Bell's theorem with and without inequalities for the three-qubit Greenberger-Horne-Zeilinger and *W* states

Adán Cabello*
Departamento de Física Aplicada II, Universidad de Sevilla, 41012 Sevilla, Spain
(Received 31 July 2001; published 5 February 2002)

A proof of Bell's theorem without inequalities valid for both inequivalent classes of three-qubit entangled states under local operations assisted by classical communication, namely Greenberger-Horne-Zeilinger (GHZ) and *W*, is described. This proof leads to a Bell inequality that allows more conclusive tests of Bell's theorem for three-qubit systems. Another Bell inequality involving both tri- and bipartite correlations is introduced which illustrates the different violations of local realism exhibited by the GHZ and *W* states.

DOI: 10.1103/PhysRevA.65.032108 PACS number(s): 03.65.Ud, 03.65.Ta

I. INTRODUCTION

In recent years, Greenberger-Horne-Zeilinger (GHZ) states of three or more qubits [1] have become ubiquitous in quantum information theory [2–8]. However, the interest in GHZ states began in connection with Bell's theorem. While Bell's proof of the impossibility of Einstein, Podolsky, and Rosen's (EPR's) ''elements of reality'' [9] was based on statistical predictions and inequalities [10], GHZ showed that a simpler proof can be achieved with perfect correlations and without inequalities [1,11,12]. Subsequently, the GHZ proof was translated into experimentally verifiable Bell inequalities [13,14] and into real experiments [15,16]. It has recently been found that not only the GHZ state but any two-qubit pure entangled state admits a proof of Bell's theorem without inequalities [17–19].

On the other hand, over the last few years the importance of quantum entanglement as a resource for unusual kinds of communication and information processing has stimulated the mathematical study of the entanglement of multiqubit systems. In particular, there has been much interest in the classification of three-qubit pure entangled states in terms of equivalences under local operations assisted by classical communication (LOCC) [20–24]. Dür, Vidal, and Cirac [21] have shown that there are only two classes of genuinely tripartite entanglement which are inequivalent under LOCC. One class is represented by the GHZ state,

$$|\mathrm{GHZ}\rangle=\frac{1}{\sqrt{2}}(|y+y+y+\rangle+|y-y-y-\rangle), \qquad (1)$$

where $\sigma_y|y\pm\rangle=\pm|y\pm\rangle$, σ_y being the corresponding Pauli spin matrix. The other class is represented by the *W* state [25],

$$|W\rangle=\frac{1}{\sqrt{3}}(|+--\rangle+|-+-\rangle+|--+\rangle), \qquad (2)$$

where $\sigma_z|\pm\rangle=\pm|\pm\rangle$. The GHZ and *W* states cannot be converted into each other by means of LOCC.

At this point, some natural questions arise. The first being in which applications does the use of the *W* state mean an improvement over previous protocols using the GHZ state. This question is partially addressed in [21], where it is pointed out that in a three-qubit system prepared in a *W* (GHZ) state, if one of the qubits is traced out then the remaining two qubits are entangled (completely unentangled). Indeed, *W* is the three-qubit state whose entanglement has the highest robustness against the loss of one qubit [21]. Moreover, from a single copy of the reduced density matrix for any two qubits belonging to a three-qubit *W* state, one can always obtain a state which is arbitrarily close to a Bell state by means of a filtering measurement [26]. This means that, if one of the parties sharing the system prepared in a *W* (GHZ) state decides not to cooperate with the other two, or if for some reason the information about one of the qubits is lost, then the remaining two parties still can (cannot) use entanglement resources to perform communication tasks.

On the other hand, it has been shown that the *W* state does not allow a GHZ-type proof of Bell's theorem [27]. Therefore, two other natural questions are whether the *W* state admits any kind of proof of Bell's theorem without inequalities and what the differences are between the violation of local realism exhibited by the GHZ and *W* states. In this paper, I will describe four related results which answer these questions. First, a proof of Bell's theorem without inequalities specific for the *W* state. Second, an extension of that proof which is also valid for the GHZ state. Such a proof leads to a Bell-type inequality for three qubits which can be experimentally useful in order to achieve more conclusive tests of Bell's theorem. Finally, a different set of Bell inequalities is considered with the purpose of illustrating some differences between the violations of local realism exhibited by the GHZ and *W* states.

