Unexpected role of excess noise in spontaneous emission

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A single inverted two-level atom is used as a theoretical model for a quantum noise detector to investigate fundamental properties of excess noise in an unstable optical resonator. For a symmetric unstable spherical mirror cavity, we develop an analytic quantum description of the field in terms of a complete set of normalizable biorthogonal quasimodes and adjoint modes. Including the interaction with a single two-level atom leads to a description analogous to the Jaynes-Cummings model with modified coupling constants. One finds a strong position and geometry-dependent atomic decay probability proportional to the square root $\sqrt{K}$ of the excess noise factor K at the cavity center. Introducing an additional homogeneous gain one recovers the K-fold emission enhancement that has been predicted before for the linewidth of an unstable cavity laser. We find that excess noise may be viewed as a spatial redistribution of the field quantum noise inside the resonator. Taking a position average of the atomic decay rate over the cavity volume leads to a cancellation of the excess noise enhancement.

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I. INTRODUCTION

The phenomenon of excess noise inside unstable resonators was first predicted by Petermann [1] and soon after, experimentally observed by measuring the enhancement of the laser linewidth [2]. The Petermann excess noise factor (K factor) was introduced to quantify the discrepancy between the expected linewidth using the Schawlow-Townes formula [3,4] and the experimentally measured linewidth. Especially in high-gain unstable lasers or in semiconductor lasers, K reaches considerable values [5,6]. A general formula connecting the K factor with the nonorthogonality of the effective oscillating laser modes was given by Siegman [7] 12 years ago. The interpretation and derivation of this formula has led to substantial controversies [8], as a simple and convincing physical picture and a clear mathematical justification of Siegman's rule was missing.

Experimentally, the validity of this rule has recently been extensively tested in a series of beautiful experiments by Woerdman and other groups [9,10]. Particularly high K factors, strongly dependent on mirror size and shape, were predicted and experimentally found for transversely unstable resonators. These findings triggered renewed theoretical interest. In a recent paper, Poizat and co-workers [11] pointed out that some of the properties of excess noise in a linear amplifier may be mimicked in a simple three-mode quantum input-output model. Later, they generalized this to a larger but finite set of coupled modes [12]. In an alternative approach based on a formal field mode expansion (modes of the universe approach), Bardroff and Stenholm found a close connection between the amount of excess noise and the difference between the spatial distributions of the gain and loss [13]. Up to some small corrections, they could also reproduce Siegman's predictions and extend their model to a nonlinear gain medium [14]. For a Fabry-Perot resonator, the longitudinal dynamics have been studied in great detail [15]. All of these approaches are, however, impractical to apply to the transverse dynamics of a geometrically unstable cavity, where particularly high K factors are found. The microscopic physical origin of the large linewidth of an unstable cavity laser thus remained unclear.

Following the standard phase diffusion model to derive the laser linewidth, the origin of the enhanced linewidth is attributed to increased spontaneous emission [1,7] of the active atoms. Alternatively amplified spontaneous emission by the gain medium [8] can explain the origin of this extra noise. In order to trace the origin of excess noise to its roots, we have reduced the system to a single, inverted two-level atom as a quantum noise detector in an unstable resonator and calculated the spontaneous emission rate [16] to lowest-order perturbation theory. In an analogy to standard cavity QED models, we derived a series expression in terms of cavity quasimodes. Taking only a single mode into account, one indeed finds a K-fold enhancement of the spontaneous emission rate into this mode. However, a subsequent and more extensive analysis of our quantum model shows the invalidity of this truncation for most physically relevant cases.

In this paper, we investigate the dynamics of a single atom in an unstable cavity in much more detail and derive an expression of wider validity for the atomic decay rate, which in many cases yields different predictions. Besides testing the limits of our and other previous treatments, the present approach also allows for a continuous transition from a stable to an unstable cavity configuration. We may also independently vary the aperture of the system, which enables us to study the origins and magnitude of quantum noise in the system in more detail. We may also directly connect the results to well-proven standard cavity QED treatments. In order to simplify the expressions and concentrate on the main effect, we make the further approximation of taking only resonator modes into account in our model. In a realistic setup, they are only responsible for a part of the spontaneous decay rate, which depends on the chosen geometry (i.e., the resonator volume and the solid angle covered by the cavity field as compared to 4π). While in macroscopic cavities this angle is normally rather small, it can be large or dominant in microscopic structures. In addition to compare the two contributions, besides the cavity geometry, the finesse of the resonator also turns out to be important [17]. Nevertheless, the omitted term is purely geometric and will be approximately constant, only weakly depending on parameters such as mirror curvature and mirror reflectivity. Hence, for the basic understanding of excess noise, this simplification will be unimportant, although it could possibly mask the excess noise effect in a practical experimental setup.

In lasers, excess noise is usually an unwanted feature that increases the laser linewidth. However, our goal here is not to find the best situations to avoid or enhance it, but to study its basic properties and trace its origin. Nevertheless, one could envisage some applications. For example, the effect of an enhanced atomic decay rate could prove very useful in situations where a fast and efficient spontaneous decay into a certain direction is desirable, as in, e.g., increasing the efficiency of light-emitting diode (LED's) or other luminescent devices. Another possibility is that enhancing the efficiency of fluorescence single-atom detection could be of practical importance. Note again, however, that the main goal of this paper is theoretical investigation and we have chosen a configuration that does not show the most spectacular values of excess noise, but that allows an analytical treatment to a large extent. In addition, our investigations may be used as a starting point to examine the role of excess noise in other quantum noise-driven processes as in, e.g., parametric down conversion.

In principle, the method used to calculate the spontaneous emission rate is straightforward. One merely has to quantize the field with the proper boundary conditions and apply perturbation theory in analogy to the derivation of Fermi's golden rule to obtain the transition probability. In practice, however, the central mathematical problem is to find a proper and useful quantum description of the electromagnetic field in a finite-sized unstable cavity, as there exists no orthonormal set of eigenmodes with the necessary boundary conditions. This is related to the fact that a geometrical optics description of such systems involves light rays escaping to infinity after only a finite number of reflections.

As mentioned above, one way to avoid this problem is to put the whole system in a huge box and expand all fields in terms of the box eigenmodes [13], often called modes of the universe [18]. It is, however, practically almost impossible to actually solve the resulting coupled equations with proper boundary conditions in a sufficiently large volume. In addition, the physical interpretation of the results obtained in this way is not transparent. For a stable 1D resonator, a strongly position-dependent spontaneous emission rate was found numerically by Bužek and co-workers [19]. Here, we choose an alternative approach in terms of effective quasimodes, i.e., field configurations that are self reproducing (up to a global factor) after one cavity round trip. Unfortunately, as mentioned above, these quasimodes are complete but not orthogonal, which raises many questions in the development of a corresponding quantum model as the associated operators will not commute in a canonical way. What is the meaning of ‘‘photons/vacuum fluctuations’’ in such modes and what is their intensity? Is it possible to reduce the system to a single-effective quasimode? How is the excess noise connected to the spontaneous emission rate?

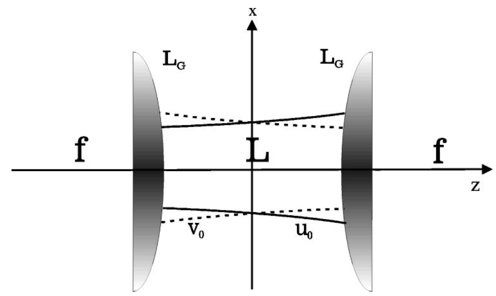

FIG. 1. Scheme of the cavity setup.

