Geometric observation for Bures fidelity between two states of a qubit

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In this Brief Report, we present a geometric observation for the Bures fidelity between two states of a qubit.

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I. INTRODUCTION

As is well known, the *trace distance* and the *Bures fidelity* are two important distance measures for quantum computation and quantum information [1–7]. A qubit is completely described by the 2×2 density matrix as

$$\rho(\mathbf{n}) = \tfrac{1}{2}(\mathbf{1}+\vec{\sigma}\cdot\mathbf{n}), |\mathbf{n}|\leqslant 1, \tag{1}$$

where $\mathbf{1}$ is the unit matrix, $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ the Pauli matrices vector, and $\mathbf{n}$ the Bloch vector. $|\mathbf{n}|=1$ corresponds to a pure state, otherwise, a mixed state. Let

$$\rho_1 = \tfrac{1}{2}(\mathbf{1}+\vec{\sigma}\cdot\mathbf{u}),$$

$$\rho_2 = \tfrac{1}{2}(\mathbf{1}+\vec{\sigma}\cdot\mathbf{v}) \tag{2}$$

be two states of a qubit. The trace distance and the Bures fidelity between ρ_1 and ρ_2 are defined by the equations

$$D(\rho_1,\rho_2) = \tfrac{1}{2}\,\mathrm{tr}|\rho_1-\rho_2|, \tag{3}$$

$$F(\rho_1,\rho_2) = [\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}]^2. \tag{4}$$

One can write Eq. (3) as

$$D(\rho_1,\rho_2) = \frac{|\mathbf{u}-\mathbf{v}|}{2}, \tag{5}$$

so that the trace distance between two single qubit states has a simple geometric interpretation as half the ordinary Euclidean distance between points on the Bloch sphere. However, no similarly clear geometric interpretation is known for the Bures fidelity between two states of a qubit [7]. The purpose of this brief report is to provide a geometric observation for the Bures fidelity for the case of a qubit. In Sec. II, a definite geometric relation is formulated for the Bures fidelity in terms of *hyperbolic parameters*. A conclusion is made in the last section.

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II. FORMALISM

Theorem. The Bures fidelity between states ρ_1 and ρ_2 is equal to

$$F(\rho_1,\rho_2) = \frac{\cosh(\phi_{\mathbf{w}}/2)}{\cosh\phi_{\mathbf{u}}}\,\frac{\cosh(\phi_{\mathbf{w}}/2)}{\cosh\phi_{\mathbf{v}}}, \tag{6}$$

where ϕ_i $(i=\mathbf{u,v,w})$ are rapidities.

Proof. Let us introduce the hyperbolic parameter "ϕ" to represent the Bloch vector as

$$\mathbf{u} = \hat{\mathbf{u}}\tanh\phi_{\mathbf{u}}, \tag{7}$$

where $\hat{\mathbf{u}}=\mathbf{u}/|\mathbf{u}|$ is a unit vector. It is easy to check $|\mathbf{u}|\leqslant 1$ because of $|\tanh\phi_{\mathbf{u}}|\leqslant 1$; $\phi_{\mathbf{u}}=0$ corresponds to $|\mathbf{u}|=0$, while $\phi_{\mathbf{u}}\to\infty$ corresponds to $|\mathbf{u}|=1$. In other words, Eq. (7) is a one-to-one mapping between $\phi_{\mathbf{u}}$ and $\mathbf{u}$.

At this moment, the density matrix $\rho(\mathbf{u})$ can be rewritten as

$$\rho(\mathbf{u}) = \tfrac{1}{2}(\mathbf{1}+\vec{\sigma}\cdot\hat{\mathbf{u}}\tanh\phi_{\mathbf{u}}). \tag{8}$$

It is not difficult to observe that the relation between the density matrix $\rho(\mathbf{u})$ and the Lorentz boost matrix

$$L(\mathbf{u}) = \exp\left(\frac{\varphi_{\mathbf{u}}}{2}\vec{\sigma}\cdot\hat{\mathbf{u}}\right) = \mathbf{1}\cosh\left(\frac{\varphi_{\mathbf{u}}}{2}\right)+\vec{\sigma}\cdot\hat{\mathbf{u}}\sinh\left(\frac{\varphi_{\mathbf{u}}}{2}\right)$$

is

$$\rho(\mathbf{u}) = \frac{L(\mathbf{u})}{2\cosh\phi_{\mathbf{u}}}, \quad \phi_{\mathbf{u}}=\varphi_{\mathbf{u}}/2. \tag{9}$$

Obviously, $\rho(\mathbf{u})$ and $L(\mathbf{u})$ are in one-to-one correspondence. For the former, the physical meaning of the vector $\mathbf{u}$ is the Bloch vector in quantum mechanics, while for the latter, the relativistic velocity. Due to the rapidity φ, i.e., the hyperbolic angle, special relativity can be formulated in terms of hyperbolic geometry. Consequently, some physical quantities have been found to have definite geometric meanings, such as the Thomas rotation angle (sometimes also called the Wigner angle) corresponds to the defect of a hyperbolic triangle [8,9]. Since $\rho(\mathbf{u})$ and $L(\mathbf{u})$ are in one-to-one correspondence, we are led to view the Bloch vector $\mathbf{u}$ as a relativistic velocity, and the angle ϕ as the rapidity. Accordingly, we will uncover a geometric interpretation for the quantum fidelity $F(\rho_1,\rho_2)$ in the framework of hyperbolic geometry.