II. BELL'S THEOREM WITHOUT INEQUALITIES FOR THE *W* STATE

First, I will show that the *W* state allows three local observers to define elements of reality which are incompatible with some predictions of quantum mechanics [28]. I will use the following notation: z_q and x_q will be the results (-1 or 1) of measuring σ_z and σ_x on qubit q ($q=1,2,3$). The first step of the proof consists of showing that, in the *W* state, all

*Electronic address: adan@us.es

the z_q and x_q satisfy EPR's criterion of elements of reality. EPR's condition reads: *"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity"* [9]. From the expression of the W state given in Eq. (2), it can be immediately seen that z_1, z_2, and z_3 are elements of reality, since the result z_i can be predicted with certainty from the results of spacelike separated measurements of z_j and z_k ($i \neq j$, $j \neq k$, and $k \neq i$). In addition, by rewriting the W state in the suitable basis, it can easily be seen that, if $z_i = -1$ then, with certainty, $x_j = x_k$. Therefore, if $z_i = -1$, then by measuring x_j (x_k) one can predict x_k (x_j) with certainty. Therefore, if $z_i = -1$, then x_j and x_k are elements of reality. If $z_i = +1$ then, using the expression (2), it can immediately be seen that $z_j = -1$. Therefore, following the previous reasoning, x_i and x_k are elements of reality (although x_i could have ceased to be an element of reality after measuring σ_z on particle i). In conclusion, z_q and x_q are EPR elements of reality and therefore, according to EPR, they should have predefined values -1 or 1 before any measurement.

However, according to quantum mechanics, such an assignment of values is impossible. The proof can be presented in a very similar way to Hardy's proof of Bell's theorem for nonmaximally entangled states of two qubits [17] by using four properties of the quantum state and a logical argument based on them. For the refutation of EPR's elements of reality, the relevant properties of the W state (2), which can easily be checked, are

$$P_W(z_i = -1, z_j = -1) = 1, \tag{3}$$

$$P_W(x_j = x_k | z_i = -1) = 1, \tag{4}$$

$$P_W(x_i = x_k | z_j = -1) = 1, \tag{5}$$

$$P_W(x_i = x_j = x_k) = \frac{3}{4}, \tag{6}$$

where $P_W(z_i = -1, z_j = -1)$ means the probability of two qubits (although we cannot tell which one) giving the result -1 when measuring σ_z on all three qubits, and $P_W(x_j = x_k | z_i = -1)$ is the conditional probability of σ_{xj} and σ_{xk} having the same result given that the result of σ_{zi} is -1. Property (3) tells us that, when measuring σ_z on all three qubits, the result -1 *always* occurs in *two* of them. Let us call these qubits i and j. Then let us suppose that we had measured σ_x on qubits j and k, instead of σ_z. Then, according to property (4), the results would have been the same. Therefore, following EPR, one reaches the conclusion that the predefined values corresponding to the elements of reality x_j and x_k are equal. Now let us suppose that we had measured σ_x on qubits i and k, instead of σ_z. Then, according to property (5), the results would have been the same. Therefore, the predefined values corresponding to the elements of reality x_i and x_k are equal. Taking these two conclusions together, one must deduce that, in the W state, the predefined values corresponding to the elements of reality x_1, x_2, and x_3 *always* satisfy $x_1 = x_2 = x_3$. However, this is in contradiction with property (6) which states that, when measuring σ_x on all three qubits, one finds results that *cannot* be explained with elements of reality in $\frac{1}{4}$ of the cases. Therefore, the conclusion is that quantum predictions for the W state cannot be "completed" with EPR's elements of reality.

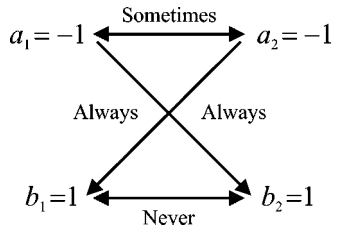

FIG. 1. Diagram for Hardy's proof of Bell's theorem without inequalities for two qubits in a nonmaximally entangled state [17,29]. a_i and b_i are alternative spin observables of qubit i.

