Stochastic initiation of superradiance in a cavity: An approximation scheme within quantum trajectory theory

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(Received 15 August 2001; published 16 January 2002)

We investigate the stochastic initiation of superradiant emission from a collection of excited two-state atoms coupled through a low-Q cavity mode. Noncollective emission into other modes of the electromagnetic field is included, allowing us to describe the transition from predominantly noncollective to collective behavior as the number of atoms is increased. We examine the dependence of the photon statistics on the number of atoms and the ratio of collective to noncollective decay rates. Numerical results for a few to 10^6 atoms are obtained using an approximation scheme developed within the framework of quantum trajectory theory. Results are compared with those for a pure collective emission process and an earlier treatment of noncollective effects.

DOI: 10.1103/PhysRevA.65.023815 PACS number(s): 42.50.Fx, 42.50.Lc, 42.50.Ar

I. INTRODUCTION

The collective interaction of atoms with light is a topic of fundamental interest in quantum optics. There exists a large literature on superradiance and superfluorescence, in particular, dating back to the seminal paper by Dicke [1]. The simplest theoretical formulation of superradiance holds for Dicke's small sample case where all atoms are contained in a volume much smaller than a cubic wavelength. In this instance, it is readily seen that the symmetric, half-inverted atomic state (precisely half the atoms excited) radiates at a rate proportional to the square of the number of atoms, indicating the presence of coherence between the radiation emitted by different atoms. A complete picture of the phenomenon requires, however, that the method of excitation be taken into account. Specifically, direct excitation by a laser pulse from the ground state leaves the atomic sample in possession of a macroscopic polarization (an uncertain number of atoms excited). Under these conditions, the subsequent radiation can be understood from the semiclassical point of view taken in the theory of magnetic resonance [2,3] (Dicke nevertheless still speaks of ''super-radiant'' states). On the other hand, when the sample is excited to the fully inverted state and then allowed to relax, it apparently passes through a cascade of states of precise, though diminishing, excitation. The quantum fluctuations in the emission process become important and the appearance of any coherence is necessarily spontaneous. It is this more interesting situation that the present paper addresses.

While the word ''superfluorescence'' is sometimes used to distinguish initial states that lack a macroscopic polarization [4], we choose to follow the older and still common practice of employing the term ''superradiance'' without regard to the initial state.

Collective emission can arise when the atomic sample is not confined within a cubic wavelength. Indeed, with one notable exception [5], all superradiance experiments deal with something more complex than the idealized small sample case [6]. A widely adopted approach to the understanding of extended systems is to assume that after the very earliest stage of the spontaneous decay process, emission into just one mode of the radiation field predominates. The assumption leads to the so-called ''single-mode'' model [7], which turns out to be formally equivalent to Dicke's small sample model. Most work on the quantum statistics of superradiance is based upon the ''single-mode'' assumption [8–13]. For most initial conditions, these treatments reproduce the semiclassical pulse structure obtained by Eberly and Rehler [14,15], but they also provide an account of fluctuations when the excitation brings the system close to the fully excited state.

We are interested in understanding collective emission in systems that do not meet the small sample or single-mode criteria. We expect that new insights can be brought to the problem by a quantum trajectory formulation. The most general situation requires the atoms to couple to all modes of the field, while in addition diffraction and propagation effects must be taken into account. We retain the Markov approximation, however, assuming the propagation time between atoms to be much shorter than the duration of the collective decay. We thus aim for the level of description provided by the general superradiance master equation [16–18]. Carmichael and Kim have obtained the quantum trajectory unraveling of this equation [19].

In his 1972 essay, Eberly speaks of treating the radiative decay process as ''a quantum-classical interpolation problem'' [20]. The imagery is particularly appropriate, since while the initiation of collective emission requires a statistical treatment within quantum mechanics, what results, shot by shot, is an essentially ''classical'' pulse of light. Apparently the single pulse emission does, indeed, interpolate, as a physical process, between quantum and classical regimes of evolution. The process, however, is manifestly stochastic and not very accessible if it is kept buried under the quantum-mechanical average. Whereas Eberly had in mind a method of extraction grounded in the physical intuition of semiclassical Bloch equations, quantum trajectory theory provides a rigorous formulation of the stochastic initiation and development of the individual pulses.

Even in quantum trajectory theory, we must nevertheless introduce simplifications if we are to go any distance beyond a mere formulation. Naive numerical simulation will not take us very far, as only a few tens of two-state atoms yield a

Hilbert space dimension well beyond the capacity of any computer. As a first step toward understanding the general stochastic process for spatially distributed atoms formulated by Carmichael and Kim [19], we study, in this paper, a model with two emission channels. We couple N two-state atoms collectively to one mode of the radiation field and each atom, independently, to all other modes of the field. The model may be realized by placing the atoms (at low density) in a low-Q resonant cavity. It was a popular model for theoretical work on the bad-cavity limit of optical bistability [21–23]. Surprisingly, despite the close approximation cavity systems make to Dicke's original proposal, there are only a few experiments to date on superradiance in resonant cavities [24–27].

The neglect of independent coupling of the atoms to noncavity modes is a major deficiency of nearly all previous quantum-statistical studies of superradiance. The stochastic initiation of emission from the fully excited state is largely a competition between collective and noncollective decay. The competing noncollective decay is especially important in the case of a relatively small number of atoms. When the atoms are too few, there is insufficient time for the collective enhancement of the emission to develop before the excitation energy is lost to noncollective modes. We show that as a consequence of the competition, on increasing the number of atoms, superradiance emerges through a broad transition region featuring large fluctuations in the delay time, width, and intensity of the pulse coupled out through the cavity mode. We make comparisons with the work of Lee [28,29], who has explicitly considered some aspects of the competition from noncollective emission.