From the addition law of velocities in special relativity

$$\mathbf{w}=\mathbf{u}\oplus\mathbf{v}=\frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left[\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right], \tag{10}$$

where $\gamma_{\mathbf{u}}=1/\sqrt{1-|\mathbf{u}|^2/c^2}$ is the Lorentz factor and c is the speed of light in vacuum space, we have

$$\gamma_{\mathbf{w}}=\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\mathbf{u}\cdot\mathbf{v}), \tag{11}$$

or

$$\cosh\phi_{\mathbf{w}}=\cosh\phi_{\mathbf{u}}\cosh\phi_{\mathbf{v}}(1+\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}\tanh\phi_{\mathbf{u}}\tanh\phi_{\mathbf{v}}), \tag{12}$$

which is the Cosin law in the hyperbolic geometry.

From Eq. (9), one obtains

$$\sqrt{\rho(\mathbf{u})}=\frac{\cosh(\phi_{\mathbf{u}}/2)}{2\cosh\phi_{\mathbf{u}}}[\mathbf{1}+\vec{\sigma}\cdot\hat{\mathbf{u}}\tanh(\phi_{\mathbf{u}}/2)]. \tag{13}$$

From $\det(\sqrt{\rho_1}\rho_2\sqrt{\rho_1}-\Lambda 1)=0$, we have

$$\Lambda^2-\frac{\gamma_{\mathbf{w}}}{2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}\Lambda+\frac{1}{16\gamma_{\mathbf{u}}^2\gamma_{\mathbf{v}}^2}=0, \tag{14}$$

so that

$$\Lambda_{\pm}=\frac{\cosh\phi_{\mathbf{w}}\pm\sinh\phi_{\mathbf{w}}}{4\cosh\phi_{\mathbf{u}}\cosh\phi_{\mathbf{v}}}. \tag{15}$$

Thus, the Bures fidelity is

$$F(\rho_1,\rho_2)=(\sqrt{\Lambda_+}+\sqrt{\Lambda_-})^2=\frac{\cosh(\phi_{\mathbf{w}}/2)}{\cosh\phi_{\mathbf{u}}}\frac{\cosh(\phi_{\mathbf{w}}/2)}{\cosh\phi_{\mathbf{v}}}. \tag{16}$$

This ends the proof.

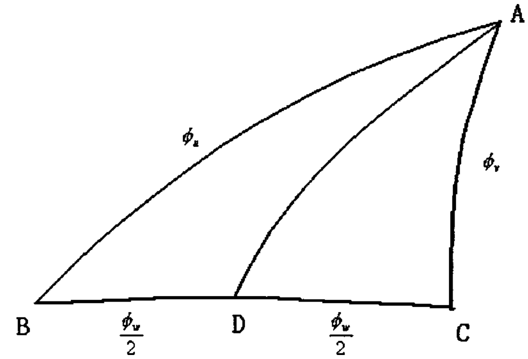

FIG. 1. The hyperbolic triangle ΔABC. Its three sides are $|AB|=\phi_{\mathbf{u}}=\tanh^{-1}|\mathbf{u}|$, $|AC|=\phi_{\mathbf{v}}=\tanh^{-1}|\mathbf{v}|$, $|BC|=\phi_{\mathbf{w}}=\tanh^{-1}|\mathbf{w}|$. D is the midpoint of the side BC. The angle between AB and AC is equal to $\pi-\arccos(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}})$.

III. CONCLUSION

In Fig. 1, we draw a hyperbolic triangle ΔABC formed by three hyperbolic angles $\{\phi_{\mathbf{u}}=|AB|,\phi_{\mathbf{v}}=|AC|,\phi_{\mathbf{w}}=|BC|\}$, where D is the midpoint of the side BC. As one can see that the trace distance $D(\rho_1,\rho_2)=|\mathbf{u}-\mathbf{v}|/2$ is related to an ordinary Euclidean triangle, whose three sides are $|\mathbf{u}|$, $|\mathbf{v}|$, and $|\mathbf{u}-\mathbf{v}|$; similarly, the Bures fidelity is related to a hyperbolic triangle, it is the product of the ratio $\cosh(\phi_{\mathbf{w}}/2)/\cosh\phi_{\mathbf{u}}$ and the ratio $\cosh(\phi_{\mathbf{w}}/2)/\cosh\phi_{\mathbf{v}}$. From Eq. (16), one easily sees that $F(\rho_1,\rho_2)$ is symmetric in its inputs, i.e., $F(\rho_1,\rho_2)=F(\rho_2,\rho_1)$, and is invariant under unitary transformations on the state space.

In conclusion, we have presented a geometric observation for the Bures fidelity between two states of a qubit. It is also interesting and significant to study the geometric meaning of the Bures fidelity for the case of a qunit (i.e., a N-dimensional quantum object, $N=2$ corresponds to a qubit) [10], since the calculation becomes much more complicated, we shall investigate it elsewhere. Nevertheless, we believe that a similar simple hyperbolic geometric relation, such as Eq. (16), is possibly held for the case of a qunit.

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