Measurement theory and interference of spinor Bose-Einstein condensates

S. Ashhab and A. J. Leggett
Department of Physics, University of Illinois, 1110 West Green Street, Urbana, Illinois 61801

(Received 21 March 2001; published 17 January 2002)

We study two aspects of measurement theory in spinor Bose-Einstein condensates of $F=1$ atoms: the probability of obtaining a certain outcome of the measurement and the evolution of the state of the condensate due to the measurement. We also study the interference patterns arising from the spatial overlap of two spinor condensates. We show that neither a measurement on a small number of escaping atoms nor an interference experiment can distinguish between an antiferromagnetic coherent state condensate, i.e., a condensate in which all the atoms have $S_z=0$ along an *a priori* unknown direction, and a spin-singlet condensate, i.e., a condensate with $S_{\rm total}=0$. We also show that a singlet-state condensate evolves into a coherent state as a result of the measurement.

DOI: 10.1103/PhysRevA.65.023604 PACS number(s): 03.75.Fi, 05.30.Jp

I. INTRODUCTION

In the past few years there have been several experimental and theoretical studies of the structure of a spinor Bose-Einstein condensate (BEC) for atoms of total spin $F=1$ [1–6]. Ho [2] and Ohmi and Machida [3] independently treated the problem of spinor BEC in the Gross-Pitaevskii (GP) approximation. In the case of antiferromagnetic interactions they found that the ground state is given by

$$|C\rangle = \frac{1}{\sqrt{N!}}(a_0^\dagger)^N|0\rangle, \qquad (1)$$

where $a_0^\dagger$ creates a particle in the lowest spatial single-particle state (which can be calculated from the GP equation) with $S_z=0$ along some arbitrary direction (we use the letter S to denote the total atomic spin). $|C\rangle$ stands for (antiferromagnetic) "coherent state" and $|0\rangle$ is the true vacuum state containing no particles. The state $|C\rangle$ is a singly-condensed state, i.e., all the particles occupy the same single-particle state. (Notice that this definition of coherence differs from the one normally used in quantum optics.)

Noticing that the state (1) breaks the SO(3) rotational symmetry of the Hamiltonian in spin space, Law, Pu, and Bigelow [4] carried out a calculation where they assumed that all atoms in the condensate occupy the same spatial wave function, but they did not impose any constraint on the spin state of the atoms. They found that in the absence of external magnetic fields, the ground state of the system is not a singly-condensed state, but rather a spin-singlet state, i.e., a state with total spin of the condensate $S_{\rm total}=0$,

$$|S\rangle = {\rm const}\times(a_0^\dagger a_0^\dagger - 2a_1^\dagger a_{-1}^\dagger)^{N/2}|0\rangle, \qquad (2)$$

where, as in the rest of this paper, N is assumed to be even for simplicity. The minus sign in Eq. (2) is in accordance with the phase convention of Refs. [4–6], which we shall follow in this paper.

In the absence of spin-dependent external fields the energies per particle of the states (1) and (2) are very close, with a difference of $\sim N^{-1/2}$. Depending on the magnitude of the external fields, either one of them can be the true ground state [6–8]. We address the question of how to devise an experiment that will be able to distinguish between the two states [6,8,9].

By examining the two expressions (1) and (2) it is obvious that there are physical differences between them. For example, the singlet state has $\langle S_{\rm total}^2\rangle=0$, while $\langle S_{\rm total}^2\rangle=2N$ for the coherent state ($\hbar=1$). Also, if one could measure the occupation numbers of the three hyperfine states n_1, n_0, and n_{-1}, a clear difference would be that in the singlet state one always gets $n_1=n_{-1}$, even though that value may vary from shot to shot. However, in the above examples the entire condensate has to be probed, and measurements of atom numbers have to be accurate at least to relative order $N^{-1/2}$ in order to distinguish between the two states. To avoid this difficulty, we shall examine measurements in which a small number of atoms is probed.

One important aspect of measurement theory is the evolution of the quantum state as a result of the measurement. An example, which is related to the present problem, is that of measuring the relative phase between two condensates [10–12]. Castin and Dalibard demonstrated how an experiment measuring the relative phase between two condensates "builds up" the phase between them [11]. A detection measurement that does not give information about which condensate the atom came from creates an uncertainty in the relative number between the two condensates. Using appropriate definitions, it can be shown that the relative phase quantum operator and the relative number quantum operator are, to a good approximation, canonically conjugate operators [13,14]. Therefore, an increase in the uncertainty in the relative number allows for a reduction in the uncertainty in the relative phase, which is the case in the Castin-Dalibard scheme. In the context of spinor BEC, the coherent states are analogous to the phase states, and the singlet state is analogous to the unbroken-symmetry number state. Therefore, we shall examine whether a similar phenomenon exists in the measurement of a spinor BEC.

In the present paper, we shall show that a measurement based on the detection of a small number of atoms from the condensate and measuring their spin structure cannot distinguish between a coherent state and a singlet state. We also show that a spinor BEC in a singlet state evolves into a coherent state after the measurement. These results will be obtained using the same method as described in Ref. [11]. Moving in a different direction, because in interference experiments the entire condensates can overlap, one might expect such experiments to distinguish between the two states. We shall study the interference patterns arising from the spatial overlap of two spinor condensates and show that the expected patterns are the same whether one starts with coherent or singlet states.

The paper is organized as follows. In Sec. II we study the outcome probabilities and the evolution of the state of the condensate following a sequence of single-particle detection measurements. In Sec. III we study two-particle detection measurements and show that the effect of the measurement process is qualitatively similar to that of a single-particle measurement. In Sec. IV we discuss some properties and implications of particle detection measurements. We also discuss the quantum-mechanical description of absorption and phase-contrast imaging. In Sec. V we outline a method to predict the emergence of an interference pattern from two overlapping scalar (i.e., spineless) condensates. In Sec. VI we generalize the method to study the interference patterns produced by the overlap of two spinor condensates and show that both coherent and singlet states give the same interference patterns, since the results for coherent states have to be averaged over all directions.