Our calculations are based on an approximation that reduces the exact quantum trajectory description to a two-channel rate process. The trajectory formulation offers a path to the approximation because it permits the use of basis states which are adapted, throughout a simulation, to the sequence of past photon emissions. The approximation allows us to calculate the statistics of the emitted radiation for a few to 10^6 atoms, and for various ratios of collective to noncollective decay rates, with only a modest investment of computer resources. Our method provides access to any number of statistical properties, including the correlations between the pulse intensity and the pulse delay time and width.

We outline the two-channel model and its quantum trajectory formulation in Sec. II. In Sec. III we describe the simulation scheme used to calculate time-dependent quantities, and we derive a difference equation for the photon number distribution of the collective pulse in Sec. IV. We present results in Sec. V, including results on the photon statistics of the emitted light throughout the transition region as well as delay time statistics and time-intensity correlations. We compare our results with previous work. Conclusions are presented in Sec. VI.

II. MODEL AND ITS QUANTUM TRAJECTORY FORMULATION

We consider N two-state atoms inside an optical cavity. The atoms couple collectively, and on resonance, to one cavity mode; either they are located at antinodes of a standing wave and couple through collective operators $\hat{J}_{\pm}=\Sigma_{i=1}^{N}\hat{\sigma}_{i\pm}$, where $\hat{\sigma}_{i\pm}$ are raising and lowering operators for an individual atom, or they reside in a ring cavity and couple through $\hat{J}_{\pm}=\Sigma_{i=1}^{N}e^{\pm ik_0 z_i}\hat{\sigma}_{i\pm}$, where k_0 is the mode wave number and z_i denotes the position of an atom along the cavity axis. The cavity decay rate is κ and the atoms have a spontaneous decay rate γ to other modes of the radiation field; it is assumed that the spontaneous emission to other modes is independent for each atom. In the bad-cavity limit (κ larger than the rate of collective emission), the cavity field adiabatically follows the atomic polarization. The master equation (valid for times larger than κ^{-1}) is then given by [7,30]

$$\dot{\rho}=\frac{\Gamma}{2}(2\hat{J}_{-}\rho\hat{J}_{+}-\hat{J}_{+}\hat{J}_{-}\rho-\rho\hat{J}_{+}\hat{J}_{-})+\frac{\gamma}{2}\sum_{i=1}^{N}(2\hat{\sigma}_{i-}\rho\hat{\sigma}_{i+}-\hat{\sigma}_{i+}\hat{\sigma}_{i-}\rho-\rho\hat{\sigma}_{i+}\hat{\sigma}_{i-}), \tag{1}$$

where the first term describes the cavity-assisted collective emission and the second describes the noncollective emission due to spontaneous decay. The collective decay rate is determined to be $\Gamma=2g^2/\kappa$, where g is the atomic dipole coupling constant.

While our model is most reliable for the cavity situation, to make a connection with a distribution of atoms in free space we might alternatively define the collective emission rate by $\Gamma=\int d\Omega\, I_0(\vec{k})F(\vec{k},\vec{k}')$, where $I_0(\vec{k})$ is the intensity of spontaneous emission from a single atom in the direction $\vec{k}$, and $F(\vec{k},\vec{k}')=|\langle\exp[i(\vec{k}-\vec{k}')\cdot\vec{r}]\rangle|$, [2] where the angular brackets denote an average over atomic positions; $\vec{k}'$ points in a preferred direction, the axis of a narrow cylindrical distribution of atoms for example [14,18,29]. Any atomic distribution that shows clear directionality of emission should yield a small Γ, since this rate is approximately proportional to the solid angle into which the emission occurs. We thus pay particular attention to the parameter regime $\Gamma/\gamma\ll 1$. In a cavity, or course, much larger values of Γ/γ are possible, even when the solid angle of the radiation coupled out through the cavity mode is small.

The usual strategy in proceeding from Eq. (1) is to assume that the collective emission dominates, at least after a very short time, and thus to solve the equation keeping only the first term on the right-hand side. A variety of methods and approximations have been used to arrive at a solution [9–13]. Our aim is to avoid the assumption. Instead, we use the quantum trajectory formalism to convert the master equation into a stochastic process which can be simulated on a computer to account for the competition between collective and noncollective decay. The difficulty with keeping both terms in Eq. (1) is that while a basis of Dicke states is convenient for treating the first term, it is a particularly inconvenient choice for treating the second. The difficulty is not removed completely in the quantum trajectory formulation. The approach does show a way around it, however, if we are prepared to make a different, though less serious, approximation.