In Sec. II, we review the empty cavity field dynamics within the paraxial approximation [20,21] for a symmetric, unstable two-mirror cavity in one transverse dimension (cylindrical mirrors). Effective apertures are modeled by introducing mirrors with a Gaussian transverse reflectivity profile. Fortunately, in this case, it is possible to analytically calculate the *normalizable* quasimode functions with well-defined frequencies and loss rates. In Sec. III, we turn to a quantum description in terms of these modes. As we are dealing with a lossy (open) system, a proper quantum description requires the inclusion of an external reservoir. A modified version of the Jaynes-Cummings Hamiltonian, where the intracavity field is coupled to a two-level atom [22], is derived in Sec. IV, and the consequences of the Petermann K factor appearing in this Hamiltonian are demonstrated using the example of the atomic decay probability. Finally, we try to connect our results to known cases such as a stable cavity or a laser [1,7].

II. MODES OF SYMMETRIC CAVITIES

In order to develop a consistent physical theory for unstable optical resonators interacting with atoms we will restrict ourselves to the simplest system that demonstrates the essential properties. With regard to analytical solvability, we consider first a 1D resonator with length L and two symmetric mirrors of focal length f, as depicted in Fig. 1. Surprisingly, as we will show below, one still finds normalizable finite-sized modes for unstable systems if the mirrors are assumed to have a Gaussian reflectivity profile with width L_G. The slowly varying amplitude of the field modes [34] calculated in the paraxial approximation reads [21]

$$\begin{aligned} u_n(x,z) = c_n \sqrt{\frac{w(0)}{w(z)}} \left(\frac{w(-z)}{w(z)}\right)^{n/2} \\ \times \exp\left\{ i\left[\frac{n+1}{2}\Psi(z) - \frac{n}{2}\Psi(-z) \right.\right. \\ \left.\left. - \frac{1}{2}\Psi(0)\right]\right\} H_n[f(z)x] \\ \times \exp\left\{ -\frac{ik}{2R(z)}x^2 - \frac{x^2}{w(z)^2}\right\} \end{aligned} \tag{2.1}$$

with the generalized z-dependent waist function w, radius of curvature R, a transverse scaling p, and Guoy phase Ψ

$$w(z)^2 = \frac{2z_0}{k_n}\left[1+\frac{(r_0+z)^2}{z_0^2}\right], \tag{2.2}$$

$$R(z) = (r_0+z)\left[1+\frac{z_0^2}{(r_0+z)^2}\right], \tag{2.3}$$

$$p(z) = \sqrt{\frac{ik_n/q}{1-z^2/q_0^2}}, \tag{2.4}$$

$$\Psi(z) = \arctan\frac{r_0+z}{z_0}. \tag{2.5}$$

The only remaining free parameter is now the complex source point $q_0 = r_0 + iz_0$, which is directly linked to the cavity parameters. Note that the case $r_0 = 0$ (minimal beam width within the symmetry plane $z=0$) corresponds to an ideal stable mirror configuration ($L/f \epsilon [0,4], L_G \to \infty$) and the modes Eq. (2.1) are merely the well-known Hermite Gaussian beams.

These quasimodes fulfill a self-reproducing condition for one full-cavity round trip, i.e., $u_n(x,2L) = \gamma_n u_n(x,0)$. Since the z propagation of a field is governed by the corresponding Huygens' integral operator [23], these modes are solutions of the eigenvalue problem

$$\int dx' \sqrt{\frac{i}{\lambda B}}\exp\left\{-\frac{ik}{2B}(Ax'^2 - 2x'x + Dx^2)\right\}, \tag{2.6}$$

where the coefficients A, B, and D are determined by the ray matrix for this cavity configuration

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1-2l+l^2/2 & L(2-l)(1-l/4) \\ -(2-l)l/L & 1-2l+l^2/2 \end{pmatrix}. \tag{2.7}$$

Here, we have introduced $l = L/f + i/N$ and the Fresnel number $N = \pi L_G^2/\lambda L$ in correspondence to a hard-edged spherical mirror. In the limiting case of the aperture going to infinity ($N \to \infty$), these modes are no longer normalizable for unstable cavities. (In fact, they correspond to unphysical eigenfunctions of an inverse oscillator potential [24] and a different set of modes has to be used). For any finite Gaussian transverse reflectivity profile of the mirrors (finite N), and in the case of a symmetric ray matrix, the eigenvalue problem can be explicitly solved in this way and yields Eq. (2.1) as eigenfunctions with eigenvalues

$$\gamma_n = (A+\sqrt{A^2-1})^{-(n+1/2)} = \left(\frac{q_0 - L/2}{q_0 + L/2}\right)^{2n+1}. \tag{2.8}$$

The lowest-order eigenvalues for large transverse mirror extensions ($N \to \infty$) are shown in Fig. 2. Within the stable parameter range ($L/f \epsilon [0,4]$), the eigenvalues γ_n are on the complex unit circle, and hence, the field only acquires a phase factor after each round trip. Outside the stable region, the magnitude of γ_n decreases very rapidly. The eigenvalues may now be used to determine the allowed wave numbers as the field $a_n = e^{-ik_n z}u_n$ must be multiplied with a real and positive factor for each round trip to ensure the correct

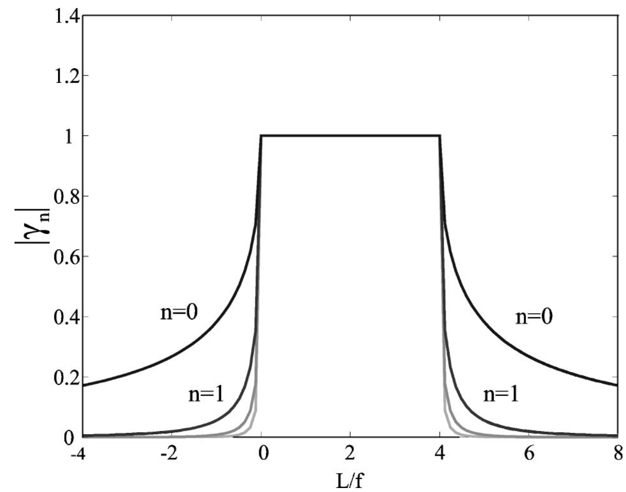

FIG. 2. The first four eigenvalues of Huygens' integral operator for different symmetric cavities, i.e., different values of L/f, and perfectly reflecting mirrors ($N \to \infty$).

boundary conditions at the mirrors. This is equivalent to the condition that the phase difference of the spatial mode between the left and right mirror must be a multiple of π, as in standard cavity calculations. For the allowed wave numbers, we find

$$k_{nm} = \frac{1}{L}\left[m\pi + \left(n+\frac{1}{2}\right)\arg\gamma_0\right]. \tag{2.9}$$

Furthermore, one finds that one complex source point is associated with any set of Hermite-Gaussian modes

$$q_0 = \frac{B}{\sqrt{A^2-1}} = \frac{L}{2}\sqrt{\left(1-\frac{4}{l}\right)}. \tag{2.10}$$

If we split this parameter into a real and imaginary part ($q_0 = r_0 + iz_0$), and making use of the normally large values of N, this result reads up to first order in $1/N$

$$r_0 = \frac{L}{2}\sqrt{\left(1-\frac{4f}{L}\right)}, \tag{2.11}$$

$$z_0 = \frac{f^2}{2Nr_0}, \tag{2.12}$$

within the unstable region. Otherwise r_0 and z_0 become imaginary and thus change their roles for stable cavity configurations. The radius of curvature r_0 indicates the localization of the beam waist (for a negative focal length always outside the cavity), and the Rayleigh length z_0 is related to the spot size that obviously becomes infinite for unstable resonators [$z_0 \to 0 \Rightarrow w(z) \to \infty$]. One may easily verify that the case $r_0 = 0$ is realized in the whole stable regime with ideal mirrors ($N \to \infty$).