II. SINGLE-PARTICLE MEASUREMENTS

In this section we study the outcome of a sequence of single-particle measurements. We use a model where in each step of the measurement one atom leaves the trap and the z component of the spin of that atom is measured [9]. In order to neglect the spin dynamics of the condensate during the measurement, we shall assume that the time over which the measurement process is completed is much smaller than the inverse of the energy difference between the two initial states in question [15]. We also assume that the spin-spin interaction energy is small enough that the escape rate of atoms is independent of their spin state. Note that in contrast to Refs. [6,9], we do not assume any *a priori* knowledge about the direction of the broken-symmetry state [Eq. (1)]. We denote the initial state of the system by $|\Psi\rangle_0$, and the state of the system after n measurements by $|\Psi\rangle_n$. Each measurement gives $S_z=+1$, 0, or -1. The probability of obtaining the value $S_z=m$ in the nth measurement is given by

$$P_n(m)=\frac{{}_{n-1}\langle\Psi|a_m^\dagger a_m|\Psi\rangle_{n-1}}{\sum_l {}_{n-1}\langle\Psi|a_l^\dagger a_l|\Psi\rangle_{n-1}}, \tag{3}$$

where the operators a_m, $a_m^\dagger$ are the annihilation and creation operators of a particle with $S_z=m$. If the outcome of the nth measurement is $S_z=m$, the state of the system is projected as follows:

$$|\Psi\rangle_n=\frac{a_m|\Psi\rangle_{n-1}}{\sqrt{{}_{n-1}\langle\Psi|a_m^\dagger a_m|\Psi\rangle_{n-1}}}. \tag{4}$$

Using Eqs. (3) and (4) we now analyze the measurement process for the coherent and singlet states. Our treatment can be straightforwardly generalized to the more general class of states with $S_{\text{total}}\ll N$.

A. Coherent state

Let us assume that the initial state of the system is given by Eq. (1). We rewrite it in a form that contains the preferred direction, defined by the angles (θ,ϕ), explicitly,

$$|\Psi\rangle_0=\frac{1}{\sqrt{N!}}[d^1_{1,0}(\theta)e^{-i\phi}a_1^\dagger+d^1_{0,0}(\theta)a_0^\dagger+d^1_{-1,0}(\theta)e^{i\phi}a_{-1}^\dagger]^N|0\rangle\equiv|N,\theta,\phi\rangle, \tag{5}$$

where $d^1_{m,0}(\theta)$ are the single-particle rotation matrix elements

$$d^1_{m,0}(\theta)=\langle m|e^{-i\theta\hat{S}_y}|0\rangle=\begin{cases}-\frac{1}{\sqrt{2}}\sin\theta & \text{if } m=1,\\ \cos\theta & \text{if } m=0,\\ \frac{1}{\sqrt{2}}\sin\theta & \text{if } m=-1.\end{cases} \tag{6}$$

Equation (5) is the state that contains N atoms, all of which occupy the same single-particle state with $S_z=0$ along the direction (θ,ϕ). Substituting Eq. (5) into Eqs. (3) and (4) one easily finds that

$$P_n(m)=(d^1_{m,0})^2(\theta) \tag{7}$$

and

$$|\Psi\rangle_n=|N-n,\theta,\phi\rangle \tag{8}$$

up to an overall phase factor. This means that the probabilities in the nth measurement are independent of the previous measurements, and that after each measurement only the number of particles in the condensate changes, while θ and ϕ are not affected. As one would expect from rotational symmetry about the z axis, the measurement is insensitive to the angle ϕ. The probability $P(m_1,\ldots,m_n|\theta)$ of finding a certain sequence of values $(m_1,\ldots,m_n)$ for a given value of θ is just the product of the single-measurement probabilities,

$$\begin{aligned}P(m_1,\ldots,m_n|\theta)=\prod_n P_n(m)&=\frac{(\sin\theta)^{2(k_1+k_{-1})}}{2^{k_1+k_{-1}}}(\cos\theta)^{2k_0}\\ &\approx\exp\left[\sin^2\theta_0\ln\left(\frac{\sin^2\theta_0}{2}\right)\right.\\ &\left.\quad+n\cos^2\theta_0\ln(\cos^2\theta_0)-2n(\theta-\theta_0)^2\right],\\ &\qquad n\gg1,\end{aligned} \tag{9}$$

where

$$\tan^2\theta_0=\frac{k_1+k_{-1}}{k_0} \tag{10}$$

and k_m is the number of times the value $S_z=m$ appears in the sequence $(m_1,\ldots,m_n)$.

Since the probability in Eq. (9) is peaked at the value $\theta_0=\theta$, the experimental procedure described above can be considered a measurement of θ. However, it does not give any information about the polar angle ϕ. A straightforward method to measure both θ and ϕ is to make measurements along three (or more) different axes, not necessarily perpendicular to one another. Assume that one makes $n_1(\gg 1)$ measurements along z_1. This will determine that the direction (θ,ϕ) lies in a ring defined by a certain value of θ relative to z_1. Making $n_2(\gg 1)$ measurements along z_2 will determine that (θ,ϕ) lies in a ring defined by a certain value of θ relative to z_2. Combining the two measurements, the direction (θ,ϕ) will be specified by the two points where the two rings intersect. (It is highly unlikely that the rings will not intersect, and therefore we do not examine that possibility.) If (θ,ϕ) is determined to be one of two directions, one can then make $n_3(\gg 1)$ measurements along z_3, and the outcome distribution will pick one of the two directions. This phenomenon is reminiscent of the uncertainty between ϕ and $-\phi$ in measuring the relative phase between two condensates [11]. This uncertainty can be removed by making additional measurements with a phase shift γ given to the atoms leaving one of the two condensates. Even if γ is different for each escaping atom, the system is driven closer and closer to a coherent (i.e., phase) state.

Since we are assuming that we do not have any *a priori* knowledge about the direction (θ,ϕ) of the coherent state, the probability of a certain outcome is given by the average of Eq. (9) over all possible directions,

$$\begin{aligned}P(m_1,\ldots,m_n)&=\int\frac{d\Omega}{2\pi}\,\frac{(\sin\theta)^{2(k_1+k_{-1})}}{2^{k_1+k_{-1}}}(\cos\theta)^{2k_0}\\&=\frac{\Gamma(k_1+k_{1-1}+1)\Gamma(k_0+1/2)}{2^{k_1+k_{-1}+1}\Gamma(n+3/2)},\end{aligned} \tag{11}$$

where $d\Omega=\sin\theta d\theta d\phi$, and the integral covers the upper hemisphere $(\theta\leqslant\pi/2)$, as we explain in the Appendix.

