PHYSICAL REVIEW A, VOLUME 65, 030301(R)

Generalized measurements by linear elements

John Calsamiglia
Helsinki Institute of Physics, PL 64, FIN-00014 Helsingin yliopisto, Finland
(Received 23 August 2001; published 5 February 2002)

I give a first characterization of the class of generalized measurements that can be exactly realized on a pair of qudits encoded in indistinguishable particles, by using only linear elements and particle detectors. Two immediate results follow from this characterization. (i) The Schmidt number of each element in the positive operator valued measure cannot exceed the number of initial particles. This rules out any possibility of performing perfect Bell measurements for qudits. (ii) The maximum probability of performing a generalized incomplete Bell measurement is one-half.

DOI: 10.1103/PhysRevA.65.030301 PACS number(s): 03.67.−a, 03.67.Hk, 42.50.−p, 42.25.Hz

I. INTRODUCTION

In the last years there has appeared very important contributions to the field of quantum information processing with linear elements (see below). Linear elements provide the means to exploit symmetry and interference effects associated with indistinguishable particles. This raises many interesting questions from the fundamental point of view, but it is also highly relevant technologically. Quantum information processing has a wide range of striking applications [1]. Many of these applications have been first implemented in optical systems, where the lack of interaction at the single-photon level [2] makes the indistinguishability of the photons a crucial feature. Photons are ideal qubit or qudit (d-dimensional counterpart of a qubit) carriers, because of their low decoherence rates, and linear optical elements are extremely simple devices which allow one to perform certain operations on the encoded photons in a controlled fashion. It is therefore very desirable to know the capabilities and limitations of those operations.

In a recent work Knill *et al.* [3] make an important breakthrough in this direction showing that fault-tolerant computation with linear optics is in principle possible. Starting from the idea of teleportation of quantum gates [4], they develop a method to perform any quantum operation with a probability that *asymptotically* approaches unity with a growing number of highly entangled auxiliary photons. While their work makes a big step forward by presenting a proof of principle, the preparation of the required auxiliary states is far beyond the current technological possibilities. Moreover, their method does not elucidate the role played by the particle symmetry and indistinguishability, nor does it exclude the idea that for specific applications one can find simpler protocols [5] with less technological restrictions. For example, recent research shows how to perform complete polarization Bell measurements on momentum-entangled photons coming, for example, from a parametric down-conversion source [6], purify entangled photons from those sources [7] or from noisy communication channels [8], reject bit flip errors in quantum communication [9], perform optimal unambiguous state discrimination [10], and efficiently eavesdrop a quantum key distribution [11], by using only a few beam splitters and particle detectors. Thus, the prospect of applications and the need for a deeper understanding warrants further research on the power of linear elements.

In this Rapid Communication I address the question of performing generalized measurements on indistinguishable particles by linear elements and particle detectors. For brevity and for reasons that I will bring out shortly, I term the qudits that are encoded in indistinguishable particles *i-qudits*, *b-qudits* for bosons, and *f-qudits* for fermions. At first sight it seems that linear elements cannot realize nontrivial generalized measurements, for they are unable to provide interaction between the particles. However, this argument has its roots in the concept of locality, which becomes vague when dealing with indistinguishable particles. Assigning a notion of locality to the *i*-qudits is only possible if each *i*-qudit occupies a different set of modes. As soon as the mixing of modes becomes possible the notion of locality vanishes, and nontrivial measurement can be realized. Thus, the term *i*-qudit emphasizes the clear-cut difference between the processing of qudits in the standard quantum information paradigm—where every qudit is represented by a distinct physical system and two-qudit operations are implemented by physical interactions between these systems—and the processing of qudits represented in indistinguishable particles—where interference and particle statistics can be used to establish quantum correlations between qubits without requiring interaction between them. To formalize these possibilities, the description of a set of *i*-qudits has to account for their indistinguishability. Hence, *i*-qudits and qudits are different mathematical objects.

