# Remote-state preparation in higher dimension and the parallelizable manifold $S^{n-1}$

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(Received 28 May 2001; published 15 January 2002)

This paper proves that the remote-state preparation (RSP) scheme in real Hilbert space can only be implemented when the dimension of the space is 2, 4, or 8. This fact is shown to be related to the parallelizability of the $(n-1)$-dimensional sphere $S^{n-1}$. When the dimension is 4 and 8 the generalized scheme is explicitly presented. It is also shown that for a given state with components having the same norm, RSP can be generalized to arbitrary dimension case.

DOI: 10.1103/PhysRevA.65.022316 PACS number(s): 03.67.−a

Remote-state preparation (RSP) [1–3] is called "teleportation of a known state." Unlike quantum teleportation [4–8], in RSP, Alice knows the state that she will transmit to Bob. Her task is to help Bob construct a state that is unknown to him by means of a prior shared entanglement and a classical communication channel. Recently, Pati has shown that a state of a qubit chosen from equatorial or polar great circles on the Bloch sphere (i.e., a state with the components of the same amplitude or with real components) can be remotely prepared with one cbit from Alice to Bob if they share one ebit of entanglement [1]. Here, qubit stands for quantum bit whose state is a superposition of two orthonormal bases $|0\rangle$ and $|1\rangle$; one cbit is one-ary classical states of communication carrying classical information; and ebit is the so-called entanglement bit usually carrying a Bell state. It is noted that in Pati's special case, to remotely prepare a state of one qubit, the entanglement cost is the same as that in teleportation but the classical information cost is only half of that in teleportation. Most recently, Lo [2] and Bennett *et al.* [3] have studied the classical information cost for general state preparation in the scheme of RSP, using the concepts of entanglement dilution [9], high-entanglement limit and low-entanglement RSP [3]. They have also investigated the trade off between entanglement cost and classical communication cost in RSP [2,3]. However, in protocols of Lo or Bennett *et al.* either the entanglement cost or the classical information cost is more than that in Pati's special case. This fact can be well understood by considering the geometry of Pati's case: Pati's states lie on the equatorial or polar great circles on a Bloch sphere. For this reason, we call the case treated by Pati the "minimum" case.

As Pati presents his result only in the qubit case, it is natural to ask whether his result can be generalized to the higher-dimension case. It is well known that as far as teleportation, which transmits an unknown state, is concerned, the generalization from the qubit case to higher-dimension case is straightforward. In fact, the first $n$-dimensional teleportation protocol is just given by Bennett *et al.* in their first paper that introduced the celebrated concept of quantum teleportation [4]. Later the $n$-dimensional case of teleportation and its mathematical background were studied in more detail by many other authors [10–13]. Even in the case concerning continuous variable [14], it can well be tackled [15]. The purpose of this paper is to seek a generalization of Pati's result to higher-dimension case. It will be shown that one can directly generalize the equatorial case. On the other hand, the generalization of the polar great circle case is highly nontrivial.

We first consider the generalization of the polar great circle case (i.e., the case that the state has real components). Precisely, we formulate our problem as follows. Suppose that Alice and Bob can share entangled state between two identical quantum systems the dimension of the state space of which is $n$. Choose an orthonormal basis $\{\phi_i | i=0,1,...,n-1\}$ of the state space. By measuring the system with respect to a certain basis, Alice wishes to prepare a quantum state of the form

$$|\Psi\rangle = \sum_{i=0}^{n-1} a_i |\phi_i\rangle$$

at Bob, where the coefficients are real numbers. Between Alice and Bob there is a classical channel capable of transmitting information carried by classical states that can take $n$ different values, say, $0,1,...,n-1$. By prior agreement, each value carried by the classical state can be corresponded to a unitary operation on the quantum system at Bob. That is to say, when Bob receives a value $i$ he will exert a certain unitary operation $U_i$ on his system. Now our question is the following: for the above minimum RSP procedure to be realizable what condition should the dimension $n$ satisfy? By convention, in the procedure of RSP the maximally entangled state shared by Alice and Bob, will be the Einstein-Podolsky-Rosen (EPR) state,

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{n}}\left(\sum_{i=0}^{n-1} |\phi_i\rangle \otimes |\phi_i\rangle\right).$$

*Remark.* In Ref. [1], the EPR state is $|\Phi\rangle_{AB} = (1/\sqrt{2}),(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle)$, which is a little different from the EPR state we use here. But there is no essential difference.