B. Singlet state

Now we analyze the same measurement process as in Sec. II A starting with the initial state $|S_{\rm total}=0\rangle$ [Eq. (2)]. The singlet state can be rewritten as (see the Appendix for derivation)

$$|S\rangle=\frac{\sqrt{N}}{2\pi}\int d\Omega|N,\theta,\phi\rangle. \tag{12}$$

This form allows us to straightforwardly evaluate the probabilities and projected state due to the measurement. If we define $c_n(\theta,\phi)$ by

$$|\Psi\rangle_n\equiv\int d\Omega\, c_n(\theta,\phi)|N-n,\theta,\phi\rangle, \tag{13}$$

we can make use of the quasiorthogonality of coherent states (see the Appendix) and find that

$$P_n(m)\approx\frac{\int d\Omega(d^1_{m,0})^2(\theta)|c_{n-1}(\theta,\phi)|^2}{\int d\Omega|c_{n-1}(\theta,\phi)|^2}, \tag{14}$$

$$|\Psi\rangle_n\approx\frac{\int d\Omega\, d^1_{m,0}(\theta)e^{-im\phi}c_{n-1}(\theta,\phi)|N-n,\theta,\phi\rangle}{\sqrt{2\pi/N\int d\Omega(d^1_{m,0})^2(\theta)|c_{n-1}(\theta,\phi)|^2}}. \tag{15}$$

This finally leads to

$$\begin{aligned}P(m_1,\ldots,m_n)&=\frac{{}_{n-1}\langle\Psi|a^\dagger_{m_n}a_{m_n}|\Psi\rangle_{n-1}}{\sum_{l_n}{}_{n-1}\langle\Psi|a^\dagger_{l_n}a_{l_n}|\Psi\rangle_{n-1}}\,\frac{{}_{n-2}\langle\Psi|a^\dagger_{m_{n-1}}a_{m_{n-1}}|\Psi\rangle_{n-2}}{\sum_{l_{n-1}}{}_{n-2}\langle\Psi|a^\dagger_{l_{n-1}}a_{l_{n-1}}|\Psi\rangle_{n-2}}\cdots\frac{{}_0\langle\Psi|a^\dagger_{m_1}a_{m_1}|\Psi\rangle_0}{\sum_{l_1}{}_0\langle\Psi|a^\dagger_{l_1}a_{l_1}|\Psi\rangle_0}\\&=\frac{(N-n)!}{N!}\,\frac{{}_0\langle\Psi|a^\dagger_{m_1}\cdots a^\dagger_{m_n}a_{m_n}\cdots a_{m_1}|\Psi\rangle_0}{{}_0\langle\Psi|\Psi\rangle_0}\\&\approx\frac{\int d\Omega\,\frac{(\sin\theta)^{2(k_1+k_{-1})}}{2^{k_1+k_{-1}}}(\cos\theta)^{2k_0}|c_0(\theta,\phi)|^2}{\int d\Omega|c_0(\theta,\phi)|^2}\end{aligned} \tag{16}$$

and

$$c_n(\theta,\phi)\propto(\sin\theta)^{k_1+k_{-1}}(\cos\theta)^{k_0}e^{i(k_{-1}-k_1)\phi}c_0(\theta,\phi). \tag{17}$$

Notice that Eqs. (14)–(17) are valid for a general state as long as $S_{\text{total}}\ll N$. For the singlet state $c_0(\theta,\phi)=\sqrt{N}/2\pi$, which gives

$$P(m_1,\ldots,m_n)\approx\int\frac{d\Omega}{2\pi}\frac{(\sin\theta)^{2(k_1+k_{-1})}}{2^{k_1+k_{-1}}}(\cos\theta)^{2k_0} \tag{18}$$

$$c_n(\theta,\phi)\approx\left(\frac{N\sqrt{n}}{2^{3/2}\pi^{5/2}\sin\theta_0}\right)^{1/2}e^{-n(\theta-\theta_0)^2}e^{i(k_{-1}-k_1)\phi}, \tag{19}$$

where θ_0 is given by Eq. (10). Corrections to Eq. (18) are of the order $O(n/N)+O((k_1-k_{-1})^2/N)$ relative to the value of P. The expression for the probability is (almost) exactly the same as that for a uniform statistical distribution of coherent states [Eq. (11)].

We notice that although the coefficient $c_n(\theta,\phi)$ becomes peaked around a certain value of θ with increasing n, the ϕ dependence is only affected through a phase factor. As was discussed in the case of a coherent state, making measurements along three different axes determines both θ and ϕ, and, therefore, in each single run it projects the singlet state into a coherent state along a well-defined direction. It is tempting to think that changing the axis of measurement for each atom would have the effect of washing out the localization of c_n in the coordinates θ and ϕ. By multiplying two factors, each of which has the form (19) in a different system of coordinates, one can see that this washing-out effect does not occur. Moreover, changing the axis of measurement rids us of having to worry about conservation of S_z of the whole condensate. In other words, if all the measurements are made along the same axis, the total S_z of the escaping atoms and the remaining condensate is equal to zero for the singlet state. This constraint is lifted if we use several different axes for different atoms. Then, the condensate is projected closer to a coherent state in both θ and ϕ directions.

In conclusion, a measurement performed on a few single atoms leaving the condensate cannot determine whether the condensate was in a coherent or singlet state before the measurement. In each single run the condensate behaves as a coherent-state condensate, even if it were in a singlet state before the measurement. This phenomenon follows immediately from realizing that the model used in this section describes a measurement of the direction of a coherent state. A collection of such measurements gives the probabilities for the system to be in each state of the basis (of coherent states). In a single realization of the measurement the system behaves as if it were in one of the basis states, regardless of its initial state.

III. TWO-PARTICLE MEASUREMENTS

Let us look at the expression for the singlet state (2). This state is formed by creating $N/2$ pairs of atoms, each in a total spin singlet state $(S=0)$. A naive argument might say that if one takes a pair of atoms out of the condensate and measures their total spin, one would always get the value $S=0$. However, the pairs of atoms are not bound molecules, and the two atoms that left the condensate could have come from two different pairs. Thus, both values $(S=0$ and $S=2)$ are possible.

We shall now carry out a calculation to show the above result in more detail. The annihilation operators of a pair of atoms are given by

$$A_{0,0}=\frac{1}{\sqrt{6}}(a_0a_0-2a_1a_{-1}),$$

$$A_{2,0}=\frac{1}{\sqrt{3}}(a_0a_0+a_1a_{-1}),$$

$$A_{2,\pm1}=a_0a_{\pm1},$$

$$A_{2,\pm2}=\frac{1}{\sqrt{2}}a_{\pm2}a_{\pm2}, \tag{20}$$

where the indices refer to the total spin and the z component of the total spin of the pair. As in Sec. II, we can now calculate the probabilities and projected state.