Until now, the measurement on *i*-qudits has only been studied in the context of unambiguous discrimination of a given set of states. Special attention has been drawn to the discrimination of Bell states, also referred to as Bell measurement. Bell states are bipartite pure states of the form $|\Psi\rangle = 1/\sqrt{d}\Sigma_{i=1}^{d}|e_i\rangle|\tilde{e}_i\rangle$ (for some local basis $\{|e_i\rangle\}$ and $\{|\tilde{e}_i\rangle\}$), that define an orthonormal basis. In [12,13] the impossibility of performing a Bell measurement on two *i*-qudits was proven. However, with certain probability, $P_{\mathrm{succ}}<1$, it is still possible to unambiguously discriminate two-qubit Bell states. The optimum efficiency of such an incomplete Bell measurement was found in [14]. These results reflect the current problems and put serious upper bounds on the efficiency of some important quantum information protocols, such as teleportation [15], entanglement swapping [16], or quantum dense coding [17]. Later, Carollo *et al.* [18] showed that discrimination without error is also impossible for a very particular set of states, that, despite being product states, they exhibit nonlocal properties as a set.

The general approach in the above papers was to feed the linear device with the states to be discriminated and check under what conditions the particle detectors at the output produce different ''click'' combinations that could identify the input state. Here I will adopt a different approach. Given that the measurement outcomes are of a known form, namely, a ''click'' pattern, we will find the positive operator valued measure (POVM) (see below) on the input *i*-qudits induced by this type of measurement. This approach is much more general and sets a suitable framework to arrive at the full characterization of the class of generalized measurements that can be implemented by linear elements.

II. *i*-QUDITS

An arbitrary one-i-qudit state $|\boldsymbol{\alpha}\rangle = \Sigma_{i=1}^{d} \alpha_i |i\rangle$ is encoded in a single excitation occupying d field modes, $|\boldsymbol{\alpha}\rangle = \Sigma_{i=1}^{d} \alpha_i a_i^\dagger |0\rangle$. Here $|\mathbf{0}\rangle$ denotes the vacuum state and $a_i^\dagger$ are bosonic (fermionic) creation operators—whenever needed, I will give the results corresponding to each of the particle statistics. In order to encode a two-qudit state $|C\rangle = \Sigma_{i,j=1}^{d} C_{ij} |i\rangle |j\rangle$, a second particle is used occupying d extra modes $\{a_{d+1}^\dagger, \ldots, a_{2d}^\dagger\}$, $|C\rangle = \Sigma_{i,j=1}^{d} C_{ij} a_i^\dagger a_{d+j}^\dagger |\mathbf{0}\rangle$. Any two-boson (-fermion) state can be defined with a bilinear form

$$|\Psi\rangle = \sum_{i,j=1}^{n} N_{ij} a_i^\dagger a_j^\dagger = \mathbf{a}^T N \mathbf{a} |\mathbf{0}\rangle, \tag{1}$$

where N is an $n \times n$ symmetric (antisymmetric) matrix and $\mathbf{a} = (a_1^\dagger, \ldots, a_n^\dagger)^T$. In particular, the bilinear form of the two-i-qudit state $|C\rangle$ is

$$N = \frac{1}{2} \begin{pmatrix} 0 & C & \\ & & 0 \\ (-)C^T & 0 & \\ 0 & & 0 \end{pmatrix}. \tag{2}$$

The $d \times d$ matrix C is defined using the correspondence between the state vectors $|C\rangle = \Sigma_{i,j=1}^{d} C_{ij} |i\rangle |j\rangle$ and the $n \times n$ complex matrix C with elements C_{ij}. Matrix analysis theory [19] renders this representation into a very convenient one for studying bipartite quantum systems. Some useful relations between both representations are

$$A \otimes B |C\rangle = |ACB^T\rangle, \quad \langle A|B\rangle = \mathrm{Tr}(A^\dagger B), \tag{3}$$

$$\mathrm{Tr}_1(|A\rangle\langle B|) = AB^\dagger \ \text{ and } \ \mathrm{Tr}_2(|A\rangle\langle B|) = A^T B^*. \tag{4}$$

Matrices and vectors are written in the canonical basis. Thereby, the correspondence between matrices and bipartite state vectors (which is obviously basis dependent) is always well defined.

Finally, the action of the linear elements is defined by a linear mapping of the input creation operators $\{a_1^\dagger \cdots a_n^\dagger\}$ to the output creation operators $\{c_1^\dagger \cdots c_n^\dagger\}$

$$c_i^\dagger = \sum_{j=1}^{n} U_{ij}^\dagger a_j^\dagger . \tag{5}$$

To implement this operation one can use a series of beam splitters and phase shifters [20] or multiports [21].