To prepare the state $|\Psi\rangle$ in a remote place, similar to the Pati's protocol in the qubit case, Alice needs to find a set of orthonormal basis $\{|\Psi_i\rangle\}_{i=0}^{n-1}$ with respect to which the measurement is done on her system. The EPR state $|\Phi\rangle_{AB}$ can be written as $|\Phi\rangle_{AB} = 1/\sqrt{n}\Sigma_i |\Psi_i\rangle \otimes |\Omega_i\rangle$. Here $|\Omega_i\rangle = \Sigma_{j,k} |\Psi_j\rangle [\langle\Psi_j|\phi_k\rangle\langle\Psi_i|\phi_k\rangle]$, $i=0,...,n-1$. We notice that

 

$\{|\Omega_i\rangle\}_{i=0}^{n-1}$ is a set of orthonormal vectors. To realize the minimum RSP task, there should exist $n$ unitary operators $U_i$ $(i=0,1,...,n-1)$ independent of $|\Psi\rangle$ such that $|\Omega_i\rangle = U_i|\Psi\rangle$. If such unitary operators do exist, then Alice can measure her system with respect to the basis $\{|\Psi_i\rangle\}_{i=0}^{n-1}$ and get a state $|\Psi_i\rangle$. Then through the classical communication channel, she can send Bob the value $i$. After receiving the message, Bob will be able to construct the target state $|\Psi\rangle$ by letting his system experience the unitary evolution $U_i$, according to their prior agreement. It turns out that the requirement that such unitary operators $U_i$'s exist imposes very strong restriction on the dimension of the state space. Before proceeding along with the discussion, let us prepare some terminology about parallelizable manifold.

Let $M$ be a manifold of dimension $n$. The tangent space $T_xM$ is well defined for every point $x\in M$ as the real vector space consisting of all tangent vectors to $M$ at $x$ [16]. A continuous vector field $v$ in $M$ is a continuous function that assigns a vector $v(x)\in T_xM$ to every $x\in M$. By a $k$-field we mean a $k$-tuple $v_1,v_2,\ldots,v_k$ of continuous vector fields on $M$, such that the vectors $v_1(x),\ldots,v_n(x)$ at each point $x\in M$ are linearly independent. The largest $k$ for which a $k$ field exists is called Span$(M)$. If Span$(M)=n$, then the manifold is said to be parallelizable. It is a difficult problem to determine Span$(M)$ for any given manifold. But we have the following deep result [17].

*Theorem*. The sphere $S^{n-1}$ is parallelizable only for $n=1,2,4,8$. We proceed to prove the following interesting result.

*Proposition*. If the minimum RSP scheme is realizable in $n$-dimensional real Hilbert space, then the sphere $S^{n-1}$ is parallelizable.

*Proof*. From the above discussion, if RSP is realizable there should exist $n$ unitary operators $U_i(i=0,1,...,n-1)$ such that $|\Omega_i\rangle = U_i|\Psi\rangle$. As pointed out above, $\{|\Omega_i\rangle | i=0,1,...,n-1\}$ is a set of orthonormal vectors. Thus we have

$$\langle\Psi|U_0^{-\dagger}U_i|\Psi\rangle=0, \quad i=1,2,...,n-1.$$

Write $U_0^{-\dagger}U_i$ as

$$U_0^{-\dagger}U_i=V_i+\sqrt{-1}W_i,$$

where $V_i$ and $W_i$ are real matrices. Then it follows that

$$\langle\Psi|V_i|\Psi\rangle=0, \quad i=1,2,...,n-1$$

as $|\Psi\rangle$ has real coefficients. If we only consider the case $W_i=0$, i.e., the minimum RSP scheme in real Hilbert space, this $\{V_i|\Psi\rangle\}$ $(i=0,1,2,...n-1)$ form an orthonormal basis of $n$-dimensional real Hilbert space. Obviously, $|\Psi\rangle$ can be regarded as a point on $S^{n-1}$. Thus the map $|\Psi\rangle\to V_i|\Psi\rangle$ defines an $(n-1)$ field on the manifold $S^{n-1}$. This means that $S^{n-1}$ is parallelizable.

Now we are prepared to present the main result of this paper.

*Main theorem*. Minimum RSP is realizable in real Hilbert space if and only if the dimension of the space is 1, 2, 4, or 8.