A. Coherent state

The probability of finding a total spin S and z component m for the nth pair is given by

$$P_n(S,m)=\frac{{}_{n-1}\langle\Psi|A^\dagger_{S,m}A_{S,m}|\Psi\rangle_{n-1}}{\sum_{L,l}{}_{n-1}\langle\Psi|A^\dagger_{L,l}A_{L,l}|\Psi\rangle_{n-1}}=r_n^2(\theta), \tag{21}$$

where

$$r(\theta)=\langle S,m|e^{-i\theta\hat{S}_y}|m_1=0,m_2=0\rangle$$
$$=\begin{cases}\frac{1}{\sqrt{3}} & \text{if } S=m=0,\\ \sqrt{\frac{2}{3}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right) & \text{if } S=2,m=0,\\ \frac{1}{2}\sin 2\theta & \text{if } S=2,m=\pm1,\\ \frac{1}{2}\sin^2\theta & \text{if } S=2,m=\pm2.\end{cases} \tag{22}$$

It follows that the probability of finding a certain sequence $(S_1,m_1,\ldots,S_n,m_n)$ is simply

$$P(S_1,m_1,\ldots,S_n,m_n|\theta)=\prod_l r_l^2(\theta). \tag{23}$$

As with single-particle measurements, the condensate remains in a coherent state regardless of the outcome of the measurement

$$|\Psi\rangle_n=|N-2n,\theta,\phi\rangle \tag{24}$$

up to an overall phase factor. Since we are assuming a uniform distribution of coherent states, the probability of finding a certain sequence is given by

$$P(S_1,m_1,\ldots,S_n,m_n)=\int \frac{d\Omega}{2\pi}\prod_l r_l^2(\theta). \tag{25}$$

B. Singlet state

Applying Eq. (21) to the singlet state gives

$$P_n(S,m)\approx \frac{\int d\Omega\, r_n^2(\theta)|c_{n-1}(\theta,\phi)|^2}{\int d\Omega |c_{n-1}(\theta,\phi)|^2}$$

$$P(S_1,m_1,\ldots,S_n,m_n)\approx \frac{\int d\Omega \Pi_l r_l^2(\theta)|c_0(\theta,\phi)|^2}{\int d\Omega |c_0(\theta,\phi)|^2} = \int \frac{d\Omega}{2\pi}\prod_l r_l^2(\theta). \tag{26}$$

And using the notation $|\Psi\rangle_n \equiv \int d\Omega\, c_n(\theta,\phi)|N-2n,\theta,\phi\rangle$,

$$c_n(\theta,\phi)\propto\left(\tfrac{3}{2}\cos\theta-\tfrac{1}{2}\right)^{k_{2,0}}(\sin 2\theta)^{k_{2,1}+k_{2,-1}}(\sin\theta)^{2(k_{2,2}+k_{2,-2})} \times\exp[(2k_{2,-2}+k_{2,-1}-k_{2,1}-2k_{2,2})\phi], \tag{27}$$

where $k_{S,m}$ is the number of times the value (S,m) appears in the measurement sequence. Obviously, $c_n(\theta,\phi)\neq c_0(\theta,\phi)$ unless all measurements give the value $S=0$, which has a vanishingly small probability for $n\gg 1$. In general, the function $c_n(\theta,\phi)$ is peaked around two values of θ, one of them between 0 and $\cos^{-1}(1/\sqrt{3})\approx 0.3\pi$ and the other between $\cos^{-1}(1/\sqrt{3})$ and $\pi/2$. [The physical significance of the angle $\cos^{-1}(1/\sqrt{3})$ is that a pair of atoms, each with spin projection 0 along a direction making this angle with the z axis, cannot be in a $S=2$, $S_z=0$ state.] However, apart from a small region in the parameter space $k_{S,m}$ that corresponds to a small probability, one of these maxima will be much greater than the other, and we find that, as in Sec. II, the singlet state evolves into a coherent state as the measurement proceeds.

IV. DISCUSSION OF THE RESULTS AND POSSIBLE EXPERIMENTAL REALIZATIONS

We have shown in Sec. III that measuring S and S_z of pairs of atoms gives the same results for both a uniform distribution of coherent states and the singlet state. Starting from the singlet state we found that the system will evolve, *because of the measurement*, into a coherent state. We found that if in a single measurement the value $S=0$ is obtained, the spin state of the system is not affected, apart from taking two atoms out of the condensate. It was the measurement of $S=2$, $m=0$, ±1, ±2 that caused the evolution from a singlet to a coherent state. One may then ask the question: Is it possible to measure only S (without measuring m) and leave the singlet state unchanged? This would be achieved if one could perform a measurement that is described by a rotationally invariant projection operator. A rotationally invariant operator cannot change $c(\theta,\phi)$. It would not change a singlet state into a coherent state. However, it can be easily shown that such an operator does not exist for $S\neq 0$. Any operator that removes two atoms from the condensate can be written as a linear superposition of the annihilation operators $A_{S,m}$ defined in Eq. (20). The operators $A_{S,m}$ transform under the same rotation group as the spherical harmonics $Y_{0,0}, Y_{2,m}$. The operator $A_{0,0}$ possesses the desired property of transforming into itself under any rotation and, therefore, does not affect $c(\theta,\phi)$. Among the other five operators there is none that transforms into itself under an arbitrary rotation. Therefore, when a pair with $S=2$ is detected, the angular dependence of the state of the condensate will, in general, change.

The impossibility of preserving the singlet state can be understood from the nature of the measurement process. In a measurement based on the detection of a particle (or group of particles), one assumes that at the moment that the particle is detected, it ceases to be part of the system. Before the measurement, the spin states of the escaping atom and the rest of the condensate are entangled, i.e., neither of them is determined independently of the other. For example, when the first atom leaves from a singlet state, the composite system is described by the $S_{\rm total}=0$ state

$$|\Psi\rangle=\frac{1}{\sqrt{6}}(|+1\rangle_a|1,-1\rangle_C+|-1\rangle_a|1,+1\rangle_C-2|0\rangle_a|1,0\rangle_C), \tag{28}$$

where the subscripts a and C refer to the atom and the condensate, respectively. If the atom hits a detector, the entanglement has to remain between the atom and the condensate, or it is transmitted to other degrees of freedom in the environment such that the total spin is conserved. This would result in an entangled state of the condensate and the degrees of freedom of the environment, provided the environment started in a state of well-defined total spin. As long as we do not allow for such entanglement, it is necessary to specify the exact state of the atom at detection, i.e., to project out the appropriate component of Eq. (28). As a result, the condensate is left with $S=1$ and a definite m value (along some direction). In the case of a pair of atoms leaving the condensate simultaneously, arguments similar to those given in Sec. III lead to the same conclusion. If one does not "read out" all the information obtained by the measurement, the condensate will be described by a mixed state, which simply reflects the ignorance of the experimenter of the well-defined outcome of the measurement. We stress, though, that one cannot speak of the difference between a mixed and a pure state unless this difference can be measured experimentally.