III. GENERALIZED MEASUREMENTS ON *i*-QUDITS

A generalized measurement is described by a positive operator valued measure (POVM) [22] given by a collection of positive operators F_k with $\Sigma_k F_k = \mathbb{1}$. Each operator F_k corresponds to one classically distinguishable measurement outcome (e.g., a given combination of ''clicks'' in the output detectors). The probability $p(k|\rho)$ for the outcome k to occur, conditioned to an input density matrix ρ, is $p(k|\rho) = \mathrm{Tr}(\rho F_k)$.

If we send an i-qudit $|\boldsymbol{\alpha}\rangle$ through a linear device, the state in the output is $|\boldsymbol{\alpha}\rangle_{out} = \Sigma_{j=1}^{n} U_{ji} \alpha_j c_i^\dagger |0\rangle = |U^T \boldsymbol{\alpha}\rangle$, where the vector $\boldsymbol{\alpha}$ is padded with extra zeros whenever $n > d$. Notice that the number of modes involved in the transformation can be larger than the number of modes occupied by the i-qudit. This provides a straightforward extension of our input Hilbert space, $\mathcal{H}_1 \oplus \mathcal{H}_{1'}$ where $\mathcal{H}_1$ is the i-qudit Hilbert space and $\mathcal{H}_{1'}$ is spanned by single-particle states occupying modes $\{a_{d+1}^\dagger \cdots a_n^\dagger\}$. According to Neumark's theorem [22] *any* POVM can be realized by performing von Neumann measurements on this extended Hilbert space. Explicitly, the event of one ''click'' in mode c_i is associated with the POVM element F^i

$$\begin{aligned} \mathrm{Tr}(F^i |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|) &= p(i|\boldsymbol{\alpha}) = |\langle 0|c_i|U^T \boldsymbol{\alpha}\rangle|^2 \\ &= \mathrm{Tr}(|\boldsymbol{v}_i\rangle\langle\boldsymbol{v}_i|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|) \forall \boldsymbol{\alpha} \rightarrow F^i = |\boldsymbol{v}_i\rangle\langle\boldsymbol{v}_i|, \end{aligned} \tag{6}$$

where the d-dimensional vector $\boldsymbol{v}_i = (U_{1i}^*, \ldots, U_{di}^*)^T$. This is the reason why protocols that rely on single-qudit POVMs—such as optimal unambiguous state discrimination [10], or some particular entanglement transformations [23]—can be successfully carried out in linear optics.

The situation is quite different in the two-i-qudit case. A two-i-qudit state $|C\rangle$ described, according to Eq. (2), by a matrix N will be transformed into a two-particle state with the following matrix representation in terms of the output modes:

$$|C\rangle = \mathbf{c}^T M \mathbf{c} |\mathbf{0}\rangle \ \text{ with } \ M = U^T N U, \tag{7}$$

where $\mathbf{c} = (c_1^\dagger, \ldots, c_n^\dagger)^T$. Notice that the only mode transformations that leave the i-qudit Hilbert space invariant are

$$U_{\mathrm{sep}} = \begin{pmatrix} U_1 & & \\ & U_2 & \\ & & U_3 \end{pmatrix}, \quad U_{\mathrm{sw}} = \begin{pmatrix} 0 & \mathbb{1}_d & \\ \mathbb{1}_d & 0 & \\ & & \mathbb{1}_{n-2d} \end{pmatrix}$$

and compositions of both (U_1 and U_2 are $d \times d$ unitary matrices). From Eq. (3) it follows that the first transformation corresponds to a separable operation in the i-qudit Hilbert space $U_1 \otimes U_2 |C\rangle$, while the second transforms the *i*-qudit $|C\rangle$ to $(-)|C^T\rangle$, i.e., performs the nonseparable swap operation.