While the structure of this proof is similar to Hardy's [17,29], the logical argument is different: in Hardy's, from a result that *sometimes* occurs, two local observers infer a result that *never* occurs (see Fig. 1); here, from a result that *always* occurs, two observers infer a result that *only sometimes* occurs (see Fig. 2). In addition, while in Hardy's proof only 9% of the runs of a certain experiment contradict local realism, here 25% of the runs of the last experiment cannot be explained by local realism. On the other hand, while in Hardy's proof we need both qubits to start the argument and in GHZ's proof [11,12] all three qubits are required, in the proof for the W state the contradiction results from inferences from only *two* of all three qubits, but we cannot tell which one.

III. BELL'S THEOREM WITHOUT INEQUALITIES FOR THE GHZ AND *W* STATES

For two-qubit pure states, the logical structure that can be obtained from Fig. 1 by changing the "never" to "fewer times" is not particularly useful, since the states which satisfy a "sometimes-always-never" structure like that in Fig. 1, namely, nonmaximally entangled states [17], are the same which satisfy the extended structure [30]. However, for three-qubit pure states, a similar extension of Fig. 2's "always-always-sometimes" structure (changing the first "always" to "sometimes" and the last "sometimes" to "fewer times") allows us to extend the proof for the W state to the GHZ state and, therefore, to obtain a common proof of Bell's theorem without inequalities for both classes of genu-

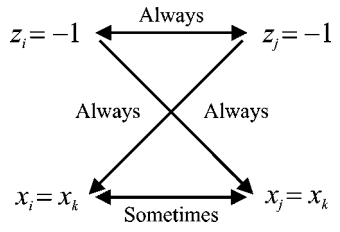

FIG. 2. Diagram for the proof of Bell's theorem without inequalities for the W state.

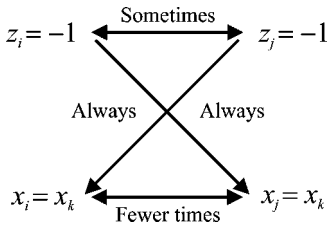

FIG. 3. Diagram for the extended proof of Bell's theorem without inequalities valid for the GHZ and *W* states. The diagram in Fig. 2 is a particular case of this one.

inely tripartite entangled states. Such a proof is illustrated in Fig. 3 and, for the GHZ state (1), it is based on its following four properties which can easily be checked:

$$P_{\mathrm{GHZ}}(z_i=-1,z_j=-1)=\frac{3}{4}, \tag{7}$$

$$P_{\mathrm{GHZ}}(x_j=x_k|z_i=-1)=1, \tag{8}$$

$$P_{\mathrm{GHZ}}(x_i=x_k|z_j=-1)=1, \tag{9}$$

$$P_{\mathrm{GHZ}}(x_i=x_j=x_k)=\frac{1}{4}. \tag{10}$$

The proof for the GHZ state is parallel to the one for the *W* state, changing only the ''always'' to ''in 75% of the cases'' and changing the percentage of events of the fourth experiment that cannot be explained with elements of reality, which now is 50%.

The previous demonstrations complete the family of simple proofs of Bell's theorem without inequalities for the main classes of two- [17–19] and three-qubit [1,11,12] pure entangled states.

IV. BELL-CH INEQUALITIES FOR THREE QUBITS

The proofs without inequalities described in Secs. II and III can easily be converted into experimentally testable Bell inequalities. As noted in [31–33], the two-qubit Bell inequalities proposed by Clauser and Horne (CH) [34] can be put into a form in which only the four probabilities of a Hardy-type argument are included. Similarly, it can easily be seen that four probabilities related to those in Eqs. (3)–(6) [or in Eqs. (7)–(10)] must satisfy the following Bell inequality:

$$-1\leqslant P(z_i=-1,z_j=-1)-P(z_i=-1,x_j\neq x_k) -P(x_i=x_k,z_j=-1)-P(x_i=x_j=x_k)\leqslant 0. \tag{11}$$

For the *W* state (2), the value of the first probability in Eq. (11) is 1, the values of the second and third probabilities are 0, and the value of the fourth is 3/4. Therefore, the middle term in Eq. (11) is 0.25. This means that the violation of the inequality (11) is higher than the maximum violation obtained from Hardy's proof, where the middle term is 0.090 [32,33], and even higher than the violation for a singlet state, where the middle term is 0.207 [32,33]. The violation exhibited by the GHZ state is even higher. By using properties (7)–(10), it can easily be seen that the middle term in Eq. (11) is 0.5, which is the maximum allowed violation of a CH-type inequality and corresponds to a value four in the corresponding Clauser-Horne-Shimony-Holt (CHSH) inequality [35].