We note that since the environment acts as a measurement device, our arguments apply to the interaction between the condensate and its environment even without any human intervention. An escaping atom will interact with the environment and its spin component along some axis will be measured. The important point here is that regardless of the direction of the measurement axis, the state of the condensate will be projected closer to a coherent state, even if a different direction of this axis is chosen for each escaping atom [16]. Another point worth mentioning is that even if the atoms remain entangled with the condensate until a measurement is performed on the condensate, tracing over the states of the escaping atoms gives the same results as above. Therefore, just by losing atoms from the condensate a singlet state will evolve into a coherent state. This looks like spontaneous symmetry breaking, except that it is not caused by residual external fields, but rather by continuous measurement of the system. The chosen direction is determined entirely by chance, just as the outcome of any measurement in quantum mechanics. In the absence of any symmetry-breaking external fields, the competition between this measurement mechanism and stochastization due to interatomic interactions [17] will determine the state of the system. Typically, the energy difference between the coherent and singlet state is of the order 100 s^{-1}, while atom loss rates are of the order 10^4 s^{-1}. This suggests a highly coherent state. Zurek, Habib, and Paz found a similar result for a harmonic oscillator in contact with its environment [18]. One should keep in mind that although the above argument is conceptually convenient, an ensemble describing a uniform distribution of coherent states is equivalent to an ensemble of definite S_{total} states with the correct weights.

Now we turn to the question of how realistic is our model for describing the measurement process. We have assumed that the condensate is not tightly bound inside the trap so that occasionally an atom will escape and will be detected. In practice, however, condensates are probed using optical-imaging techniques. In order to be able to give the correct description of the measurement, one has to understand the quantum-mechanical description of the interaction between the condensate and the imaging laser beam [19,20]. Near-resonance absorption imaging is the closest to our model in the sense that an atom is removed from the condensate upon detection. When imaging a scalar (i.e., spinless) condensate a scattered photon projects the state of the condensate $|\Psi\rangle$ into $\hat{a}|\Psi\rangle$, and a nonscattered photon projects it into $\sqrt{1-\gamma\hat{N}}e^{i\delta(\hat{N})}|\Psi\rangle$, where γ is the absorption probability per photon per atom and normalization constants are implicitly understood. This expression can be obtained by considering the probability for a photon not to be scattered by a condensate with N atoms (probability$=1-\gamma N$ in the dilute limit) and the phase shift $\delta(N)$ given to such a photon, which is proportional to N in the dilute limit. Here we are assuming that an excited atom leaves the trap before the next photon arrives at the condensate. When imaging a spinor condensate, using appropriate probe beam frequency and polarization one could choose to image each of the three hyperfine states separately. The initial absorption rate is proportional to the number of atoms in the imaged hyperfine state and cannot be used to distinguish between a coherent and a singlet state, in agreement with our result. One has to measure the population of the $+1$ and -1 states to relative accuracy better than $N^{-1/2}$ in order to distinguish between the coherent and singlet states, since in the singlet state $n_1=n_{-1}$. However, that measurement requires counting essentially all the atoms in the different hyperfine states, which is exactly the difficulty we are trying to avoid.

Phase-contrast imaging differs from our model in a more fundamental way: the projection operator associated with this type of measurement, which does not remove atoms from the condensate, is $a_m^\dagger a_m$ rather than a_m. Another less fundamental difference occurs if the linewidths are comparable to or greater than the hyperfine splitting energy scale. In that case it is not possible to image each one of the three states separately. Assuming the linewidths are small enough, we find that, as in absorption imaging, in order to distinguish between the coherent and singlet states, one has to measure the different n's to relative accuracy better than $N^{-1/2}$ [21]. The difference between the two methods is that in absorption imaging the condensate is destroyed after the measurement due to heating, whereas in phase-contrast imaging the condensate ends up in a state of definite n_1, n_0, and n_{-1} without losing atoms from the condensate (neglecting heating effects).

V. INTERFERENCE BETWEEN TWO SCALAR CONDENSATES

Having failed to find a few-particle measurement scheme to distinguish between the coherent and singlet states, we now take a different direction. It appears as if interference experiments involve all the atoms of the overlapping condensates and could, therefore, serve the desired purpose. We now examine this possibility. We begin by outlining an analytical method to predict the emergence of an interference pattern when two scalar condensates are made to overlap. The results of this section are well known from analytical [22,12], experimental [23], and numerical [10] studies. It mainly serves as an introduction to Sec. VI, where we generalize the calculation to the case of spinor condensates.

For simplicity we shall treat the problem in one dimension. We shall neglect interatomic interactions [24]. We shall also assume that the two condensates have equal numbers of atoms and are pushed towards each other such that at the time of imaging they occupy the single-particle wave functions $\theta(1/2+x)\,\theta(1/2-x)e^{\pm ipx}$, respectively, where $\theta(x)$ is the Heaviside step function and p is a small multiple of π, so that at most a few interference fringes are formed. The creation operators for the two condensates are $a^\dagger(\pm p)=\int_{-1/2}^{1/2}dx\,e^{\pm ipx}\psi^\dagger(x)$. The state of the system can be written in the form [10]

$$\begin{aligned}|\Psi\rangle_0&=\frac{1}{(N/2)!}[a^\dagger(p)]^{N/2}[a^\dagger(-p)]^{N/2}|0\rangle\\&=\left(\frac{\pi N}{2}\right)^{1/4}\int_{-\pi}^{\pi}\frac{d\chi}{2\pi\sqrt{N!}}\left(\frac{e^{i\chi/2}}{\sqrt{2}}a^\dagger(p)\right.\\&\quad\left.+\frac{e^{-i\chi/2}}{\sqrt{2}}a^\dagger(-p)\right)^N|0\rangle\\&=\left(\frac{\pi N}{2}\right)^{1/4}\int_{-\pi}^{\pi}\frac{d\chi}{2\pi\sqrt{N!}}\left(\sqrt{2}\int_{-1/2}^{1/2}dx\right.\\&\quad\left.\times\cos(px+\chi/2)\psi^\dagger(x)\right)^N|0\rangle,\end{aligned}\tag{29}$$

which means that a product of two Fock states, which have a well-defined number of atoms in each condensate, is a linear superposition of all possible phase states between $-\pi$ and π, each of which would produce interference fringes. The imaging device is modeled by a large number of adjacent atom detectors covering the entire spatial extent of the condensates. In each detection event an atom is removed from the condensate at the position of the respective detector. The interference pattern is then obtained by plotting a histogram of the number of detected atoms as a function of the position. In a real experiment it is the time of shining the imaging laser beam that is controlled and not the number of detected atoms. However, our method demonstrates the emergence of the interference pattern without going through the details of how the image is produced.