Define

$$U = \begin{pmatrix} A \\ B \\ D \end{pmatrix}, \tag{8}$$

where A and B are $d\times n$ matrices. From Eqs. (2) and (7) it is clear that the output state M will not depend on the values of matrix elements of D, i.e., on how the initially unoccupied modes transform. Now, we are in a position to calculate the resulting POVM on the *i*-qudits when particle detectors are placed in the output modes. Linear elements preserve the number of particles; therefore each measurement outcome is associated with the absorption of two particles at modes c_i, c_j. Given an arbitrary two-*i*-qudit state $|C\rangle$, the probability amplitude of such an event is (for $i\neq j$)

$$\begin{aligned}\langle 0|c_i c_j|C\rangle &= \langle 0|M_{kl}c_i c_j c_k^\dagger c_l^\dagger|0\rangle = 2\langle i|M|j\rangle \\ &= \langle i|A^T CB \pm B^T C^T A|j\rangle \\ &= \mathrm{Tr}[A^T CB(|i\rangle\langle j| \pm |j\rangle\langle i|)] \\ &= \mathrm{Tr}[CB(|i\rangle\langle j| \pm |j\rangle\langle i|)A^T] = \mathrm{Tr}(CP^{ij\,\dagger}),\end{aligned} \tag{9}$$

where the $d\times d$ matrix P^{ij} is defined as

$$P^{ij} = A^* \Delta^{ij} B^\dagger \quad \text{with} \quad \Delta^{ij}_{kl} = \delta_{ki}\delta_{lj} \pm \delta_{kj}\delta_{li}, \tag{10}$$

and the $+$ $(-)$ refers to the *b*-qudit (*f*-qudit) result. In the bosonic case a normalizing factor $1/\sqrt{2}$ should be added to Eqs. (9) and (10) when $i=j$. Equation (3) allows us to write the probability amplitude (9) as

$$\langle 0|c_i c_j|C\rangle = \langle P^{ij}|C\rangle. \tag{11}$$

From Eq. (11) we find that the POVM associated with this detection event is $F^{ij} = |P^{ij}\rangle\langle P^{ij}|$. Making use of Eq. (3), we arrive at the central result: the class of generalized measurements that one can realize with linear elements is defined by the POVM elements

$$F^{ij} = |P^{ij}\rangle\langle P^{ij}| \quad \text{with} \quad |P^{ij}\rangle = \sqrt{2}A^* \otimes B^* |\psi^{ij}\rangle \tag{12}$$

or

$$|P^{ij}\rangle = |\boldsymbol{a}_i\rangle|\boldsymbol{b}_j\rangle \pm |\boldsymbol{a}_j\rangle|\boldsymbol{b}_i\rangle, \tag{13}$$

where we have introduced the normalized states $|\psi^{ij}\rangle \propto (|i\rangle|j\rangle \pm |j\rangle|i\rangle)$, and $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are the ith columns of A^* and B^*, respectively. For *b*-qudits, double clicks, i.e., $i=j$, correspond to separable POVM elements, while for *f*-qudits, the Pauli exclusion principle prohibits these events. In the last equations we see that each $|P^{ij}\rangle$ is the superposition of, at most, two terms; thus, their Schmidt rank [24] is at most 2. Since every possible POVM element is a convex combination of those defined in Eq. (12), we conclude that all POVM elements on two *i*-qudits realized with linear elements will have a Schmidt number [24], at most, equal to 2 [25]. This means that for *i*-qudits with $d>2$ no analog of the incomplete Bell measurement [26] can exist—nor can it be used in teleportation [15], entanglement swapping [16], quantum dense coding [17], or probabilistic implementation of nonlocal gates [27]. Also, from this result and a theorem by Carollo and Palma [28], it follows that, even with the aid of auxiliary photons and conditional dynamics, it is not possible to implement a never failing Bell measurement for *i*-qudits [13].

Leaving *i*-qudits $(d>2)$ aside, in the remainder of this Rapid Communication we will study the efficiency of a generalized form of Bell measurement on *i*-qubits $(d=2)$. The appeal of a Bell measurement is not only its ability to discriminate unambiguously between the specific four Bell states, but that it can project an unknown state into a maximally entangled state. Any generalized measurement in which every POVM element is maximally entangled, would have much the same appeal. Trivial modifications, consisting only in local operations, could make the teleportation, entanglement swapping, or the probabilistic nonlocal gates function with such a generalized Bell measurement.