We notice that the "only if" part of the theorem is a direct consequence of the above proposition and the cited theorem preceding it. To prove the "if part" of the theorem we only need to show that when $n=1$, 2, 4, or 8 there exist real unitary matrices $V_i(i=0,1,...,n-1)$ such that for any $\Psi$ with real coefficients $\{V_i|\Psi\rangle | i=0,1,...,n-1\}$ is an orthonormal basis of the state space. Indeed if such unitary matrices exist then the EPR state can be rewritten as

$$|\Phi\rangle_{AB}=\frac{1}{\sqrt{n}}\left(\sum_{i=0}^{n-1}|\Psi_i\rangle\otimes|\Psi_i\rangle\right),$$

where $|\Psi_i\rangle=V_i|\Psi\rangle$. Then it is clear that RSP can be realized. Since the one-dimensional case is trivial and the two-dimensional case have been dealt with by Pati [1], in the following we only consider the cases of $n=4$ and $n=8$.

We observe that the existence of the above mentioned $V_i$ is closely related to the existence of $(n-1)$-field on the manifold $S^{n-1}$. So at this point it is enlightening to recall the marvelous method of relating the dimension $n$ of a division algebra over the real number field $R$ to the parallelizability of the manifold $S^{n-1}$. It turns out that by this method we can find the $V_i$'s we need.

It is noticed that if $A$ is a division algebra of dimension $n$, one can choose a vector space isomorphism to $A$ onto $R^n$ and transfer the multiplication defined on $A$ to $R^n$ [17]. Let $e_1,e_2,\ldots,e_n$ be the standard basis vectors of $R^n$ and let $y\in S^{n-1}$. Then the vectors $e_1\cdot y$, $e_2\cdot y,\ldots,e_n\cdot y$ are linear independent. If we orthonormalize them we obtain $n$ vectors $V_0(y)$, $V_1(y),\ldots,V_{n-1}(y)$. The vectors $V_1(y),\ldots,V_{n-1}(y)$ are tangential to $S^{n-1}$ at the point $V_0(y)$. They define an $(n-1)$ field on $S^{n-1}$. Now it is not difficult to see that when we take $A$ to be the quarternion algebra and the ontonion algebra, whose dimension is 4 and 8, respectively, these $V_i$'s are exactly what we need. This finishes the proof of the main theorem.

To illustrate the above procedure we explicitly calculate the $V_i$'s as follows.

When $n=4$, we consider the quaternion field H. A quaternion in H can be expressed as $A=a_0e_0+a_1e_1+a_2e_2+a_3e_3$, where $\{e_i\}_{i=0}^3$ form the standard basis of quaternion. According to the rules of Hamilton multiplication [17], two quaternion's Hamilton multiplication can be calculated as follows:

$$\begin{aligned}A\cdot B&=(a_0e_0+a_1e_1+a_2e_2+a_3e_3)\cdot(b_0e_0+b_1e_1+b_2e_2+b_3e_3)\\&=(a_0b_0-a_1b_1-a_2b_2-a_3b_3)e_0+(a_0b_1+a_1b_0+a_2b_3-a_3b_2)e_1+(a_0b_2-a_1b_3+a_2b_0+a_3b_1)e_2\\&\quad+(a_0b_3+a_1b_2-a_2b_1+a_3b_0)e_3.\end{aligned}$$

Of course, with the usual addition and scalar product, $H$ can be considered as a vector space over $R$, which is isomorphic to $R^4$ and $e_0$, $e_1$, $e_2$, $e_3$ form a set of natural basis of this linear space. The inner product in $H$ can be defined as $\langle e_i, e_j\rangle = \delta_{ij}$, i.e., $\langle A,B\rangle = \Sigma_{i=0}^{3} a_i b_i$. For an arbitrary unit vector A that satisfies $\langle A,A\rangle = 1$, we can define a set of vectors $\{A_i = e_i \cdot A\}_{i=0}^{3}$. Using the property of division algebra [17], we have $\langle A_i, A_j\rangle = \langle \epsilon_j, e_i\rangle\langle A,A\rangle = \langle e_j, e_i\rangle = \delta_{ij}$. Therefore, $\{A_i\}_{i=0}^{3}$ is a set of orthonormal basis. It is easy to see that $A_0 = A$ and the orthonormal transformations $\{V_i\}_{i=0}^{3}$ that transform $A$ to $\{A_i\}_{i=0}^{3}$ are independent of $A$. Therefore, $\{V_i\}_{i=0}^{3}$ are just what we want to find.