The probability density of finding the nth detected atom at position x is given by

$$\rho_n(x)=\frac{{}_{n-1}\langle\Psi|\psi^\dagger(x)\psi(x)|\Psi\rangle_{n-1}}{{}_{n-1}\langle\Psi|N-n+1|\Psi\rangle_{n-1}} \approx\frac{2\int d\chi\cos^2(px+\chi/2)|c_{n-1}(\chi)|^2}{\int d\chi|c_{n-1}(\chi)|^2}, \quad (30)$$

where $c_n(\chi)$ is defined implicitly by $|\Psi\rangle_n\equiv\int d\chi c_n(\chi)[\int dx\sqrt{2}\cos(px+\chi/2)\psi^\dagger(x)]^{N-n}|0\rangle$, and we have used the quasiorthogonality of phase states [11]. The state of the system after n measurements is

$$|\Psi\rangle_n=\frac{\psi(x_n)|\Psi\rangle_{n-1}}{\sqrt{{}_{n-1}\langle\Psi|\psi^\dagger(x_n)\psi(x_n)|\Psi\rangle_{n-1}}} \propto\int d\chi\cos(px_n+\chi/2)c_{n-1}(\chi) \times\left[\int dx\sqrt{2}\cos(px+\chi/2)\psi^\dagger(x)\right]^{N-n}|0\rangle. \quad (31)$$

Thus, the probability density of finding the first n detected atoms at $x_1,\ldots,x_n$ is

$$\rho(x_1,\ldots,x_n)\approx\int\frac{d\chi}{2\pi}2^n\prod_l\cos^2(px_l+\chi/2). \quad (32)$$

This expression is what one would expect to find for a uniform distribution of initial states with well-defined relative phases. Thus, Eq. (32) is sufficient to show that the interference pattern that arises from the spatial overlap of two condensates is the same whether the initial state is a product of two independent Fock states or a uniform statistical distribution of phase states. However, we shall go on and find explicit expressions for the probability of finding a certain density distribution.

A given density distribution is defined by the occupation of K cells of width $1/K$ and centred at $X_l=l/K$ (corresponding to the detectors). Some manipulation of Eq. (32) gives the probability of finding the distribution $\{k_l\}$ of atoms in the cells labeled by l,

$$P(k_{-K/2},\ldots,k_{K/2})\approx\int\frac{d\chi}{2\pi}F(k_{-K/2},\ldots,k_{K/2}|\chi), \quad (33)$$

where

$$F(k_{-K/2},\ldots,k_{K/2}|\chi)=\frac{n!}{K^n}\prod_l\frac{R(pX_l+\chi/2)^{k_l}}{k_l!}, \quad (34)$$

$$R(pX_l+\chi/2)=2\cos^2(pX_l+\chi/2)+O\left(\frac{p}{K}\right)^2. \quad (35)$$

The function $R(pX_l+\chi/2)$ is the interference pattern one finds for a well-defined value of the relative phase χ. For large K the second term in Eq. (35) can be neglected for most values of X_l, but we keep it in mind to avoid divergences that would arise in our approximate expressions if $R(pX_l+\chi/2)=0$. A straightforward variational calculation shows that for a given value of χ, the function F takes its maximum value when

$$k_l=\frac{n}{K}R(pX_l+\chi/2). \quad (36)$$

We also find that at the point of maximum value

$$F\approx\left(\frac{n}{(2\pi)^{K-1}\Pi_l k_l}\right)^{1/2}, \qquad \frac{d^2\ln F}{dk_l dk_m}\approx-\frac{K}{nR(pX_l+\chi/2)}\delta_{l,m}. \quad (37)$$

It should be noted that these expressions are good approximations only when all the k_l's are large. We shall make that assumption, since it does not affect the essential physics of the interference process. Thus, $F(k_{-K/2,\ldots,}k_{K/2}|\chi)$ can be approximated by

$$F(k_{-K/2},\ldots,k_{K/2}|\chi)\approx\left(\frac{n}{(2\pi)^{K-1}\Pi_l k_l}\right)^{1/2}e^{-nG}, \quad (38)$$

where

$$G=\frac{K}{n^2}\sum_l\frac{1}{R(pX_l+\chi/2)}\left(k_l-\frac{n}{K}R(pX_l+\chi/2)\right)^2 \approx\int dx\frac{[f(x)-R(px+\chi/2)]^2}{R(px+\chi/2)} \quad (39)$$

and $f(x)$ is a smooth function defined between $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ such that $f(X_l)=Kk_l/n$.

The probability of finding a certain (normalized) density distribution $f(x)$ takes the maximum value if $f(x)=R(px+\chi/2)$ for some value of χ, and drops to negligibly small values when $G\gtrsim K/n$ for all values of χ. Thus, in any single interference experiment one expects to find

$$f(x)=R(px+\chi/2)+\sqrt{(K/n)}\epsilon(x), \tag{40}$$

where $\epsilon(x)$ is a random function with $|\epsilon(x)|\lesssim 1$. The function $R(px+\chi/2)$ describes a smooth sinusoidal density distribution, whereas $\epsilon(x)$ describes shot-to-shot fluctuations around that distribution. (Notice that these fluctuations exist even if the initial state is a phase state.) The visibility of the function $R(px+\chi/2)$ is

$$V\equiv\frac{R^{\max}-R^{\min}}{R^{\max}+R^{\min}}=1-O\left(\frac{p}{K}\right)^2. \tag{41}$$

In any single run, an interference pattern with almost 100% visibility is observed (corresponding to a randomly chosen value of the relative phase).

VI. INTERFERENCE BETWEEN TWO SPINOR CONDENSATES

We now generalize the method of the preceding section to the case of spinor condensates. In spinor condensates, atoms belonging to different hyperfine states do not interfere. Therefore, the appearance or absence of interference fringes depends on the spin structure of the condensates. Here we shall study the interference patterns that would arise from the spatial overlap of two spinor condensates, and show that such an experiment gives the same results whether we start with coherent or singlet states.

At the time of imaging we assume that the state can be approximated using the following states.

(1) Coherent states:

$$|\Psi\rangle_0=\frac{1}{(N/2)!}[a^\dagger(p,\theta_1,\phi_1)]^{N/2}[a^\dagger(-p,\theta_2,\phi_2)]^{N/2}|0\rangle$$
$$=\left(\frac{\pi N}{2}\right)^{1/4}\int\frac{d\chi}{2\pi\sqrt{N!}}\left(\frac{e^{i\chi/2}}{\sqrt{2}}a^\dagger(p,\theta_1,\phi_1)+\frac{e^{-i\chi/2}}{\sqrt{2}}a^\dagger(-p,\theta_2,\phi_2)\right)^N|0\rangle. \tag{42}$$

(2) Singlet states. From Eq. (12) we can see that

$$|\Psi\rangle_0\propto\int d\Omega_1 d\Omega_2[a^\dagger(p,\theta_1,\phi_1)]^{N/2}[a^\dagger(-p,\theta_2,\phi_2)]^{N/2}|0\rangle$$
$$\propto\int d\Omega_1 d\Omega_2 d\chi\left(\frac{e^{i\chi/2}}{\sqrt{2}}a^\dagger(p,\theta_1,\phi_1)+\frac{e^{-i\chi/2}}{\sqrt{2}}a^\dagger(-p,\theta_2,\phi_2)\right)^N|0\rangle, \tag{43}$$

where

$$a^\dagger(\pm p,\theta,\phi)=-\frac{\sin\theta e^{-i\phi}}{\sqrt{2}}a_1^\dagger(\pm p)+\cos\theta a_0^\dagger(\pm p)+\frac{\sin\theta e^{i\phi}}{\sqrt{2}}a_{-1}^\dagger(\pm p) \tag{44}$$

and the creation operators $a_m^\dagger(\pm p)$ are the (obvious) generalization of the $a^\dagger(\pm p)$'s used in Sec. V [25].