In the following, I will show that the optimal generalized Bell measurement fails to project on a maximally entangled state in half of the trials. For this purpose let us first notice that the POVM elements (13) that have a contribution from the detection in a mode c_i can be written (for both, bosons and fermions) as

$$|\tilde{P}^{ij}\rangle \equiv W_i^\dagger \otimes V_i^\dagger |P^{ij}\rangle = |\boldsymbol{a}_i||1\rangle|\boldsymbol{x}\rangle + |\boldsymbol{b}_i||\boldsymbol{y}\rangle|1\rangle, \tag{14}$$

where $\boldsymbol{x} = V_i^\dagger|\boldsymbol{b}_j\rangle$ and $\boldsymbol{y} = W_i^\dagger|\boldsymbol{a}_j\rangle$, and V_i and W_i are unitary transformations. The matrix representation of this state is

$$\tilde{P}^{ij} = \begin{pmatrix} |\boldsymbol{a}_i|x_1 + |\boldsymbol{b}_i|y_1 & |\boldsymbol{a}_i|x_2 \\ |\boldsymbol{b}_i|y_2 & 0 \end{pmatrix}. \tag{15}$$

It is characteristic of maximally entangled states that each of its subsystems has a reduced density matrix proportional to the identity matrix. By Eq. (4), this implies that, in the matrix representation, maximally entangled states are proportional to unitary matrices. Thus, if the POVM elements are to be maximally entangled, $\tilde{P}^{ij}$ has to be unitary (up to a constant κ_i), and these conditions have to hold,

$$|\boldsymbol{a}_i|x_1 + |\boldsymbol{b}_i|y_1 = 0 \quad \text{and} \quad |\boldsymbol{a}_i||x_2| = |\boldsymbol{b}_i||y_2| = |\kappa_i|. \tag{16}$$

Enforcing these conditions, we have $\tilde{P}^{ij} = \kappa_i \left(\begin{smallmatrix} 0 & 1 \\ e^{i\phi} & 0 \end{smallmatrix}\right)$, which, after switching back to the state representation, allows us to conclude that a detection in mode c_i can only contribute to maximally entangled POVM elements of the form

$$|P^{ij}\rangle = \kappa_i W_i \otimes V_i (|1\rangle|2\rangle + e^{i\phi}|2\rangle|1\rangle). \tag{17}$$

On the other hand, after some simple algebra and using $AB^\dagger = 0$, one can find the total contribution, in the resolution of the identity, of all the POVM elements where a detection in the c_i mode is involved,

$$\mathbb{1}=\sum_{i\geqslant j=1}^{n}|P^{ij}\rangle\langle P^{ij}|=\sum_{i=1}^{n}\frac{\kappa_i}{2}W_i\otimes V_i(|\boldsymbol{a}_i|^2|1\rangle\langle 1|\otimes\mathbb{1}_2+|\boldsymbol{b}_i|^2\mathbb{1}_2\otimes|1\rangle\langle 1|)W_i^{\dagger}\otimes V_i^{\dagger}. \tag{18}$$

The factor $\frac{1}{2}$ comes from the symmetry $P^{ij}=P^{ji}$ and compensates the double counting of the terms with $i\neq j$. Comparing this result with Eq. (17) it is clear that not all POVM elements involving a detection in c_i can be maximally entangled; the space spanned by the POVM elements defined in Eq. (17) does not cover the whole support of the ith term in the sum in Eq. (18). An upper bound on the total weight of the maximally entangled POVM elements in this term fixes the maximum probability of successfully projecting an unknown input state $\rho=\frac{1}{4}\mathbb{1}_2\otimes\mathbb{1}_2$ onto a maximally entangled state

$$\begin{aligned} p_{\mathrm{succ}}^{i}&\leqslant\frac{1}{2}\mathrm{Tr}[W_i\otimes V_i(|\boldsymbol{a}_i|^2|1\rangle\langle 1|\otimes|2\rangle\langle 2|\\ &\quad+|\boldsymbol{b}_i|^2|2\rangle\langle 2|\otimes|1\rangle\langle 1|)W_i^{\dagger}\otimes V_i^{\dagger}\rho]\\ &=\frac{1}{8}(|\boldsymbol{a}_i|^2+|\boldsymbol{b}_i|^2)=\frac{1}{8}\sum_{k=1}^{4}|U_{ik}|^2, \end{aligned} \tag{19}$$

where we employed the definition in Eq. (8). By adding up the contributions from all the detectors we obtain the total probability of success,

$$P_{\mathrm{succ}}=\sum_{i=1}^{n}p_{\mathrm{succ}}^{i}\leqslant\frac{1}{8}\sum_{k=1}^{4}\sum_{i=1}^{n}|U_{ik}|^2=\frac{1}{8}\sum_{k=1}^{4}1=\frac{1}{2}. \tag{20}$$

This also sets to one-half the ultimate efficiency of teleportation, entanglement swapping, and the probabilistic implementation of nonlocal gates on i-qubits.