Furthermore, the modes $u_n(x,z)$ are complete and biorthogonal to their adjoint modes

$$v_n(x,z)=\tilde{c}_n \sqrt{\frac{w(0)}{w(-z)}}\left(\frac{w(z)}{w(-z)}\right)^{n/2} \times\exp\left\{-i\left[\frac{n+1}{2}\Psi(z)-\frac{n}{2}\Psi(-z)-\frac{1}{2}\Psi(0)\right]\right\} \times\exp\left\{\frac{ikn}{2R(-z)}x^2-\frac{x^2}{w(-z)^2}\right\}. \tag{2.13}$$

The normalization factors c_n and $\tilde{c}_n$ are chosen such that

$$\int dx u_n^*(x,z)u_n(x,z)=1, \tag{2,14}$$

$$\int dx v_n^*(x,z)u_m(x,z)=\delta_{nm}. \tag{2.15}$$

We thus have found a countable and normalizable basis set for our cavity field at the expense of introducing additional losses through finite mirrors. This will of course lead to substantial changes in the quantum model.

We remark here that the Petermann K factor as defined by Siegman [7] is simply given by the norm of the adjoint modes

$$K_n=\int dx v_n^*(x,z)v_n(x,z), \tag{2.16}$$

which is fixed by Eqs. (2.14), (2.15). For symmetric cavities, one may show that the modes are just proportional to the complex conjugate of their adjoint modes, at least at $z=0$: $v_n(x)=e^{i\varphi_n}\sqrt{K_n}u_n^*(x)$, with a given phase φ_n that may be chosen to be zero. This property plays a key role leading to an enhanced atomic spontaneous emission rate in the corresponding quantum model.

Let us now look at some special properties of these modes and compare the decay rate

$$\kappa_n=-\frac{c}{2L}\log|\gamma_n|=\frac{c}{L}\log\frac{w(L/2)}{w(-L/2)}\left(n+\frac{1}{2}\right) \tag{2.17}$$

with the transverse mode spacing

$$\Delta\psi_\perp=-\frac{c}{L}\arg\gamma_0=\frac{c}{L}[\Psi(L/2)-\Psi(-L/2)] \tag{2.18}$$

as is illustrated in Fig. 3. Within the stable regime κ_0 becomes arbitrarily small for large Fresnel numbers whereas $\Delta\omega_\perp$ is localized somewhere between zero (planar mirrors, $L/f=0$) and π (concentric mirrors $L/f=4$). In the unstable regime, the losses become more and more dominant. Furthermore, the edges at the two critical points are washed out for a smaller aperture size but the general dependence is only weakly influenced by the Fresnel number. Note that for an unstable cavity setup, the decay rates for all modes are much larger than their energy separation so that we may consider them degenerate.

For unstable resonators, the mode spacing is approximately zero or equal to the longitudinal mode spacing (π). On the other hand, the loss rate rapidly increases within the unstable region. Thus, a single-mode approximation is reasonable for stable cavities not too close to the planar or concentric case. Near these two degeneracy points (also for unstable cavities) the effective mode spacing is about twice as large as the lowest-loss rate. Except for the lowest longitudinal mode with $k=(\pi+[\Psi(L/2)-\Psi(-L/2)]/2)/L$ in the right unstable regime ($L/f>4$) is really isolated for moderate loss rates since there exists no lower transverse set of modes that could give an additional contribution. Hence, in general, a single-mode treatment for unstable resonators is doubtful due to the strong overlap of the spectral lines. Atoms inside the cavity interact not only with one single-mode/adjoint-mode pair, but are substantially coupled to a whole set of modes. In particular, to calculate a spontaneous emission rate, a large set of modes will turn out to be important. As we will see later, a gain medium as in a laser or amplifier may actively select a single mode, so that a single K may play a dominant role in the dynamics.

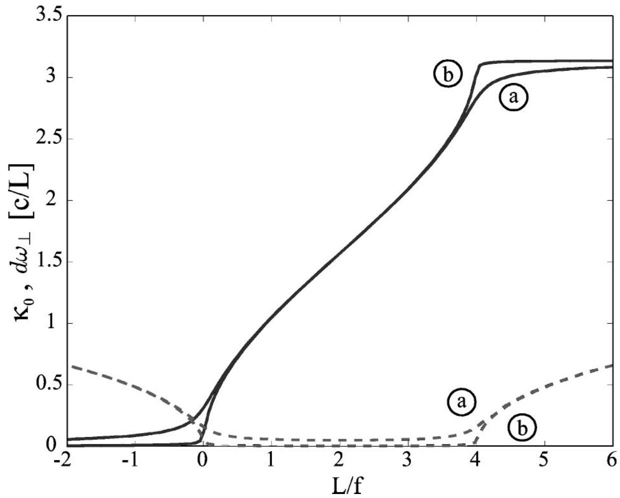

FIG. 3. The loss rate κ_0 (dashed line) and the transverse mode spacing $\Delta\omega_\perp$ (solid line) for different transverse extensions. The Fresnel number is chosen as (a) $N=5$, (b) $N=50$.

III. QUANTUM DYNAMICS IN TERMS OF QUASIMODES

Having, at least in principle, solved the classical problem for unstable optical resonators, we now try to develop an approximate quantum description based on a non-Hermitian cavity QED model. First, let us look for an appropriate Hamiltonian describing the dynamics and derive a corresponding generalized photon concept. We will apply this model to study the interaction of a single atom with the cavity field.

For the free electromagnetic field confined to a volume with partly absorbing boundaries, it is possible to find a complete set of quasimodes $\{u_n(x)\}$, as, e.g., outlined in the previous section for symmetric unstable resonators with a Gaussian reflectivity profile. These modes are not necessarily orthogonal, but are biorthogonal to a second set of adjoint modes $\{v_n(x)\}$, such that $\int dx\ v_n^*(x)u_m(x)=\delta_{nm}$ and $\langle v_n,v_n\rangle=K_n$, with K_n being the Petermann excess noise factor [cf. Eqs. (2.14)–(2.16)]. In the case of symmetric mirror configuration (with respect to forward and backward propagation), the adjoint modes are proportional to the complex conjugates of the cavity modes and may be written as

$$v_n(x)=\sqrt{K_n}u_n^*(x). \tag{3.1}$$

This defines the phase of modes and adjoint modes. Note that for stable cavities, one has $v_n(x)=u_n(x)$ and $K_n=1$. Since these mode pairs fulfill a completeness relation

$$\sum_n v_n^*(x)u_n(x')=\delta(x-x'), \tag{3.2}$$

in principle, every field distribution may be expanded uniquely either in the modes or in the adjoint modes. For our purpose, we expand the field operators in the following way:

$$A(x,t)=\sum_n \sqrt{\frac{\hbar}{2\epsilon_0\omega_n}}[a_n(t)u_n(x)+b_n^\dagger(t)v_n^*(x)], \tag{3.3}$$