Using an unraveling of the master equation (1) based on counting of the emitted photons, we decompose the density operator into a sum over pure states,

$$\rho(t)=\sum_{\mathrm{REC}} P_{\mathrm{REC}}|\psi_c(t)\rangle\langle\psi_c(t)|, \tag{2}$$

where REC denotes a particular record of photon emissions, up to time t (type—collective or noncollective—and time of emission), P_{REC} is the probability for that record, and $|\psi_c(t)\rangle$ is the state of the atoms conditioned on the sequence of emissions [31]. The *unnormalized* state conditioned on a particular record of n emissions is given by

$$|\overline{\psi}_c(t)\rangle=\hat{B}(t-t_n)\hat{C}_n\cdots\hat{B}(t_2-t_1)\hat{C}_1\hat{B}(t_1)|\psi(0)\rangle, \tag{3}$$

where $\hat{C}_k$ and t_k are the jump operator and time for the kth emission, with $\hat{C}_k=\sqrt{\Gamma}\hat{J}_-$ or $\hat{C}_k=\sqrt{\gamma}\hat{\sigma}_{i-}$ (some i) for collective or noncollective emissions, respectively; $\hat{B}(\tau)=\exp(-i\hat{H}\tau/\hbar)$ generates the continuous time evolution in between emissions, where the non-Hermitian Hamiltonian is

$$\hat{H}=-i\hbar\frac{\Gamma}{2}\hat{J}_+\hat{J}_- -i\hbar\frac{\gamma}{2}\sum_{i=1}^{N}\hat{\sigma}_{i+}\hat{\sigma}_{i-}. \tag{4}$$

The probability for a particular record is

$$P_{\mathrm{REC}}=\langle\overline{\psi}_c(t)|\overline{\psi}_c(t)\rangle dt_n\cdots dt_1, \tag{5}$$

where the photon emissions occur in time intervals (t_k,t_k+dt_k), $k=1,\ldots,n$.

If we introduce an approximation within this scheme, an extremely efficient numerical algorithm is made available for simulating the emission sequences.

III. AN APPROXIMATION SCHEME ENABLING LARGE-*N* SIMULATIONS

We consider a set of basis states that are products of Dicke states and single-atom ground states. A general basis state $|j,m\rangle_{2j}|-\rangle_{i_1}\cdots|-\rangle_{i_{N-2j}}$ has atoms $i_1,\ldots,i_{N-2j}$ in their ground states and the remaining $2j$ atoms in a Dicke state of maximum cooperation number, where $j=(0,\frac{1}{2}),\ldots,N/2$, $m=-j,\ldots,j$. The fully inverted initial state is denoted $|N/2,N/2\rangle_N$. The actions of the atomic operators on these states are

$$\hat{J}_-|j,m\rangle_{2j}|-\rangle_{i_1}\cdots|-\rangle_{i_{N-2j}} =\sqrt{(j+m)(j-m+1)}|j,m-1\rangle_{2j}|-\rangle_{i_1}\cdots|-\rangle_{i_{N-2j}}, \tag{6a}$$

$$\hat{\sigma}_{l-}|j,m\rangle_{2j}|-\rangle_{i_1}\cdots|-\rangle_{i_{N-2j}} =\sqrt{\frac{j+m}{2j}}|j-\tfrac{1}{2},m-\tfrac{1}{2}\rangle_{2j-1}|-\rangle_{i_1}\cdots|-\rangle_{i_{N-2j}}|-\rangle_l \tag{6b}$$

These actions are the actions of the jump operators $\hat{C}_k$. Collective emission jumps preserve the cooperation number for the $2j$ Dicke-state atoms while leaving the labeled atoms $i_1,\ldots,i_{N-2j}$ alone. A noncollective emission jump puts one more atom in the ground state, leaving the remaining $2j-1$ atoms in a Dicke state of reduced cooperation number. Thus, with every noncollective emission one atom is identified (labeled) through spontaneous emission and hence removed from the population of collective atoms. A particular sequence of photon emissions produces a cascade down the ladder of the basis states according to these rules.

The story of the evolution along a quantum trajectory is not yet complete, however. There is also the evolution in between jumps, generated by the non-Hermitian Hamiltonian (4), and while the jumps move the system from one basis state to another, between them the continuous evolution creates a superposition of states; more importantly, new states are brought in from outside the defined basis through pseudospin exchange. To see how this works, we may divide the collective operators $\hat{J}_\pm$ into two pieces, writing

$$\hat{J}_\pm=\hat{L}_\pm+\hat{S}_\pm, \tag{7}$$

where $\hat{L}_\pm$ sums over all atoms that have not yet been identified through spontaneous emission and $\hat{S}_\pm$ sums over the labeled atoms, $i_1,\ldots,i_{N-2j}$. The Hamiltonian is then the sum of two pieces, $\hat{H}=\hat{H}_0+\hat{H}_1$, with

$$\hat{H}_0=-i\frac{\Gamma}{2}(\hat{L}_+\hat{L}_-+\hat{S}_+\hat{S}_-)-i\hbar\frac{\gamma}{2}\sum_{i=1}^{N}\hat{\sigma}_{i+}\hat{\sigma}_{i-}, \tag{8a}$$

$$\hat{H}_1=-i\frac{\Gamma}{2}(\hat{L}_+\hat{S}_-+\hat{L}_-\hat{S}_+). \tag{8b}$$

Now after $n=j-m$ collective emissions and $k-n=N-2j$ noncollective emissions (k emissions in total), we assume the state reached is the one obtained by applying the jump operators alone, i.e., neglecting the evolution in between the jumps; the state is thus assumed to be $|j,m\rangle_{2j}|-\rangle_{i_1}\cdots|-\rangle_{i_{N-2j}}$, which we write, alternatively, as a product of two Dicke states,

$$|j,m\rangle_{2j}|\tfrac{1}{2}N-j,-\tfrac{1}{2}N+j\rangle_{(i_1,\ldots,i_{N-2j})}. \tag{9a}$$

This state is an eigenstate of $\hat{H}_0$; however, it is coupled by $\hat{H}_1$ to

$$|j,m-1\rangle_{2j}|\tfrac{1}{2}N-j,-\tfrac{1}{2}N+j+1\rangle_{\{i_1,\ldots,i_{N-2j}\}}. \tag{9b}$$

At higher orders the pseudo-spin exchange introduces still more Dicke state products.

