x})$, see Sec. II A]. Different energy eigenfunction will lead to a different $W_1(F_1)$. Higher-order joint probability densities ($n\geqslant2$) are given by

$$W_n(F_1,t_1;F_2,t_2;\ldots;F_nt_n)=W_1(F_1)\,\delta(F_2-F_1)\,\delta(F_3-F_1)\cdots\delta(F_n-F_1). \tag{31}$$

B. Nonstationary states

Two nonchaotic Hamiltonian systems are studied here: harmonic oscillator and free particle.

For the one-dimensional harmonic oscillator, for the nonstationary state given by Eq. (12), the Bohmian position of each member of the ensemble is [6]

$$x^{(j)}(t)=x^{(j)}(0)+a\cos\omega t-a. \tag{32}$$

Therefore, Eq. (5) gives a time-independent quantum force experienced by each member of the ensemble at its Bohmian position,

$$F^{(j)}(t)=m\omega^2(x-a\cos\omega t)|_{x=x^{(j)}(t)}=m\omega^2[x^{(j)}(0)-a]. \tag{33}$$

The stochastic process $\{F^{(j)}(t)\}$ is thus stationary. The first-order probability density $W_1(F_1,t_1)=W_1(F_1)$ is just the probability density of $m\omega^2[x^{(j)}(0)-a]$. Given that the probability density of the random position $x^{(j)}(0)$ is

$|\psi(x^{(j)}(0),0)|^2$, i.e., a Gaussian centered at a with variance $\hbar/2m\omega$ [see Eq. (12)], it is easy to show using the transformation of variable technique that $W_1(F_1)$ is also a Gaussian but centered at 0 with variance $\hbar m\omega^3/2$. Higher-order joint probability densities $(n\geqslant 2)$ are given by

$$W_n(F_1,t_1;F_2,t_2;\ldots;F_n,t_n)=W_1(F_1)\,\delta(F_2-F_1)\,\delta(F_3-F_1)\cdots\delta(F_n-F_1). \quad (34)$$

For the one-dimensional free particle, Eq. (5) gives, for the nonstationary state in Eq. (15), a time-dependent quantum force experienced by each member of the ensemble at its Bohmian position given by Eq. (16),

$$F^{(j)}(t)=\frac{\hbar^2}{4m\sigma_t^4}x\bigg|_{x=x^{(j)}(t)}=b(t)x^{(j)}(0), \quad (35)$$

where

$$b(t)=\frac{\hbar^2}{4m\sigma_t^3\sigma_0}. \quad (36)$$

The stochastic process $\{F^{(j)}(t)\}$ is thus nonstationary. The first-order probability density $W_1(F_1,t_1)$ is just the probability density of $b(t_1)x^{(j)}(0)$. Given that the probability density of the random position $x^{(j)}(0)$ is $|\psi(x^{(j)}(0),0)|^2$, i.e., a Gaussian centered at 0 with variance σ_0^2 [see Eq. (15)], $W_1(F_1,t_1)$ is also a Gaussian centered at 0 but with variance $b^2(t_1)\sigma_0^2$. Higher-order joint probability densities $(n\geqslant 2)$ are given by

$$\begin{aligned} W_n(F_1,t_1;F_2,t_2;\ldots;F_n,t_n) &= W_1(F_1,t_1)\,\delta\left(F_2-\frac{b(t_2)}{b(t_1)}F_1\right)\\ &\quad\times\delta\left(F_3-\frac{b(t_3)}{b(t_1)}F_1\right)\cdots\delta\left(F_n-\frac{b(t_n)}{b(t_1)}F_1\right). \end{aligned} \quad (37)$$

For the one-dimensional free particle, for the nonstationary state given by Eq. (22), Eq. (5) gives a time-independent quantum force experienced by each member of the ensemble at its Bohmian position [17],

$$F^{(j)}(t)=\frac{B^3}{2m}\bigg|_{x=x^{(j)}(t)}=\frac{B^3}{2m}. \quad (38)$$

The stochastic process $\{F^{(j)}(t)\}$ is thus stationary with the following nth-order joint probability density

$$W_n(F_1,t_1;F_2,t_2;\ldots;F_n,t_n)=\delta\left(F_1-\frac{B^3}{2m}\right)\delta\left(F_2-\frac{B^3}{2m}\right)\cdots\delta\left(F_n-\frac{B^3}{2m}\right). \quad (39)$$

IV. CONCLUSIONS AND A CONJECTURE

Section II A shows that the stochastic process defined by Bohm's momentum is the same for all stationary energy eigenstates, independent of whether the system is classically nonchaotic or chaotic. In particular, the process is a stationary, Dirac-delta process [see Eq. (11)].

Section II B 1 shows that, for nonstationary states, the stochastic process defined by Bohm's momentum is nongeneric in classically nonchaotic systems [compare Eqs. (14), (20), and (24)].

The numerical results in Sec. II B 2 show that, for nonstationary states, the stochastic process defined by Bohm's momentum is stationary in the wide sense with first-order and second-order stable probability densities in the classically chaotic kicked pendulum. It is, however, not possible in practice to numerically check that the process is stationary and stable for all orders, i.e., in the strict sense. But verification that the process is stationary and stable in the wide sense strongly suggests that it is so in the strict sense. I conjecture that, for nonstationary states, the stochastic process defined by Bohm's momentum is also a stationary, stable, brown (f^{-2} power spectrum) process in other classically chaotic Hamiltonian systems. This conjecture is motivated by the fact that this class of system exhibits, for nonstationary states, generic quantum signatures in: the variances of the wave function [18–21], the real and imaginary parts of the wave function [22], and, more relevantly, the amplitude and phase of the wave function [23]. Recall that the phase of the wave function essentially determines the momentum of each member of an ensemble through Eq. (4). Thus the generic behavior of the phase should lead to a generic behavior in the stochastic process defined by the momentum.

Section III A shows that, for stationary energy eigenstates, the stochastic process defined by Bohm's quantum force is nongeneric in both classically nonchaotic and chaotic systems [see Eq. (31)].

Section III B shows that, for nonstationary states, the stochastic process defined by Bohm's quantum force is nongeneric in classically nonchaotic systems [compare Eqs. (34), (37), and (39)].

Work in progress, to be reported elsewhere, involves replacing the ordinary differential equation (1) and Eqs. (2a) and (2b) by their appropriate stochastic versions and deriving from them the time-evolution equation for, respectively, the position probability density and phase-space probability density. The aim is to determine whether these time-evolution equations agree with the corresponding ones from Bohm's theory.

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