$$E(x,t)=i\sum_n \sqrt{\frac{\hbar\omega_n}{2\epsilon_0}}[a_n(t)u_n(x)-b_n^\dagger(t)v_n^*(x)], \tag{3.4}$$

where

$$a_n(t)=-i\sqrt{\frac{\epsilon_0}{2\hbar\omega_n}}\int dx v_n^*(x)[E(x,t)+i\omega_n A(x,t)], \tag{3.5}$$

$$b_n^\dagger(t)=i\sqrt{\frac{\epsilon_0}{2\hbar\omega_n}}\int dx u_n(x)[E(x,t)-i\omega_n A(x,t)] \tag{3.6}$$

are generalized creation or annihilation operators for the corresponding mode/adjoint mode pairs. This becomes obvious if we rewrite the canonical commutation relations [25] for the field in the form

$$[a_n,b_m^\dagger]=\frac{\omega_n+\omega_m}{2\sqrt{\omega_n\omega_m}}\int dx\, v_n^*(x)u_m(x)=\delta_{nm}, \tag{3.7}$$

$$[a_n,a_m^\dagger]=\frac{\omega_n+\omega_m}{2\sqrt{\omega_n\omega_m}}\int dx\, v_n^*(x)v_m(x)\approx B_{nm}, \tag{3.8}$$

$$[b_n,b_m^\dagger]=\frac{\omega_n+\omega_m}{2\sqrt{\omega_n\omega_m}}\int dx\, u_n^*(x)u_m(x)\sim A_{nm}, \tag{3.9}$$

where A_{nm} and B_{nm} are the overlap matrices between each of the cavity modes, respectively, the adjoint modes. Note that the commutation relation for a single-mode pair reads $[b_n,b_n^\dagger]=1$ and $[a_n,a_n^\dagger]=K_n$. We could stop at this point and calculate the noise of a single-mode field distribution $X=a_nu_n(x)+a_n^\dagger u_n^*(x)$, as is also shown in [13], and immediately find an excess noise enhancement: $\langle\Delta X\rangle=\langle[a_n,a_n^\dagger]\rangle\,|u_n|^2=K_n|u_n|^2$. But as we will see, a gain medium, which allows us to consider a single mode separately, is a necessary condition for this result. Interestingly, one may not directly conclude from the excess noise enhanced-laser linewidth an enhancement of the spontaneous emission rate of the individual atoms. Usually various quasimodes may interfere and the effect of the K factor on the spontaneous emission rate is more subtle than is usually assumed.

We should mention here that this field expansion is only exact for frequency degeneracy $\omega_n=\omega_m$ (also mentioned in [13]), because otherwise the biorthogonality and completeness relations between the cavity modes and adjoint modes are no longer true. These relations are generally given for the eigenfunctions of Huygens' integral operator at *one* distinct frequency. In fact, we are dealing with many operators (many frequencies) where, respectively, only one (no degeneracy) mode pair survives due to the boundary conditions. But for a huge range of physically relevant resonators (e.g., in the infrared or optical domain) the frequency differences are negligibly small compared to their absolute value. In addition, in the limit of a large transverse extension of the mirrors of an unstable resonator, this frequency degeneracy is exactly fulfilled.

Using this field expansion and assuming that the mode functions identically fulfill the Helmholtz equation (paraxial approximation), one may write the free-field Hamiltonian in a very canonical form [35]

$$H_F=\frac{1}{2}\int dx{:}\left(\epsilon_0E^2(x,t)+\frac{1}{\mu_0}B^2(x,t)\right){:} \tag{3.10}$$

$$=\sum_n \hbar\omega_n b_n^\dagger a_n. \tag{3.11}$$

For unstable systems where $v_n\neq u_n$ and hence, $a_n\neq b_n$ the individual contributions to this Hamiltonian are obviously no longer explicitly Hermitian, but with these definitions the non-Hermitian parts cancel approximately within the sum, since the overlap matrices in Eqs. (3.8), (3.9) are inverses as a consequence of the completeness relation, i.e., $\Sigma_kA_{nk}B_{km}=\Sigma_kB_{nk}A_{km}=\delta_{nm}$. This formally Hermitian nature gives rise to a degeneracy between the left and right eigenstates

$$|n_1,n_2,\ldots\rangle,=\frac{b_1^{\dagger n_1}}{\sqrt{n_1!}}\frac{b_2^{\dagger n_2}}{\sqrt{n_2!}}\ldots|0\rangle, \tag{3.12}$$

$$\langle\widetilde{n_1,n_2}|,=\langle 0|\ldots\frac{a_2^{n_2}}{\sqrt{n_2!}}\frac{a_1^{n_1}}{\sqrt{n_1!}}, \tag{3.13}$$

which are biorthogonal to each other, in the sense that

$$\langle\tilde{n}|m\rangle=\delta_{nm}, \tag{3.14}$$

where n,m is shorthand for $\{n_1,n_2,\ldots,\},\{m_1,m_2,\ldots,\}$. These eigenstates are the non-Hermitian analogous to the n-photon Fock states containing the energy quanta $E_n=\hbar(\omega_1n_1+\omega_2n_2+\cdots)$. These eigenstates are not mutually orthogonal for the standard scalar product (SP) $\langle\cdot|\cdot\rangle$. We would like to mention that it is possible to find a different

SP $(\cdot|\cdot)$ such that the eigenstates are mutually orthogonal and the corresponding adjoint operation $\sim$ has the property that

$$\tilde{a}_n = b_n^\dagger, \tag{3.15}$$

which means that $(\cdot|a_n\cdot) = (b_n^\dagger\cdot|\cdot)$. But one has to be careful, since this SP operates also in position space. This means that a Hermitian operator [with respect to $(\cdot|\cdot)$!] may yield complex eigenvalues. Only by integration over position space does one get real eigenvalues. Explicitly, the adjoint relation $\sim$ maps $a_n \to b_n^\dagger$ and simultaneously $u_n \to v_n^*$. A further convenient consequence of the SP is that the left eigenstates become identical to the adjoint right eigenstates, i.e.,

$$\langle n_1, \widetilde{n_2}, \ldots, | = (n_1, n_2, \ldots, |. \tag{3.16}$$

The Hamiltonian could also be rewritten in a formally more symmetric form. Nevertheless, we will retain the standard notation with the asymmetric SP to maintain visible the important differences between the stable and unstable geometry.