In this paper, we make the approximation of neglecting the pseudospin exchange. In this way, we reduce the quantum trajectory evolution to a two-channel rate process. Treating $\hat{H}_1$ as a perturbation, one can show that the validity of the approximation requires

$$\frac{\Gamma(n+1)}{\gamma+\Gamma(n+1)}\frac{k-n}{(N-k)(n+1)}\ll 1. \tag{10}$$

The inequality can be satisfied by making Γ/γ arbitrarily small, but this leads to the dominance of noncollective emission, an uninteresting regime. The second ratio of the product is the most important one, and it sets a more and more restrictive condition as the photon-emission sequence progresses. In the interval between the second to last and the very last emission $k=N-1$; here the requirement is $[N-(n+1)]/(n+1)\ll 1$. Thus the majority of the jumps must be collective in order for the approximation to be good throughout the entire decay process; collective emission must be well established and dominate any noncollective emission. This result goes counter to the aim of our paper. Nevertheless, halfway through the decay, with $k=N/2$, the requirement is relaxed to $(N-2n)/[N(n+1)]\ll 1$. With N sufficiently large, this can be satisfied even if a majority of the jumps are noncollective.

We expect, then, that our approximation can describe the initiation phase of superradiance, but will not, in general, be accurate in describing the second half of the collective radiation pulse, unless during the first half the amount of noncollective emission is insignificant. These statements are supported by exact simulations carried out for up to 20 atoms. Results will be presented in a separate publication.

It is interesting to note that Dicke makes precisely the same approximation in Eq. (84) of his paper [1]. There, the conditional state after the emission of $s-1$ photons is expressed in terms of an initial density operator, ρ_0, surrounded by $s-1$ nested jump operators, which account for the sequence of previous emissions. Notably, there is no continuous evolution in between the jumps. The omission is justified in this case under Dicke's assumption of ''widely separated molecules.'' Generally, though, as we see here, superradiance is not describable as a pure rate process. There is in addition this pseudospin exchange in between jumps. It is the most difficult piece of the entire decay process to accommodate.

Accepting the rate approximation leads to a dramatic reduction in the computational resources needed for a numerical simulation of the photon-emission sequences. The algorithm consists of evolving a pure state in time steps, dt, where in a given step, the state either develops under $\hat{B}_0(\tau)=\exp(-i\hat{H}_0\,\tau/\hbar.)$ or a jump (photon emission) of one type or the other takes place. Since the state under this scheme is preserved as an eigenstate of $\hat{H}_0$, $\hat{B}_0(\tau)$ generates a trivial scaling which is removed on renormalization. In effect, there are only the jumps and the evolving rates for the jumps to account for. The branch to be taken in a given time step is decided by comparing a random number with the branching probabilities determined from the record probability (5): the probability for a noncollective emission jump,

$$P_{\text{noncoll}}=(\gamma dt)\sum_{i=1}^{N}\langle\psi_c(t)|\hat{\sigma}_{i+}\hat{\sigma}_{i-}|\psi_c(t)\rangle, \tag{11a}$$

for a collective emission jump,

$$P_{\text{coll}}=(\Gamma dt)\langle\psi_c(t)|\hat{J}_+\hat{J}_-|\psi_c(t)\rangle, \tag{11b}$$

and a probability $P_B=1-P_{\text{noncoll}}-P_{\text{coll}}$ for no effective evolution.

We can explicitly evaluate P_{noncoll} and P_{coll} in terms of the numbers of previous emissions without regard to their order. After k emissions with n of them collective, the state is given by Eq. (9a) and the rates of emission out of this state are

$$R^{k,n}_{\text{noncoll}}=\gamma(N-k) \tag{12a}$$

and

$$R^{k,n}_{\text{coll}}=\Gamma(N-k)(n+1). \tag{12b}$$

The time to the next emission is then determined by sampling the conditional waiting time distribution

$$w(\tau)=(R^{k,n}_{\text{noncoll}}+R^{k,n}_{\text{coll}})\exp[-(R^{k,n}_{\text{noncoll}}+R^{k,n}_{\text{coll}})\tau]. \tag{13}$$

The type of emission is decided by a random selection, either noncollective or collective in the ratio $R^{k,n}_{\text{noncoll}}:R^{k,n}_{\text{coll}}$.

Numerical results obtained with this algorithm are reported in Sec. V. Before discussing them, we derive a difference equation satisfied by the photon counting distribution.