Assuming that the detectors measure the spin of the atom as well as its position, we find that the probability density for finding n atoms at $x_1,\ldots,x_n$ with S_z values $m_1,\ldots,m_n$ is as follows.

(1) Coherent states:

$$\rho(x_1,m_1,\ldots,x_n,m_n)\approx\int\frac{d\chi}{2\pi}\prod_l R_{m_l}\times(px_l+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2). \tag{45}$$

(2) Singlet states:

$$\rho(x_1,m_1,\ldots,x_n,m_n)\approx\int\frac{d\chi d\Omega_1 d\Omega_2}{(2\pi)^3}\prod_l R_{m_l}(px_l+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2), \tag{46}$$

where, to zeroth order in p/K,

$$R_m(px_l+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2)=\frac{(d^1_{m,0})^2(\theta_1)+(d^1_{m,0})^2(\theta_2)}{2}+d^1_{m,0}(\theta_1)d^1_{m,0}(\theta_2)\times\cos[2px_l+\chi+m(\phi_2-\phi_1)]. \tag{47}$$

By averaging Eq. (45) over all directions for both condensates, the indistinguishability between the coherent and the singlet states becomes obvious. What that means is that in a single run of the experiment, the interfering condensates behave as if they were in coherent states. If the initial state is a product of two singlet states, a (coherent-state) direction is chosen randomly for each condensate, and from the arguments of Sec. V a relative phase between the condensates is also chosen randomly.

Now we calculate the probability of finding a certain density distribution (which is now a three-component quantity). The results apply for both a uniform distribution of coherent states and a product of two singlet states. The calculation parallels that of Sec. V. Therefore, we shall skip some of the intermediate steps. The probability of finding the density distribution $\{k_{l,m}\}$ (where l and m are spatial and hyperfine indices, respectively) is given by

$$P(\{k_{l,m}\})=\int\frac{d\chi d\Omega_1 d\Omega_2}{(2\pi)^3}F(\{k_{l,m}\}|\chi,\theta_1,\theta_2,\phi_1,\phi_2). \tag{48}$$

where

$$F(\{k_{l,m}\}|\chi,\theta_1,\theta_2,\phi_1,\phi_2) \approx \left(\frac{n}{(2\pi)^{3K-1}\Pi_{l,m}k_{l,m}}\right)^{1/2} \exp\left[-n\sum_m G_m\right], \tag{49}$$

$$G_m = \frac{K}{n^2}\sum_l \frac{\left[k_l - \frac{n}{K}R_m(pX_l+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2)\right]^2}{R_m(pX_l+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2)} \approx \int dx \frac{[f_m(x)-R_m(px+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2)]^2}{R_m(px+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2)}, \tag{50}$$

and $f_m(x)$ is a smooth function defined between $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ such that $f_m(X_l)=Kk_{l,m}/n$. The probability is maximized when $G_1=G_0=G_{-1}=0$, i.e., when $f_m(x)=R_m(px+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2)$ for some χ, θ_1, θ_2, ϕ_1 and ϕ_2 [26].

Unlike scalar condensates where interference fringes are obtained in every run of the experiment, the appearance or absence of interference fringes has a probabilistic nature in the case of spinor condensates. From Eq. (47) we can see that unless $\theta_1=\theta_2$, the visibility of the interference fringes will be less than 100%. The visibility of the m component of the interference pattern is given by

$$V_m \equiv \frac{R_m^{\max}-R_m^{\min}}{R_m^{\max}+R_m^{\min}} = \frac{2d^1_{m,0}(\theta_1)d^1_{m,0}(\theta_2)}{(d^1_{m,0})^2(\theta_1)+(d^1_{m,0})^2(\theta_2)}. \tag{51}$$

In particular, if $\theta_1=0$, $\theta_2=\pi/2$ (or vice versa), the visibility is zero and there are no interference fringes. Notice, however, that if the z axis of the detectors is rotated to a direction perpendicular to both (θ_1,ϕ_1) and (θ_2,ϕ_2), one sees interference fringes with 100% visibility. Thus, the interference pattern depends not only on the relative angle between (θ_1,ϕ_1) and (θ_2,ϕ_2), but also on the quantization axis of the detectors. This is a well-known phenomenon in neutron interference experiments [27].

The total density $\rho(x)=(n/K)\Sigma_m R_m(px+\chi/2,\theta_1,\theta_2,\phi_1-\phi_2)$, however, must be rotationally invariant (under rotations in spin space),

$$V_\rho \equiv \frac{\rho^{\max}-\rho^{\min}}{\rho^{\max}+\rho^{\min}} = |\cos\theta|, \tag{52}$$

where θ is the angle between (θ_1,ϕ_1) and (θ_2,ϕ_2). If $\theta=0$, the total density shows interference fringes with vanishing minima (100% visibility), whereas if $\theta=\pi/2$, the total density is constant in space (0% visibility). In an ensemble of measurements, one finds a uniform distribution of all values between 0% and 100%.

VII. CONCLUSIONS

We have shown that a measurement based on the detection of a small number of atoms leaving a Bose-Einstein condensate cannot be used to distinguish between an antiferromagnetic coherent state and a spin-singlet state. A singlet state will be projected, *because of the measurement*, closer and closer to a coherent state. Atom loss from the condensate has the same effect as a detection measurement, and therefore it provides a mechanism for spontaneous symmetry breaking. We have neglected the effect of spin-dependent interatomic interactions on the dynamics of the condensate. Those effects can be significant if the relevant energies are large compared to the inverse of the measurement time. However, they do not affect our result that one needs to measure occupation numbers to a relative accuracy better than $N^{-1/2}$ in order to distinguish between a coherent and a singlet state.

We have also shown that interference experiments cannot be used to distinguish between the two states in question. We have studied the possible interference patterns produced by two overlapping spinor condensates and shown that the appearance or absence of interference fringes has a probabilistic nature. One amusing result is that the interference pattern depends not only on the relative orientation of the spin states of the overlapping condenstes, but also on the quantization axis of the atom detectors.

ACKNOWLEDGMENTS

We would like to thank M. Holzmann, P. G. Kwiat, and E. J. Mueller for useful discussions, and C. M. Elliott for help in preparing the manuscript. We acknowledge correspondence with J. Javanainen on the results obtained in Ref. [9]. This work was supported by the National Science Foundation through Grant No. DMR-9986199.