The free dynamics governed by the above Hamiltonian with its eigenstates and energies may now be formally written down in the usual manner. However, as we are dealing with a lossy system, the mode amplitude decays exponentially with a mean rate κ_n [Eq. (2.17)]. Physically, a fraction of the energy is scattered into the continuum modes outside the cavity, which in a proper quantum treatment, has to be included by an input-output coupling [13,26]. However, the procedure in this case is rather involved, since the diffraction losses are indistinguishable in this picture from the losses due to mirror transmission. (Even for perfect mirrors the loss rate is still finite.) Although an exact derivation is, to our knowledge, not yet known [15], the free-field dynamics may be consistently described by the following master equation

$$\dot{\rho} = \frac{-i}{\hbar}(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger) + i\sum_{nm} A_{nm}(\tilde{\omega}_m - \tilde{\omega}_n^*) a_m \rho a_n^\dagger, \tag{3.17}$$

where we have introduced complex frequencies $\tilde{\omega}_n = \omega_n - i\kappa_n$ and an effective Hamiltonian [27] including the damping

$$H_{\text{eff}} = \hbar \sum_n \tilde{\omega}_n b_n^\dagger a_n. \tag{3.18}$$

To guarantee self-consistency the obtained master equation is of the Lindblad form, preserves the trace of the density operator, preserves the commutation relations for all mode operators (as for a_n, $b_n^\dagger$), and guarantees the damped oscillation of a_n and a_n known from the classical model, i.e.,

$$\langle \dot{a}_n \rangle = -(\kappa_n + i\omega_n)\langle a_n \rangle, \tag{3.19}$$

$$\langle \dot{a}_n^\dagger \rangle = -(\kappa_n - i\omega_n)\langle a_n^\dagger \rangle. \tag{3.20}$$

IV. ATOMIC DYNAMICS IN UNSTABLE CAVITIES

A. Generalized Jaynes-Cummings model

Finally, let us now introduce an atom interacting with the intracavity electromagnetic field. This is described in a canonical way by a minimal coupling Hamiltonian

$$H = \sum_n \hbar\omega b_n^\dagger a_n + \frac{1}{1m}[P - qA(X)]^2 + V(X), \tag{4.1}$$

where m,q are the atomic mass and charge and $V(X)$ gives rise to the internal atomic structure. For quantum optical applications, this expression may be substantially simplified by making various approximations. Usually they are known as the ‘‘dipole approximation,’’ the ‘‘rotating wave approximation’’ and the ‘‘two-level approximation.’’ This procedure has been extensively discussed in literature (see, e.g., Ref. [28]). Hence, we will reduce the following summary to the physical motivations of these approximations and investigate them with respect to the applicability to a nonorthogonal quasimode description. The dipole approximation makes use of the different length scales of an optical wavelength (typical 100 nm) and the atomic size (typical 1 Å). This argument is of course completely unaffected by the changed mode properties. Hence, the field may be treated as approximately constant when evaluated at the position of the atom. Taking into account that the atomic momentum may be transformed in terms of the position operator, i.e.,

$$\frac{P}{m} = \frac{i}{\hbar}[H_A, X], \tag{4.2}$$

it is easy to see that essentially only the atomic dipole moment $d = qX$ survives within this approximation, i.e.,

$$H_d \sim -d \cdot E(X). \tag{4.3}$$

The rotating-wave approximation reduces the tractable processes close to resonance. Here, the field frequencies ω_n are of the same order as the atomic transition frequencies ω_A. Within the interaction picture terms containing both atomic and field excitations (or de-excitations, respectively) are oscillating as $e^{\pm i(\omega_n + \omega_A)}$. Compared to the time-average effect of processes where energy quanta are transfered from the atom to the field (or vice versa), oscillating as $e^{\pm i(\omega_n - \omega_A)}$, these processes may be neglected. Once again, this approximation does not make use of the orthogonality of the field modes and is hence applicable for our purposes. At last, reducing the atom to two significant levels, g and e, separated by ω_A, this extended formalism gives rise to a Hamiltonian very similar to a multimode Jaynes-Cummings-model

$$H_{AF} = \sum_n \hbar\omega_n b_n^\dagger a_n + \frac{\omega_A}{2}\sigma_z - i\hbar\sum_n (g_n\sigma_+ a_n - \tilde{g}_n b_n^\dagger \sigma_-). \tag{4.4}$$

Formally, everything looks completely familiar except that for the coupling, we find $\tilde{g}_n \neq g_n^*$, or explicitly,

$$g_n = \sqrt{\frac{\omega_n}{2\hbar\epsilon_0}} u_n \cdot d_{eg}; \quad \tilde{g}_n = \sqrt{\frac{\omega_n}{2\hbar\epsilon_0}} v_n^* \cdot d_{eg}, \quad (4.5)$$

with d_{eq} being the atomic dipole matrix element. Again, in the special case of stable cavities, we have $v_n = u_n$ and $\tilde{g}_n = g_n^*$. For symmetric unstable cavities, we have $v_n = \sqrt{K_n} u_n^*$ and $\tilde{g}_n = \sqrt{K_n} g_n$, which invokes the K factors. We should remark here that this Hamiltonian, although not appearing to be formally Hermitian, again is in practice, since $\Sigma_n \tilde{g}_n b_n^\dagger \equiv \Sigma_n g_n^* a_n^\dagger$. We prefer the given form because it clearly shows the asymmetry between photon creation and annihilation in the case of nonorthogonal modes.

B. Spontaneous emission: perturbative approach

Let us now explore the dynamics of the modified atom-field coupling. We will use an initially excited atom as a quantum noise detector [29] and calculate the spontaneous emission rate. As the field is composed of a large set of nonorthogonal modes, the significance of a single-photon state is not completely obvious. Hence, we will use an operational definition of the spontaneous emission rate. We assume a single-excited atom and set the cavity field to the (unique) vacuum state as a well-defined initial condition. Starting from this state, we then calculate the probability $p(t)$ for the atom to be in the ground state, which is equivalent to having emitted a photon into the field as we have no nonradiative decay in our model. The derivative of this probability may then be used to define a decay rate. This way we have a clear definition of a spontaneous emission rate irrespective of the momentary state and the dynamics of the field. Of course, in general, we will not simply find a linear time dependence of $p(t)$ (indeed it may even be nonmonotonic), as is well known from the standard Jaynes Cummings model. This problem may, however, be neglected for short enough times, where we may use perturbation theory to evaluate the transition rate. As mentioned above, we are only considering the emission rate into the cavity modes and neglect other transverse modes, so that we miss an approximately constant background contribution to the transition rate that may even dominate the effect. Nevertheless, for our purposes of discussing the effect of excess noise on the decay rate, these modes are not relevant.

To carry out the calculations, we define a time-dependent interaction Hamiltonian [27] including the dissipative part of the time evolution

$$H_{Int}(t) = -i\hbar \sum_n \left(g_n \sigma_+ a_n e^{-(\kappa_n + i\Delta_n)t} - \tilde{g}_n \sigma_- b_n^\dagger e^{-(\kappa_n - i\Delta_n)t} \right), \quad (4.6)$$

where $\Delta_n = \omega_n - \omega_A$ denotes the detuning of the nth mode from the atomic frequency.

Using first-order time-dependent perturbation theory (e.g., [30]) the initial state $|0,e\rangle$, with the atom being excited and the field in the vacuum state, evolves as follows:

$$|\Psi(t)\rangle = \frac{-i}{\hbar} \int_0^t dt' H_{Int}(t') |0,e\rangle \quad (4.7)$$

$$= \sum_n \tilde{g}_n \delta_n(t) |1_n, g\rangle, \quad (4.8)$$

with

$$\delta_n(t) = \frac{e^{-(\kappa_n - i\Delta_n)t} - 1}{\kappa_n - i\Delta_n}. \quad (4.9)$$

It is easy to see that the amplitude of each contribution is enhanced by the excess noise factor since $\tilde{g}_n = \sqrt{K_n} g_n$. The probability $p(t)$, for the excited atom to make a transition to the ground state at a time t, is given by the expectation value of the projector $P_g = |g\rangle\langle g|$, yielding

$$p(t) = \sum_{nm} \tilde{g}_n \delta_n(t) A_{mn} \tilde{g}_m^* \delta_m^*(t) \quad (4.10)$$

$$= \sum_{nm} g_n \delta_n(t) B_{nm} g_m^* \delta_m^*(t). \quad (4.11)$$

The spontaneous emission rate in this operational definition is then

$$2\gamma = \dot{p}(t \to 0). \quad (4.12)$$

Considering the field dynamics, the corresponding one-photon projector would be $P_1 = \Sigma_n |1_n\rangle\langle \widehat{1_n}|$, rather than $W_1 = \Sigma_n |1_n\rangle\langle 1_n|$ or $\tilde{W}_1 = \Sigma_n |\widehat{1_n}\rangle\langle \widehat{1_n}|$. For our system where only one excitation is present, both approaches are equivalent, i.e., $\langle P_1\rangle = \langle P_g\rangle$, and represent an independent test of self consistency. Note that we have $P_1^2 = P_1$ and $[H_F, P_1] = 0$, so that P_1 has to be identified with the proper projector to the one-photon subspace. Of course, to get the probability for a photon in a particular field mode, one has to calculate $|\langle 1_n|\Psi(t)\rangle|^2$ (compare W_1), as in the case of orthogonal modes. However, due to nonorthogonality, this state also implies finite amplitudes for the photon being in different quasimodes as well.