A direct calculation following the above steps gives the following result in the four-dimension case:

$$V_0 = I, \quad V_1 = \begin{bmatrix} -i\sigma_y & 0 \\ 0 & -i\sigma_y \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{bmatrix}.$$

When $n=8$, using the rules of Cayley multiplication [17], similarly we can get $\{V_i\}_{i=0}^{7}$. The result is as follows:

$$V_0 = I, \quad V_1 = \begin{bmatrix} -i\sigma_y & 0 & 0 & 0 \\ 0 & -i\sigma_y & 0 & 0 \\ 0 & 0 & -i\sigma_y & 0 \\ 0 & 0 & 0 & -i\sigma_y \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0 & -\sigma_x & 0 & 0 \\ \sigma_z & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} 0 & -\sigma_x & 0 & 0 \\ \sigma_x & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_x \\ 0 & 0 & -i\sigma_y & 0 \end{bmatrix},$$

$$V_4 = \begin{bmatrix} 0 & 0 & -\sigma_z & 0 \\ 0 & 0 & 0 & I \\ \sigma_z & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{bmatrix},$$

$$V_5 = \begin{bmatrix} 0 & 0 & -\sigma_x & 0 \\ 0 & 0 & 0 & i\sigma_y \\ \sigma_x & 0 & 0 & 0 \\ 0 & i\sigma_y & 0 & 0 \end{bmatrix},$$

$$V_6 = \begin{bmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & -\sigma_x & 0 \\ 0 & \sigma_z & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix},$$

$$V_7 = \begin{bmatrix} 0 & 0 & 0 & -i\sigma_y \\ 0 & 0 & -\sigma_z & 0 \\ 0 & \sigma_x & 0 & 0 \\ -i\sigma_y & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, in general the case that $W_i \neq 0$, i.e., minimum RSP scheme for states with real components in complex Hilbert space should be taken into account. We conjecture that even in this case, minimum RSP scheme can only be implemented when the dimension of the space is 2, 4, or 8.

Now we consider the generalization of RSP scheme of the equatorial case. In this case, the state to be remotely prepared can be written in the form

$$|\Psi\rangle = \sum_{\alpha=0}^{n-1} \frac{1}{\sqrt{n}} e^{i\theta_\alpha}|\alpha\rangle.$$

Without loss of generality, we set $\theta_0 = 0$. We will show the RSP scheme for such states is realizable whatever the dimension $n$ is. It is easily seen that $\{|\Psi_\alpha\rangle | |\Psi_\alpha\rangle = 1/\sqrt{n}\Sigma_{\beta=0}^{n-1} e^{(2\pi_i/n)\alpha\beta} e^{i\theta_\alpha}|\beta\rangle\rangle\}_{\alpha=0}^{n-1}$ is an orthonormal basis in the $n$-dimensional case, and that the unitary transformation $U_\alpha$: $U_\alpha|\Psi\rangle = |\Psi_\alpha\rangle$ is independent of $|\Psi\rangle$. As the first step to remotely prepare $|\Psi\rangle$, Alice needs to do a local unitary transformation $U_A(|\Psi\rangle)$ on her particle. Here, $U_A(|\Psi\rangle)$ is defined as

$$U_A(|\Psi\rangle) = \sum_{\alpha=1}^{n-1} |\alpha\rangle\langle n-\alpha| \exp[i(\theta_\alpha + \theta_{n-\alpha})] + |0\rangle\langle 0|.$$

Thus we have

$$U_A(|\Psi\rangle) \otimes I_B |\Phi\rangle_{AB} = \sum_{\alpha=0}^{n-1} \frac{1}{\sqrt{n}} |\Psi_\alpha\rangle \otimes |\Psi_\alpha\rangle.$$

Here $|\Phi\rangle_{AB}$ is the EPR state $|\Phi\rangle_{AB} = \Sigma_{\alpha=0}^{n-1} 1/\sqrt{n}|\alpha\rangle \otimes |\alpha\rangle$ of the entangled pair that is priorly shared by Alice and Bob. After the transformation, Alice can measure her particle with respect to the basis $\{|\Psi_\alpha\rangle\}_{\alpha=0}^{n-1}$ and tell her result to Bob. Then Bob can do the unitary transformation $U_\alpha^{-1}$ to get the state $|\Psi\rangle$. This implements the RSP task.

In summary, this paper generalizes Pati's minimum RSP scheme to the case of higher dimension. We have shown that the minimum RSP scheme in real Hilbert space can be implemented only when the dimension is 2, 4, or 8, while the equatorial case can be generalized without restriction on the dimension. However, whether the minimum RSP scheme for the states with real components in complex Hilbert space is realizable in spaces other than 2, 4, and 8 dimension needs further investigation.

The authors gratefully thank Professor C. P. Sun for initial discussion on RSP. The authors are also grateful to Dr. X. N. Wu, Dr. Y. Q. Li, and especially Professor X. F. Liu for discussions with them. This work is partially supported by the NSF of China.

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