A similar situation occurs in Mermin's inequality [13]

$$-2\leqslant\langle A_1A_2A_3\rangle-\langle A_1B_2B_3\rangle-\langle B_1A_2B_3\rangle-\langle B_1B_2A_3\rangle\leqslant 2, \tag{12}$$

where A_i and B_i are observables of qubit *i*. By choosing $A_i=\sigma_{zi}$ and $B_i=\sigma_{xi}$, for the GHZ state (1) we obtain four for the middle term in Eq. (12), four being the maximum allowed violation of inequality (12). For the *W* state (2), considering only local spin observables on plane *x*-*z*, and that $A_1=A_2=A_3$ and $B_1=B_2=B_3$, the maximum violation is 3.046 [for instance, by choosing $A_i=\cos(0.628)\sigma_x-\sin(0.628)\sigma_z$ and $B_i=\cos(1.154)\sigma_x+\sin(1.154)\sigma_z$]. Alternatively, by choosing $A_i=\sigma_{zi}$ and $B_i=\sigma_{xi}$ (which satisfy EPR's criterion of elements of reality), the *W* state (2) gives the value three for the middle term in Eq. (12).

Two reasons suggest that, for the three-qubit GHZ state, inequality (11) could lead to a more conclusive clear-cut experimental test of Bell's theorem than inequality (12). As in CH's, inequality (11) can be put into a form which does not involve the number of undetected particles, thereby rendering unnecessary the assumption of fair sampling [34]. On the other hand, since CH and CHSH inequalities are equivalent [33], and Eq. (11) and CH [Eq. (12) and CHSH] inequalities have the same bounds, the ratio between the maximum violations shown by the GHZ and singlet states for inequality (11) over CH's [Eq. (12) over CHSH's], $1+\sqrt{2}$ [$\sqrt{2}$], suggests that Eq. (11) reveals a higher violation of local realism than Eq. (12).

V. BELL INEQUALITIES INVOLVING TRI- AND BIPARTITE CORRELATIONS

As a final remark, the results in [21] point out that bipartite correlations are relevant to the *W* state but not to the GHZ state. Therefore, it would be interesting to consider Bell inequalities involving both tripartite and bipartite correlations. The simplest way of obtaining such an inequality would be by adding genuinely bipartite correlations to the tripartite correlations considered in Mermin's inequality. For instance, a straightforward calculation would allow us to prove that any local realistic theory must satisfy the following inequality:

$$-5\leqslant\langle A_1A_2A_3\rangle-\langle A_1B_2B_3\rangle-\langle B_1A_2B_3\rangle-\langle B_1B_2A_3\rangle -\langle A_1A_2\rangle-\langle A_1A_3\rangle-\langle A_2A_3\rangle\leqslant 3. \tag{13}$$

Assuming that A_i and B_i are local observables on plane *x*-*z*, and that $A_1=A_2=A_3$ and $B_1=B_2=B_3$, a numerical calculation shows that both the GHZ and *W* states give a maximum value four for the middle term in Eq. (13) (for instance, by choosing $A_i=\sigma_{zi}$ and $B_i=\sigma_{xi}$ in both cases). Therefore, both states lead to the *same* maximal violation of the inequality

(13). However, if we assign a higher weight to the bipartite correlations appearing in the inequality, then we can reach a Bell inequality such as

$$-8 \leqslant \langle A_1 A_2 A_3 \rangle - \langle A_1 B_2 B_3 \rangle - \langle B_1 A_2 B_3 \rangle - \langle B_1 B_2 A_3 \rangle - 2\langle A_1 A_2 \rangle - 2\langle A_1 A_3 \rangle - 2\langle A_2 A_3 \rangle \leqslant 4, \quad (14)$$

which is violated by the W state [for instance, by choosing $A_i = \sigma_{zi}$ and $B_i = \sigma_{zi}$, state (2) gives the value five for the middle term in Eq. (14)] but *not* by the GHZ state. Therefore, there are scenarios involving both tripartite and bipartite correlations in which the quantum predictions for the GHZ state can be reproduced with a local model while those for the W state cannot. A more general study of multipartite Bell inequalities is presented in [36,37].