Obviously each term of Eq. (4.11) is enhanced by excess noise ($B_{nm} = \sqrt{K_n} A_{mn} \sqrt{K_m}$). However, in general, the various amplitudes interfere and the result is different to the naive expression used in [16], where we summed independently over all possible one-photon states. An alternative expression for this probability was implicitly derived earlier by Siegman [36].

Evaluating this double sum, we may now calculate the atomic decay probability and its derivative, the spontaneous emission rate. Unfortunately, for significantly unstable geometries (large K), this sum is very hard to evaluate numerically, since the individual terms first grow exponentially with the mode index (n,m) and have a rapidly changing phase (similar to the series expansion of e^{-x} for $x \gg 1$). Although one may show that the sum converges in principle, in practice, this requires summing very large complex numbers with strongly varying phase. For large $K \gg 1$, this requires very high-numerical accuracy and a very large number of terms for convergence, but it may be done for small values of K. Fortunately, it turns out that for a given finite number of modes using the numerically inverted matrix A^{-1} is more adequate than the direct analytical calculation of B. Nevertheless, the need to invert the overlap matrix A correctly strongly limits the tractable region of parameters. Similar behavior was found quite generally by Siegman [31] for expanding a given field distribution in a biorthogonal set of modes [cf. Eqs. (3.3), (3.4)]. Using the matrix A^{-1} rather than B gives the field expansion with minimal error. For the explicit practical calculations, we will thus use a different and more reliable method to find $p(t)$, as presented in the next section.

C. Spontaneous emission: adiabatic approach

In order to avoid the numerical problems mentioned above, we will make use of the fact that the cavity decay rate is the fastest time scale in the problem. Hence, we may first adiabatically eliminate the field dynamics, such that we are only left with the time evolution of the atomic operators. This approach assumes sufficiently large-field decay rates compared to the coherent atom-field coupling.

Let us again consider an excited atom coupled to the intracavity field. The time evolution of the probability of the atom being in the excited state is $p(t)=\langle P_e\rangle$ with $P_e=|e\rangle\langle e|$. Using the Hamiltonian [Eq. (4.4)] and including the effective field losses in the way discussed above [Eq. (3.17)] the Heisenberg equations of motion for the slowly varying operators take the form [15]

$$\dot{P}_e=-\sum_n (g_n\sigma_+ a_n+\tilde{g}_n b_n^\dagger\sigma_-), \tag{4.13}$$

$$\dot{a}_n=-(\kappa_n+i\Delta_n)a_n+\tilde{g}_n\sigma_-+\xi_n, \tag{4.14}$$

where ξ_n are noise operators (compare [26]) describing the influence of the reservoir. Usually the field dynamics are much faster than the atomic dynamics, and we assume that the field reaches a steady state, adiabatically following the atomic operators

$$a_n=\frac{1}{\kappa_n+i\Delta_n}(\tilde{g}_n\sigma_-+\xi_n). \tag{4.15}$$

This approximation is especially well justified in unstable resonators, where the typical loss rates are reasonably large and the field relaxes very quickly. If we substitute a_n with the adiabatic expressions of Eq. (4.13), we find an exponential decay for the atomic excitation probability $\dot{p}(t)=-2\gamma p(t)$ with

$$\gamma=\mathrm{Re}\left\{\sum_n \frac{g_n\tilde{g}_n^*}{\kappa_n+i\Delta_n}\right\}. \tag{4.16}$$

Here, we have taken into account the fact that the expectation value of the noise operators are zero. Please note that this expression strongly depends on the atomic position and the excess noise factors, since the coupling constants include the matched and adjoint quasimodes [cf. Eq. (4.5)]. This formula may now be evaluated for the specific model of a 1D symmetric resonator with Gaussian apertures, as outlined above. In fact, this sum converges rather rapidly without any further approximations. The price one has to pay is that we have no good quantitative estimate for the error introduced by the nonadiabaticity. Nevertheless, for small K where both methods may be applied, we find that the two agree very well. Far off the optical axis, where higher-order K factors dominate, it may again become necessary to use the approximation $B\to A^{-1}$ for faster convergence.

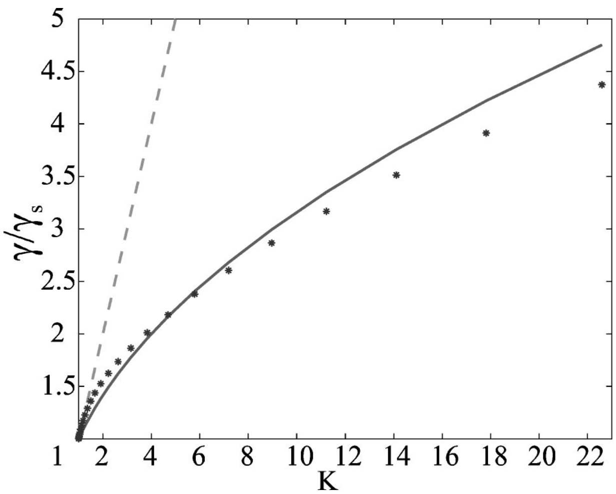

FIG. 4. The enhancement factor of the spontaneous emission rate at the cavity center $(x=0)$; $\gamma/\gamma_s(*)$ clearly deviates from the prediction K (dashed line) in [1,7] for strong excess noise. We find an enhancement approximately given by $\sqrt{K}$ (solid line).

Let us now turn to the results. In Fig. 4, we show the spontaneous emission rate into the cavity modes at the cavity center $\gamma(x=0)$ (depicted as $*$) as a function of the excess noise factor K, relative to the result for a stable single-mode cavity

$$\gamma_s=\frac{|g_0|^2}{\kappa_0}. \tag{4.17}$$

We find that, for small excess noise factors, the enhancement factor is linear in K_0, while for increasing excess noise, the sum is dominated by the first contribution of Eq. (4.16), yielding an enhancement by $\sqrt{K_0}$. Taking into account the fact that, for symmetric resonators we have $\tilde{g}_n=\sqrt{K_n}u_n$, an enhancement factor of $\sqrt{K_0}$ becomes quite obvious from the form of the sum in Eq. (4.16), which starts as

$$\gamma\approx\mathrm{Re}\left\{\frac{\sqrt{K_0}g_0^2}{\kappa_0}\right\}+\cdots. \tag{4.18}$$

To obtain Fig. 4, we have set the Fresnel number as $N=20$, the detuning as $\Delta_0=0$, and changed the mirror curvature from the stable to the unstable regime, i.e., $-0.5\leqslant L/f\leqslant 0.5$. The horizontal axis was than rescaled to linearly depend on K_0.