APPENDIX: OVERCOMPLETENESS AND QUASIORTHOGONALITY OF COHERENT STATES

In this appendix we demonstrate two useful properties of coherent states of spinor condensates, namely, overcompleteness and quasiorthogonality. The former means that any state of the condensate containing N particles can be expressed as

$$|\Psi\rangle = \int d\Omega\, c(\theta,\phi)|N,\theta,\phi\rangle, \tag{A1}$$

where $c(\theta,\phi)$ is some function of θ and ϕ. For example, the complete basis defined by the total spin and the z component of the total spin of the condensate [28] can be written as

$$|N,S,S_z\rangle = \text{const}\times\int d\Omega\, Y_{S,S_z}(\theta,\phi)|N,\theta,\phi\rangle. \tag{A2}$$

The states (A2) transform in a manner similar to the spherical harmonics Y_{S,S_z} under an arbitrary rotation, and they posses a nonzero norm. Thus, completeness (or overcompleteness) is proved. Quasiorthogonality follows from

$$\langle N,\theta_1,\phi_1|N,\theta_2,\phi_2\rangle = (\cos\theta)^N, \tag{A3}$$

where θ is the angle between two vectors defined by (θ_1,ϕ_1) and (θ_2,ϕ_2). For large N the inner product is nonvanishing at two points: when the two directions are parallel or antiparallel. It can be approximated by

$$\langle N,\theta_1,\phi_1|N,\theta_2,\phi_2\rangle \approx \frac{2\pi}{N\sin\theta_1}[\delta(\theta_1-\theta_2,\phi_1-\phi_2)+\delta(\theta_1+\theta_2-\pi,\phi_1-\phi_2\pm\pi)]. \quad (A4)$$

The plus sign in front of the second δ function in Eq. (A4) would be replaced by a minus sign if N were odd. The state obtained by rotating (θ,ϕ) to its antiparallel direction is the same state (except for a possible change of sign). Therefore we shall, with no loss of generality, restrict the basis states and the integrals to the upper hemisphere, i.e., θ_1, θ_2 range from 0 to $\pi/2$. One has to be careful that Eq. (A4) applies only when the coefficient $c(\theta,\phi)$ varies slowly in θ and ϕ. It cannot be applied to states with large S_{total} (e.g., the ferromagnetic ground state), since the function Y_{S,S_z} in Eq. (A2) changes more and more rapidly with increasing S. On the other hand, $S^2=S_z=0$ for the singlet state and the coefficient $c(\theta,\phi)$ in Eq. (A1) is independent of θ and ϕ

$$c(\theta,\phi)=\frac{\sqrt{N}}{2\pi}. \quad (A5)$$

[1] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, and W. Ketterle, Nature (London) **396**, 345 (1998).
[2] T. L. Ho, Phys. Rev. Lett. **81**, 742 (1998).
[3] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. **67**, 1822 (1998).
[4] C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. **81**, 5257 (1998).
[5] M. Koashi and M. Ueda, Phys. Rev. Lett. **84**, 1066 (2000).
[6] T. L. Ho and S. K. Yip, Phys. Rev. Lett. **84**, 4031 (2000).
[7] E. J. Mueller *et al.* (unpublished).
[8] Y. Castin and C. Herzog, C.R. Acad. Sci., Ser. IV **2**, 419 (2001).
[9] J. Javanainen, J. Phys. B **33**, 5493 (2000).
[10] J. Javanainen and S. M. Yoo, Phys. Rev. Lett. **76**, 161 (1996).
[11] Y. Castin and J. Dalibard, Phys. Rev. A **55**, 4330 (1997).
[12] R. A. Hegstrom, Chem. Phys. Lett. **288**, 248 (1998).
[13] J. Javanainen and M. Y. Ivanov, Phys. Rev. A **60**, 2351 (1999).
[14] A. J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).
[15] S. Kohler and F. Sols, Phys. Rev. A **63**, 053605 (2001).
[16] Although we have not demonstrated this result explicitly, it is overwhelmingly plausible from the analogy with the phase measurement [11].
[17] H. Pu, C. K. Law, S. Raghavan, J. H. Eberly, and N. P. Bigelow, Phys. Rev. A **60**, 1463 (1999).
[18] W. H. Zurek, S. Habib, and J. P. Paz, Phys. Rev. Lett. **70**, 1187 (1993).
[19] H. D. Politzer, Phys. Rev. A **55**, 1140 (1997).
[20] E. V. Goldstein and P. Meystre, Phys. Rev. Lett. **80**, 5036 (1998); E. V. Goldstein, O. Zobay, and P. Meystre, Phys. Rev. A **58**, 2373 (1998).
[21] An amusing idea is that if one could find a laser beam that is phase shifted by δ per $S_z=1$ atom, by $-\delta$ per $S_z=-1$ atom and is not affected by $S_z=0$ atoms, one would be able to measure n_1-n_{-1}, without measuring n_1 and n_{-1} separately. In that case the difficulty of relative accuracy of $\sim N^{-1/2}$ would be avoided. Unfortunately, we do not have a realistic experimental suggestion that would realize the above description.
[22] M. Naraschewski, H. Wallis, A. Schenzle, J. I. Cirac, and P. Zoller, Phys. Rev. A **54**, 2185 (1996); J. I. Cirac, C. W. Gardiner, M. Naraschewski, and P. Zollar, *ibid.* **54**, R3714 (1996).
[23] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, Science **275**, 637 (1997).
[24] Interatomic interactions add quantitative complications to the problem. However, they do not affect the existence or absence of interference fringes. See H. Wallis, A. Rohrl, M. Naraschewski, and A. Schenzle, Phys. Rev. A **55**, 2109 (1997).
[25] In the MIT interference experiment [23] the condensates were not pushed towards each other, but rather let to expand until they overlapped. A theoretical description of such an experiment requires an understanding of the hydrodynamics of expansion, which to our knowledge has not been carried out explicitly for spinor condensates. A related problem was treated by A. Sinatra, P. O. Fedichev, Y. Castin, J. Dalibard, and G. V. Shlyapnikov, Phys. Rev. Lett. **82**, 251 (1999).
[26] The prefactor in Eq. (49) is largest for those interference patterns having $R_m^{\min}\cong 0$. However, this does not mean that these patterns have higher probability than others, because that factor is canceled out by another factor corresponding to the spread in $k_{l,m}$ around its optimum value.
[27] G. Badurek, H. Rauch, and J. Summhammer, Phys. Rev. Lett. **51**, 1015 (1983).
[28] T. L. Ho, L. Yin, Phys. Rev. Lett. **84**, 2302 (2000).