IV. A DIFFERENCE EQUATION FOR PHOTON STATISTICS

If we neglect the pseudospin exchange, the quantum-trajectory formalism leads us directly to a difference equation for the photon number distribution, $P^k(n)$, for n collective emissions in the first k emissions, regardless of the times of the emissions or what happens following the kth emission. The probability is given by summing Eq. (5) over all records having n collective emissions among the first k emissions:

$$P^{(k)}(n)=\sum_{S[n,k]}\int_0^\infty dt_1\cdots \times\int_{t_{k-1}}^\infty dt_k\langle N/2,N/2|\hat{D}^\dagger_{[n,k]}\hat{D}_{[n,k]}|N/2,N/2\rangle, \tag{14}$$

where $\hat{D}_{[n,k]}\equiv\hat{C}_k\hat{B}(t_k-t_{k-1})\cdots\hat{C}_1\hat{B}(t_1)$ and $\Sigma_{S[n,k]}$ denotes the sum over all sequences of n collective jumps among k jumps. The sum over records following the kth emission is already taken in this expression. We substitute $\hat{B}_0(\tau)$ for $\hat{B}(\tau)$ and explicitly write out the sum over the two possibilities for the kth jump, to find

$$P^{(k)}(n)=\gamma(N-k+1)\sum_{S[n,k-1]}\int_0^\infty dt_1\cdots \times\int_{t_{k-1}}^\infty dt_k e^{-\Lambda_n^{k-1}(t_k-t_{k-1})}\langle N/2,N/2|\hat{D}^\dagger_{[n,k-1]} \times\hat{D}_{[n,k-1]}|N/2,N/2\rangle+\Gamma(N-k+1)n$$

$$\times \sum_{S[n-1,k-1]} \int_0^\infty dt_1 \cdots \int_{t_{k-1}}^\infty dt_k e^{-\Lambda_{n-1}^{k-1}(t_k - t_{k-1})}$$
$$\times \langle N/2,N/2|\hat{D}^\dagger_{[n-1,k-1]}\hat{D}_{[n-1,k-1]}|N/2,N/2\rangle, \tag{15}$$

where $\Lambda_n^{k-1}=(N-k+1)[\gamma+\Gamma(n+1)]$ is twice the eigenvalue of the Hamiltonian $\hat{H}_0$ following n collective emissions out of $k-1$ emissions. Performing the t_k integral then yields the difference equation

$$P^{(k)}(n)=\frac{\gamma}{\gamma+\Gamma(n+1)}P^{(k-1)}(n)+\frac{\Gamma n}{\gamma+\Gamma n}P^{(k-1)}(n-1). \tag{16}$$

Equation (16) is similar to an equation given by Lee [28], who has also made an analysis of superradiance with noncollective emission processes included. Lee's analysis is more ambitious as it deals with spatially distributed atoms and no cavity. We are unable, however, to obtain agreement with his rates even in a limiting case. We return to Lee's work in the next section. It will be seen that his solution of the rate equation uses an approximation that renders a detailed comparison moot.

We are able to solve our rate equation iteratively for $k=10^6$ using only moderate computational resources. The formal solution for $k=N$ is

$$P^{(N)}(n)=n!\,\gamma^{N-n}\Gamma^n\frac{1}{\gamma+\Gamma}\sum_{k_2=\max(1,1+n-N)}^{\min(1,n)}\cdots$$
$$\times\sum_{k_N=\max(k_{N-1},n-1)}^{\min(k_{N-1}+1,n)}\left(\prod_{j=2}^{N}\frac{1}{[\gamma+\Gamma(k_j+1)]}\right), \tag{17}$$

where the limits on the sums account for all possible orderings of the emissions.

V. RESULTS

We have generated photon-emission sequences by Monte Carlo simulation based on Eqs. (12a)–(13) and solved Eq. (16) by explicit numerical iteration. We compute a number of statistical properties of the emitted light. Our results demonstrate the presence of additional fluctuations over the single-mode model due to the competition between collective and noncollective decay. We find, in particular, that the decay is primarily noncollective (collective) when $N\Gamma/\gamma$ is sufficiently small (large); there is no sharp transition, however, as at the threshold of a laser. For $\Gamma/\gamma\sim 10^{-3}$, the transition point (point of maximum fluctuations) occurs at $N\Gamma/\gamma\sim 10$, but there is a transition region extending over more than an order of magnitude. The transition region broadens significantly as Γ/γ is reduced although the transition point itself shows only a weak Γ/γ dependence.

First, let us illustrate some of the changes brought to the collective emission pulses within the transition region. Figure 1 illustrates the effect of the noncollective emission in reducing the average intensity, $I(t)/N\Gamma \equiv \langle\psi_c(t)|\hat{J}_+\hat{J}_-|\psi_c(t)\rangle/N$, and increasing the width of the collective emission pulse; the peak of the pulse is also shifted a little. The dashed curve reproduces the result from the single-mode model. It is calculated in our treatment by setting $\gamma=0$. The quantum fluctuations included in the single-mode model already give some reduction in peak intensity compared with the hyperbolic secant pulse obtained in the semiclassical approximation [14,20]; Bonifacio *et al.*, for example, find a 23% reduction for $N=200$ [9]. As curve (ii) shows in particular, far greater reductions can come from the noncollective emission.

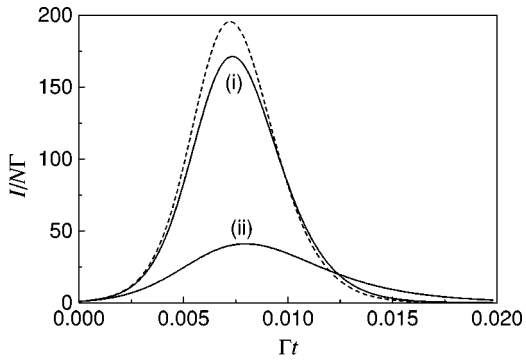

FIG. 1. Mean intensity of the collective emission pulse for $N=1000$ atoms and $\Gamma/\gamma=0.1$ (i) and 0.01 (ii). The dashed curve is the result from the single-mode model ($\Gamma/\gamma\to\infty$).