In contrast to this behavior, it has been experimentally confirmed that for unstable resonator lasers the linewidth, which is attributed to the spontaneous emission rate of the radiating atoms, is enhanced by the excess noise factor K. From this, one could draw the conclusion that spontaneous emission in general should be linearly enhanced by the excess quantum noise. From a different point of view, one might just as well conclude that a very unstable cavity with large K is almost the same as no cavity, so that there should be almost no effect on spontaneous emission at all [32]. In our calculation, we find that the enhancement factor is more or less in between these suggestions, i.e., $\gamma \propto \sqrt{K}$ as shown in Fig. 4. Hence, both of the preceding arguments are only partly correct for our case. To some extent, this can be understood from the specific model we have chosen. First, as we have neglected transverse noncavity modes, we will miss a significant fraction of the decay rate. In a physical setup with a macroscopic cavity, these transverse modes will dominate the decay and mask the excess noise effect on the total decay rate. At the same time, one could observe only the light emitted into the cavity modes (a given solid angle) to directly observe the calculated enhancement.

Second, the Gaussian reflectivity profile of our mirrors implies a very high reflectivity near the mirror centers allowing for multiple reflections of the emitted light. This gives strong feedback and may imply significant modification of the fluorescence, as one may find for, e.g., an atom close to a curved surface such as a microsphere [33]. This makes our significant enhancement factor plausible even for rather unstable geometries (large K).

The considerations above suggest that the effect should be concentrated close to the cavity axis and be more related to a redistribution and not simply a global enhancement of spontaneous emission (quantum noise) in the resonator. Hence, in the following, we consider spontaneous emission as a function of the distance from the optical axis. As a first step, we analytically calculate the transverse average of the decay rate γ and find that the excess noise factors in this limit cancel exactly, i.e., we have

$$\overline{\gamma} \propto \int dx\ \gamma(x) \propto \sum_n \frac{\kappa_n}{\kappa_n^2 + \Delta_n^2}. \tag{4.19}$$

This is a direct consequence of the biorthogonality of the matched and adjoint coupling. Hence, it is clear that quantum noise enhancement near the axis has to be accompanied by an off-axis reduction.

As the next step, we will now look at the transverse dependence in more detail. As mentioned before, to numerically calculate the transverse dependence of $\gamma(x)$, it is necessary to truncate the mode expansion at some point and substitute B with A^{-1}. We obtain good convergence for excess noise factors up to $K_0 \approx 3$. Of course, the K factors of higher-order modes are much larger (up to $\sim 10^5$) representing the origin of the numerical difficulties. We found that a good indication of the numerical inaccuracy is the deviation of the excess noise factor of the ground mode, calculated from the truncated A matrix $\tilde{K} = A_{00}^{-1}$, compared to its known analytical value $K = B_{00}$.

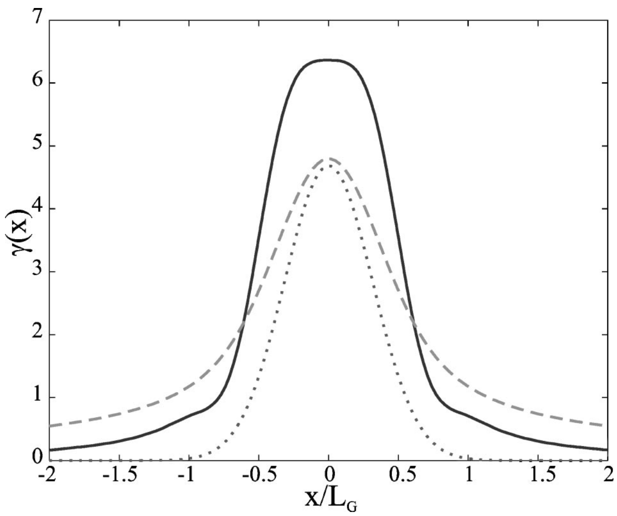

FIG. 5. Comparing the spontaneous emission rate $\gamma(x)$ (solid line) with a naive expression assuming orthogonal modes $\gamma_N(x)$ (dashed line) and the stable cavity result $\gamma_s(x)$ (dotted line) we find a clear enhancement especially at the optical axis. Here, we have chosen $L/f = -100$, $N = 15$, which gives rise to an excess noise factor of $K \approx 1.5$.

The transverse dependence of the spontaneous emission rate is depicted in Fig. 5, where we have chosen $L/f = -100$, and $N = 15$, giving rise to an excess noise factor of $K \approx 1.5$ and summed over 30 modes. Compared with the naive expression for an orthogonal mode expansion (Fermi's golden rule), i.e.,

$$\gamma_N = \sum_n \frac{|g_n|^2 \kappa_n}{\kappa_n^2 + \Delta_n^2}, \tag{4.20}$$

we clearly find a significant enhancement of spontaneous emission near the axis and a suppression further off axis, as suggested by our previous calculations. We would like to point out here that for moderate values of K, the perturbative calculation [Eq. (4.12)] gives the same results. For increasing excess noise (as considered, e.g., in Fig. 5) the agreement between the methods lessens. While the $\sqrt{K}$ enhancement for the atom on the optical axis is more or less reproduced in both models, the physically well-motivated result that averaging over the atomic position cancels the enhancement, is only approximately fulfilled within the perturbative method. This, and the analytical comparability along the axis, gives us strong confidence in the numerical results of the adiabatic calculations.

As we have seen, the excess noise factor enters approximately as $\sqrt{K}$ into the expression of the spontaneous emission rate as compared to a naive application of Fermi's golden rule. In this sense, one could call the result an enhancement as compared to a calculation in a geometry with orthogonal modes. However, this is only part of the story. A somewhat different answer is found by looking at the transition from a stable to an unstable cavity configuration as shown in Fig. 6, where we smoothly change the mirror curvature of our resonator from the stable to the unstable region. Note that along the cavity axis, the effective spontaneous

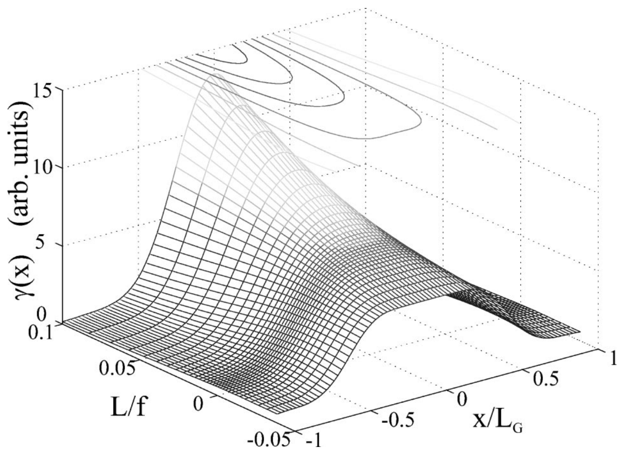

FIG. 6. The spatial distribution of the spontaneous emission rate $\gamma(x)$ is plotted at the transition from a stable to an unstable cavity configuration by changing the curvature of the mirrors L/f. The transverse extension of the mirrors L_G is fixed with the Fresnel number $N=15$.