VI. SUMMARY

In brief, we have completed the family of proofs without inequalities for two- and three-qubit pure entangled states with a proof for the W state that can also be extended to the GHZ state and we have then obtained two Bell inequalities for three qubits. The first could lead to more conclusive tests of Bell's theorem. The second, involving both tri- and bipartite correlations, illustrates some differences between the violations of local realism exhibited by the GHZ and W states.

ACKNOWLEDGMENTS

I thank J. L. Cereceda and C. Serra for their comments, and the organizers of the Sixth Benasque Center for Science, where this work was begun, the Junta de Andalucía Grant No. FQM-239, and the Spanish Ministerio de Ciencia y Tecnología Grant No. BFM2000-0529 for their support.

[1] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69.
[2] R. Cleve and H. Buhrman, Phys. Rev. A **56**, 1201 (1997).
[3] M. Żukowski, A. Zeilinger, M. A. Horne, and H. Weinfurter, Acta Phys. Pol. A **93**, 187 (1998).
[4] S. Bose, V. Vedral, and P. L. Knight, Phys. Rev. A **57**, 822 (1998).
[5] A. Karlsson and M. Bourennane, Phys. Rev. A **58**, 4394 (1998).
[6] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).
[7] C.-W. Zhang, C.-F. Li, Z.-Y. Wang, and G.-C. Guo, Phys. Rev. A **62**, 042302 (2000).
[8] J.-C. Hao, C.-F. Li, and G.-C. Guo, Phys. Rev. A **63**, 054301 (2001).
[9] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[10] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[11] N. D. Mermin, Phys. Today **43**(6), 9 (1990); Am. J. Phys. **58**, 731 (1990).
[12] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[13] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[14] S. M. Roy and V. Singh, Phys. Rev. Lett. **67**, 2761 (1991).
[15] D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **82**, 1345 (1999).
[16] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature (London) **403**, 515 (2000).
[17] L. Hardy, Phys. Rev. Lett. **71**, 1665 (1993).
[18] A. Cabello, Phys. Rev. Lett. **86**, 1911 (2001).
[19] A. Cabello, Phys. Rev. Lett. **87**, 010403 (2001).
[20] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Phys. Rev. Lett. **85**, 1560 (2000).
[21] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[22] A. Higuchi and A. Sudbery, Phys. Lett. A **273**, 213 (2000).
[23] H. A. Carteret and A. Sudbery, J. Phys. A **33**, 4981 (2000).
[24] T. A. Brun and O. Cohen, Phys. Lett. A **281**, 88 (2001).
[25] The name "*W*" is for Wolfgang (Dür), one of the authors of Ref. [21].
[26] N. Gisin, Phys. Lett. A **210**, 151 (1996).
[27] A. Cabello, Phys. Rev. A **63**, 022104 (2001).
[28] Pitowsky [I. Pitowsky, Phys. Lett. A **156**, 137 (1991)] observed that the W states allow three observers to make predictions which "taken together contradict quantum mechanics." The proof in this paper is simpler and is based solely on EPR's elements of reality.
[29] D. Boschi, S. Branca, F. De Martini, and L. Hardy, Phys. Rev. Lett. **79**, 2755 (1997).
[30] L. Hardy (private communication).
[31] L. Hardy, Phys. Rev. Lett. **73**, 2279 (1994).
[32] N. D. Mermin, Phys. Today **47**(6), 9 (1994); **47**(11), 119 (1994); Am. J. Phys. **62**, 880 (1994).
[33] N. D. Mermin, Ann. N.Y. Acad. Sci. **755**, 616 (1995).
[34] J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).
[35] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[36] I. Pitowsky and K. Svozil, Phys. Rev. A **64**, 014102 (2001).
[37] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).