Going beyond the mean intensity, there are pulse-to-pulse intensity fluctuations and correlated fluctuations in the pulse delay time and width. We illustrate the correlation in Figs. 2 and 3. In Fig. 2, we plot some examples of conditionally averaged pulse intensities with the average conditioned on the total number of collective emissions per pulse; here the distribution in the number of emissions is relatively broad, similar to curve (ii) in Fig. 4. Notice the variations in the time of the peak intensity and the pulse width. Fluctuations of this sort are completely absent in the single-mode model, which always has exactly N collective emissions per pulse.

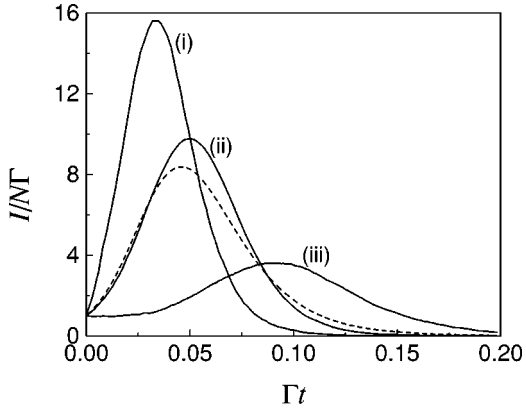

FIG. 2. Mean intensity of some collective emission pulses of given integrated photon number: for $N=100$, $\Gamma/\gamma=0.1$, and given an integrated photon number of $n=75$ (i), 57 (ii), and 31 (iii). The dashed curve is the average over n and the most probable photon number is $n=57$.

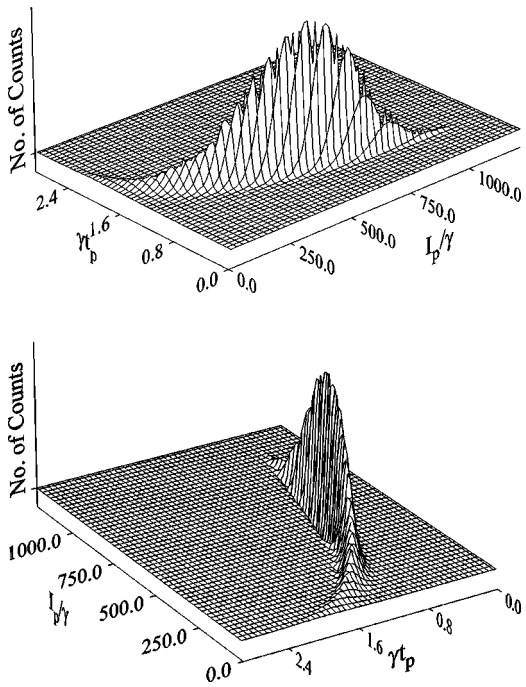

FIG. 3. Correlation between collective pulse delay time t_p and peak intensity $I_p \equiv \Gamma\langle\psi_c(t_p)|\hat{J}_+\hat{J}_-|\psi_c(t_p)\rangle$ for $N=1000$ and $\Gamma/\gamma = 0.01$.

Figure 3 shows the correlation between intensity and delay time in more detail, for a larger number of atoms but the same $N\Gamma/\gamma$. The narrowness of the distribution, shown clearly in the perspective of the lower plot, indicates that the correlation is a strong one, except when the peak intensity is small. Thus, the fluctuations in delay time that track the variation in peak intensity are much larger than the fluctuations at a fixed intensity. A similar correlation can be demonstrated for the pulse width, with the width and delay time linearly correlated.

The photon-number fluctuations in the collective pulse provide a global picture of the transition from noncollective to collective emission. In the single-mode model, the collective emission photon-number distribution is a δ function, $P^{(N)}(n)=\delta_{nN}$; since there is only one output channel, every excited atom necessarily contributes one photon to the collective pulse. For our two-channel model, this is no longer the case. When $N\Gamma/\gamma$ is too small, most, if not all, of the

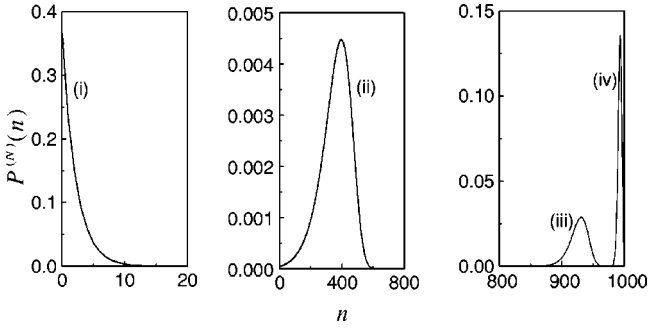

FIG. 4. The collective emission photon number distribution for $N=1000$ and $\Gamma/\gamma=10^{-3}$ (i), 10^{-2} (ii), 10^{-1} (iii), and 1 (iv).