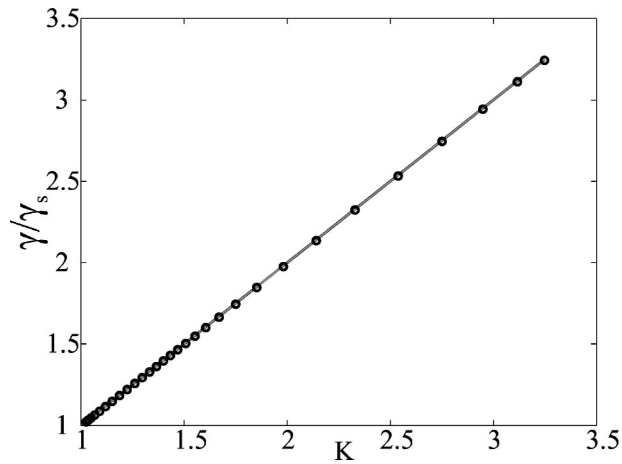

FIG. 7. The induced atomic transition rate $\gamma/\gamma_s(\mathrm{o})$ shows the expected K-fold enhancement (solid line) for an overall gain of $\Gamma = 2\kappa_0$.

emission rate decreases going from the stable to the unstable side, although the excess noise factor K is strongly increasing. This sounds somewhat contradictory, but it may be traced back to the fact that the mode amplitude at the axis decreases as a function of the mirror curvature due to the broadened beam width. This overcompensates for the increase of the excess noise factor. Looking at the spatial dependence of the spontaneous emission rate $\gamma(x)$ for such a varying cavity geometry, as in Fig. 6, we find a widening transverse plateau for increasing instability. Furthermore, the decay rate well off axis is almost independent of the mirror curvature. Note that the contour lines are approximately flat. For this graph, the excess noise factors are all below $K_0 \leqslant 2$ and we have chosen $N=15$.

It seems that the instability flattens the dependence of the decay rate close to the optical axis. This means that, for small displacements, the atomic decay probability is nearly independent of the atomic position. To some extent, this may be understood in terms of geometrical optics. If the atom is localized at $x=0$, only the axial ray is reflected onto the atom itself. If the atom sits off the axis, no closed optical path exists. On the other hand, there are a large number of rays that return to the atomic position after several reflections, before leaving the resonator. Of course, the finite transverse size of the mirror restricts the number of possible reflections, serving to limit the size of this axial plateau.

D. Siegman's law

As we have seen, the total atomic spontaneous emission rate into all cavity modes is enhanced by approximately $\sqrt{K}$. On the other hand, from previous calculations for the laser linewidth, one expects a K-fold enhancement [1,7]. In the following we will mimic a single-mode situation and consider the case of one mode being actively selected, e.g., by some extra gain mechanism. In this case, we cannot use adiabatic elimination of the selected mode and we have to use the perturbative formula [Eq. (4.12)]. In this limit, the mode selective gain simply shifts the individual loss rates of the selected modes by an overall value Γ. For only one mode above threshold, the factor

$$\delta_0(t) = \frac{e^{(\Gamma-\kappa_0+i\Delta_0)t}-1}{\kappa_0 - i\Delta_0}, \tag{4.21}$$

is now growing exponentially in time, whereas the other mode contributions stay negligibly small. Hence, after a sufficiently large time, Eq. (4.11) reduces to one single term, which is indeed proportional to the excess noise factor of this particular amplified mode $n=0$, i.e., $B_{00}=K_0$. This is also depicted in Fig. 7, where we have chosen an additional gain of $\Gamma=2\kappa_0$, shifting the ground mode towards threshold and leading to a K dependence for the spontaneous emission rate. In this way, our calculations also reconcile some of the previous controversies on this subject [1,8]. The enhancement from $\sqrt{K}$ to K via the introduced gain mechanism may be viewed as active noise amplification or amplified spontaneous emission. Hence, the total excess noise in lasers can be traced back to two origins, a spatial enhancement of the quantum noise as well as amplified spontaneous emission via the gain. Note that this result has to be treated with caution as we have not included any gain saturation.

V. CONCLUSIONS

Using our quasimode description, we have demonstrated that the concept of excess quantum noise in an unstable optical cavity also emerges in the context of spontaneous emission of a single two-level atom. Quantitatively, our result is different from previous suggestions in the context of lasers [1,7], where the spontaneous emission rate was predicted to be proportional to K instead of $\sqrt{K}$. As a limiting case, Siegman's law reappears when through some external mechanism (tailored loss or gain medium), a single quasimode is actively selected. If, e.g., in a laser, only one quasimode oscillates above threshold, spontaneous emission noise into that mode is enhanced by the excess noise factor K. In this sense, our results agree with the claim of previous authors [8,12,13] that gain is a necessary condition to obtain the *full* excess noise factor.

However, our calculations show that one has to be careful with the term enhancement as the effective spontaneous emission rate at the center of a stable cavity configuration with the same aperture and mirror reflectivity is even larger, although we clearly have no excess noise enhancement (i.e., $K \approx 1$). This is compensated by larger-field mode amplitudes and a modified mode density. Note that the enhancement only refers to contributions to the emission rate within the solid angle covered by the resonator, which in many cases may be small compared to the total decay rate into all modes. The effect becomes more significant the smaller the resonator volume and the better the reflectivity of the mirrors is. Hence, it should be important in microoptic setups.

Instability seems to play an important role in obtaining very high K factors, but only in connection with finite geometries. Interestingly, excess noise is enhanced by soft apertures. In fact, for an unstable cavity with a very large hard aperture, one finds relatively small excess noise in the center, while on the other hand, even a geometrically stable cavity configuration with Gaussian apertures may imply a significant K factor. Considering an unstable cavity with a large hard aperture going to infinity, one may find a different continuous set of orthogonal modes with rather peculiar properties [24], from which a spontaneous decay rate could, in principle, be deduced. This does not involve any excess noise arguments, but one finds an unexpected result for the mode density, which enters in the spontaneous emission rate. It is interesting to note here that these mode functions do not coincide with our solutions in the limit of infinitely large soft apertures due to the differently prescribed asymptotics. Nevertheless, the mode functions behave similarly near the cavity axis.

Let us finally point out here that further interesting results could be expected from a discussion of other quantum noise-driven systems placed in unstable resonators such as, e.g., the optical parametric oscillator. Especially below threshold we would expect to see quantum and excess noise effects.

Note added. Recently the authors became aware of some related work by Dalton and co-workers [37] underlining the keypoints of the proposed field quantization.

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[34] If only one index is written, the transverse mode number is meant and the longitudinal one is some fixed integer n_0.
[35] In principle, there may appear cross terms $a_n a_m$ or $b_n^\dagger b_m^\dagger$ oscillating with $\pm(\omega_n+\omega_m)$. But for symmetric paraxial systems, where $u_n(x) \propto v_n^*(x)$, these terms cancel exactly. And for arbitrary systems, these terms oscillate so rapidly that their average effect vanishes.

[36] Note that in Ref. [7], Eq. (27), the sums in the noise power term proportional to $G-1$ in their equation can be explicitly carried out $(\Sigma_m \tilde{A}_{nm} \tilde{B}_{hm} = 1)$, which leads to a spontaneous emission rate independent of K. Although looking rather unexpected this result is very reasonable since the equation corresponds to the undamped free space result without resonator. The correct expression is given later in Eq. (29) where the resonator including all loss rates is considered.

[37] B. Dalton *et al.*, J. Mod. Opt. (to be published).