As a last remark, even if the results for fermions and bosons are apparently similar, there is actually a large difference that is manifest in the asymptotic method that also uses auxiliary photons. In [29] it is proved that the analogue of the photonic efficient quantum computation [3] cannot exist for fermions.

In this Rapid Communication, I have introduced a formalism to study the characterization of the generalized measurements on two i-qudits implementable by linear elements. Two non-trivial results concerning maximally entangled POVMs followed from the general POVM characterization, Eq. (13). The formalism should also be very helpful in determining the viability or efficiency of other relevant POVMs in quantum information.

ACKNOWLEDGMENTS

I would like to thank N. Lütkenhaus, K.-A. Suominen, and M. Mackie for reading the manuscript and M. Dušek for the motivation on Bell measurements for qudits [13]. This work was supported by the Academy of Finland (Project No. 43336) and the EU IST EQUIP program.

[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[2] See, however, M.D. Lukin and A. Imamoǧlu, Phys. Rev. Lett. **84**, 1419 (2000); Y.-H. Kim *et al.*, *ibid.* **86**, 1370 (2001).
[3] E. Knill *et al.*, Nature (London) **409**, 46 (2001).
[4] D. Gottesman and I.L. Chuang, Nature (London) **402**, 390 (1999).
[5] T.C. Ralph *et al.*, Phys. Rev. A **65**, 012314 (2002).
[6] P.G. Kwiat and H. Weinfurter, Phys. Rev. A **58**, R2623 (1998).
[7] C. Simon and J.-W. Pan, e-print quant-ph/010863.
[8] J.-W. Pan *et al.*, Nature (London) **410**, 1067 (2001).
[9] D. Bouwmeester, Phys. Rev. A **63**, R040301 (2001).
[10] B. Huttner *et al.*, Phys. Rev. A **54**, 3783 (1996); R.B.M. Clark *et al.*, *ibid.* **63**, 040305(R) (2001); Y. Sun *et al.*, *ibid.* **64**, 022311 (2001).
[11] J. Calsamiglia *et al.*, Phys. Rev. A **65**, 012312 (2002).
[12] N. Lütkenhaus *et al.*, Phys. Rev. A **59**, 3295 (1999); L. Vaidman and N. Yoran, *ibid.* **59**, 116 (1999).
[13] M. Dušek, e-print quant-ph/0107119.
[14] J. Calsamiglia and N. Lütkenhaus, Appl. Phys. B: Lasers Opt. **72**, 67 (2001).
[15] C.H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993); D. Bouwmeester *et al.*, Nature (London) **390**, 575 (1997).
[16] M. Zukowski *et al.*, Phys. Rev. Lett. **71**, 4287 (1993); Jian-Wei Pan *et al.*, *ibid.* **80**, 3891 (1998).
[17] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992); K. Mattle *et al.*, *ibid.* **76**, 4656 (1996).
[18] A. Carollo *et al.*, Phys. Rev. A **64**, 022318 (2001).
[19] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, Cambridge, England, 1991).
[20] M. Reck *et al.*, Phys. Rev. Lett. **73**, 58 (1994).
[21] J.C. Howell and J.A. Yeazell, Phys. Rev. A **61**, 052303 (2000).
[22] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).
[23] C. Zhang, e-print quant-ph/0104054.
[24] B.M. Terhal and P. Horodecki, Phys. Rev. A **61**, 040301 (2000).
[25] It is a straightforward extension of this result that any ‘‘click’’ pattern corresponds to a POVM element with Schmidt rank smaller or equal to the number of involved particles (i-qudits +auxiliary).
[26] H. Weinfurter, Europhys. Lett. **25**, 559 (1994); S.L. Braunstein and A. Mann, Phys. Rev. A **51**, R1727 (1995); M. Michler *et al.*, *ibid.* **53**, R1209 (1996).
[27] W. Dür and J.I. Cirac, Phys. Rev. A **64**, 012317 (2001); M. Koashi *et al.*, *ibid.* **63**, 030301 (2001).
[28] A. Carollo and G.M. Palma, e-print quant-ph/0106041.
[29] E. Knill, e-print quant-ph/0108033.