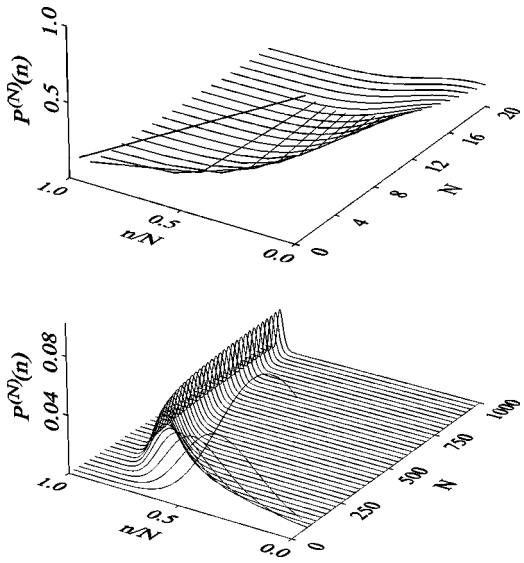

FIG. 5. The collective emission photon-number distribution as a function of N for $\Gamma/\gamma=10^{-1}$. The distribution is discrete and a continuous line connects the discrete points. The first curve in the lower plot is for $N=25$.

photons are emitted as noncollective spontaneous emission. For there to be strong collective emission, $N\Gamma/\gamma$ must be increased. The development of the collective emission photon-number distribution with increasing Γ/γ, at fixed $N=1000$, is shown in Fig. 4. We observe a transition from a sharp distribution peaked around $n=0$ through a series of broad distributions peaked at increasing values of n to a sharp distribution once again peaked at a value of n close to N. Previous treatments of quantum fluctuations in superradiance assume the extreme values of $N\Gamma/\gamma$ that take $P^N(n)$ towards the delta function δ_{nN} [8–13].

The transition region is conveniently traced by solving Eq. (16) for fixed Γ/γ. The solution $P^{(k)}(n)$ is the final collective emission photon-number distribution for $N=k$ atoms, and also the distribution reached after k emissions—of either type—for any N larger than k. We show one result obtained by numerically iterating Eq. (16) in Fig. 5. The transition, illustrated in Fig. 4 as a function of Γ/γ, at fixed N, appears here in similar form as a function of N at fixed Γ/γ.

From the $P^{(N)}(n)$ for various values Γ/γ we obtain the results for the mean photon number and Fano factor ($F=\langle\Delta n^2\rangle/\langle n\rangle$) plotted in Fig. 6. Here we see that the collective emission ''turns on'' where the photon-number distribution is most broad, in all cases around $N\Gamma/\gamma=10$, although there is a weak dependence on Γ/γ. The peak in the Fano factor is, however, extremely broad, and appears to increase inversely with Γ/γ.

We turn finally to a comparison of our results with the one previous treatment we have found of noncollective emission effects in superradiance. We compare with the work of Lee [28,29], which is based on an elaborate calculation of transition matrix elements between Dicke states for the more difficult problem of atoms distributed in free space [32]. The model in this work is also a two-channel model, with the transitions characterized as either r-conserving or

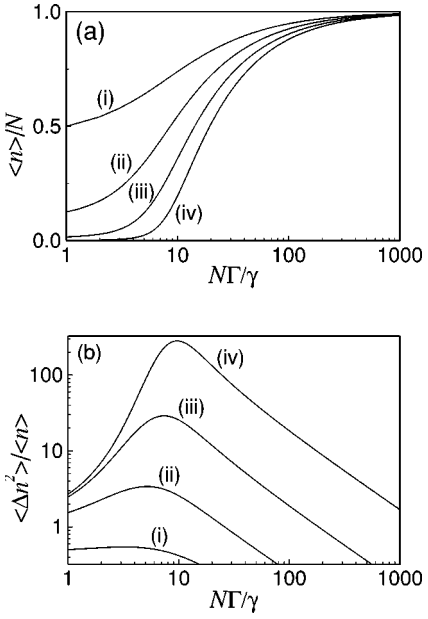

FIG. 6. Mean photon number (a) and the Fano factor (b) for the collective emission, as a function of the number of atoms: for $\Gamma/\gamma=1$ (i), 10^{-1} (ii), 10^{-2} (iii), and 10^{-3} (iv).

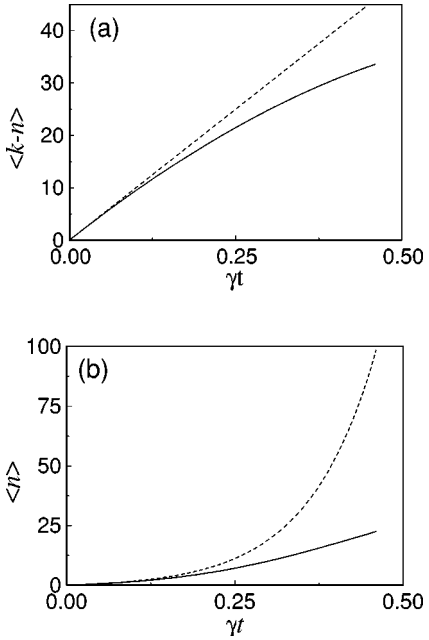

FIG. 7. Mean numbers of noncollective (a) and collective (b) photon emissions as a function of time for $N=100$ and $\Gamma/\gamma=0.1$. The dashed lines show the linear and exponential growth, respectively, from Ref. [29] and the solid lines are the results from our model.

r-nonconserving, where *r* is the cooperation number; there is a loose identification with our collective and noncollective transitions. Lee applies his derived matrix elements to an analysis of the ''transition point of superradiance'' [29].

The transition point considered by Lee is quite different, however, from that shown in Figs. 5 and 6. In fact, Lee makes an approximation that amounts to neglecting the depletion of the atomic excitation. Its effect within our model is to replace $\sqrt{(j+m)(j-m+1)}=\sqrt{(N-k)(n+1)}$ by $\sqrt{N(n+1)}$ in Eq. (6a), and $\sqrt{(j+m)/2j}=\sqrt{(N-k)/N}$ by unity in Eq. (6b). This results in an elementary description in which the average number of noncollective emissions grows linearly in time,

$$\langle k-n\rangle = N\gamma t, \tag{18}$$

while there is an exponential growth,

$$\langle n\rangle = e^{N\Gamma t}-1, \tag{19}$$

in the number of collective emissions. Thus, in Lee's model the collective emission will always dominate for long enough times; his ''transition point'' refers to the time at which the collective emission becomes dominant. We, on the other hand, treat the competition between collective and noncollective emission without any presumption that the former will ultimately dominate.

In Fig. 7, we compare the behavior given by Eqs. (18) and (19) with our results for $N=100$ atoms and $\Gamma/\gamma=0.1$. There is good agreement for short times, as one would expect. Lee's approximation cannot, however, describe the phenomenon of a superradiant *pulse*, nor can it describe the fluctuations in pulse characteristics illustrated in Figs. 1–6.

VI. CONCLUSIONS

We have investigated a model of the initiation of superradiance in a cavity in which we account for the effects of noncollective spontaneous emission. Formulating the problem within quantum trajectory theory led us to the approximation of neglecting pseudospin exchange. We thus reduced the stochastic Schrödinger evolution to a two-channel rate process, and efficient Monte Carlo simulations of the resulting photon-emission sequences have been carried out.

Our approach proves the foundation of an approximation that can be traced back to the earliest papers on superradiance. We plan to develop it in future work into a general multichannel scheme for treating arbitrary free-space distributions of atoms.

In the present context, we have computed photon-number distributions for the collective emission pulses and shown how superradiance turns on as the number of atoms is increased. We found that there is a broad transition region throughout which large fluctuations occur. We calculated the Fano factor to characterize the fluctuations in photon number and also illustrated the correlations that exist between pulse photon number, or intensity, and pulse delay time and width.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Grant No. PHY-0099576. The authors wish to thank Hyunchul Nha for helpful discussions.

[1] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[2] N. Bloembergen and R. V. Pound, Phys. Rev. **95**, 8 (1954).
[3] S. Bloom, J. Appl. Phys. **27**, 785 (1956); **28**, 800 (1967).
[4] R. Bonifacio and L. A. Lugiato, Phys. Rev. A **11**, 1507 (1975).
[5] R. G. DeVoe and R. G. Brewer, Phys. Rev. Lett. **76**, 2049 (1996).
[6] M. Gross and S. Haroche, Phys. Rep. **93**, 302 (1982).
[7] R. Bonifacio, P. Schwendimann, and F. Haake, Phys. Rev. A **4**, 302 (1971).
[8] R. Bonifacio and G. Preparata, Phys. Rev. A **2**, 336 (1970).
[9] R. Bonifacio, P. Schwendimann, and F. Haake, Phys. Rev. A **4**, 854 (1971).
[10] V. Degiorgio and F. Ghielmetti, Phys. Rev. A **4**, 2415 (1971).
[11] F. Haake and R. J. Glauber, Phys. Rev. A **5**, 1457 (1972).
[12] L. M. Narducci, C. A. Coulter, and C. M. Bowden, Phys. Rev. A **9**, 829 (1974).
[13] R. Glauber and F. Haake, Phys. Rev. A **13**, 357 (1976).
[14] J. H. Eberly and N. E. Rehler, Phys. Lett. **29A**, 142 (1969).
[15] N. E. Rehler and J. H. Eberly, Phys. Rev. A **3**, 1735 (1971).
[16] A. A. Belavin, B. Ya. Zeldovich, A. M. Perelomov, and V. S. Popov, Zh. Eksp. Teor. Fiz. **29**, 145 (1969) [Sov. Phys. JETP **56**, 264 (1969)].
[17] R. H. Lemberg, Phys. Rev. A **2**, 883 (1970).
[18] G. S. Agarwal, Phys. Rev. A **2**, 2038 (1970).
[19] H. J. Carmichael and K. Kim, Opt. Commun. **179**, 417 (2000).
[20] J. H. Eberly, Am. J. Phys. **40**, 1374 (1972).
[21] R. Bonifacio and L. A. Lugiato, Phys. Rev. Lett. **40**, 1023 (1978).
[22] L. M. Narducci, R. Gilmore, D. H. Feng, and G. S. Agarwal, Opt. Lett. **2**, 88 (1978).
[23] H. J. Carmichael, D. F. Walls, P. D. Drummond, and S. S. Hassan, Phys. Rev. A **27**, 3112 (1983).
[24] M. Gross, P. Goy, C. Fabre, S. Haroche, and J. M. Raimond, Phys. Rev. Lett. **43**, 343 (1979).
[25] J. M. Raimond, P. Goy, M. Gross, C. Fabre, and S. Haroche, Phys. Rev. Lett. **49**, 1924 (1982).
[26] Y. Kaluzny, P. Goy, M. Gross, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **51**, 1175 (1983).
[27] C. Greiner, B. Boggs, and T. W. Mossberg, Phys. Rev. Lett. **85**, 3793 (2000).
[28] C. T. Lee, Phys. Rev. A **13**, 1657 (1976).
[29] C. T. Lee, Phys. Rev. A **14**, 1926 (1976).
[30] P. Schwendimann, Z. Phys. **265**, 267 (1973).
[31] H. J. Carmichael, *An Open Systems Approach to Quantum Optics*, Lecture Notes in Physics: New Series m: Monographs, Vol. m18 (Springer, Berlin, 1993).
[32] C. T. Lee, Phys. Rev. A **12**, 575 (1975).