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General scheme for superdense coding between multiparties

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Dense coding or superdense coding in the case of high-dimension quantum states between two parties and multiparties is studied in this paper. We construct explicitly the measurement basis and the forms of the single-body unitary operations corresponding to the basis chosen, and the rules for selecting the one-body unitary operations in a multiparty case.

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Quantum dense coding or superdense coding [1] is one of the important branches of quantum-information theory. It has been widely studied both in theory and in experiment [1,2]. The basic idea of quantum dense coding is that quantum mechanics allows one to encode information in the quantum states that is denser than classical coding. Bell-basis states

$$
\begin{aligned}
|\Psi^+\rangle &= (|00\rangle + |11\rangle)/\sqrt{2},\\
|\Psi^-\rangle &= (|00\rangle - |11\rangle)/\sqrt{2},\\
|\phi^+\rangle &= (|01\rangle + |10\rangle)/\sqrt{2},\\
|\phi^-\rangle &= (|01\rangle - |10\rangle)/\sqrt{2},
\end{aligned}
\tag{1}
$$

are used in dense coding. Bell-basis states are in the Hilbert space of two particles, each with two dimensions, and they are the maximally entangled states. Suppose Alice and Bob share the maximally entangled state $|\Psi^+\rangle$. Bob then operates locally on the particle he shares with Alice, one of the four unitary transformations $I, \sigma_x, i\sigma_y, \sigma_z$, and this will transform $|\Psi^+\rangle$ into $|\Psi^+\rangle$, $|\phi^+\rangle$, $|\phi^-\rangle$, and $|\Psi^-\rangle$, respectively. Bob sends his particle back to Alice. Because the four manipulations result in four orthogonal Bell states, four distinguishable messages, i.e., 2 bits of information, can then be obtained by Alice via collective measurement on the two particles. The scheme has been experimentally demonstrated by Mattle *et al.* [3].

With the realization of preparing high-dimension quantum state [4], it is of practical importance to study the high-dimensional aspects of various topics in quantum information. For example, a multiparticle high-dimensional quantum teleportation has been constructed recently [5]. The teleportation and quantum dense coding are closely related. In this paper, we present a quantum dense coding scheme between multiparties in an arbitrary high-dimensional Hilbert space. As two-party dense coding is of primary importance, we first present the two-party dense coding scheme in arbitrary high dimensions. Then we present the general scheme for dense coding between multiparties using a high-dimensional state.

To present our scheme more clearly, let us first begin with dense coding between two parties in three dimensions. The general Bell basis of the Hilbert space of two particles with three dimensions is [5,6]:

$$
|\Psi_{nm}\rangle = \sum_j e^{2\pi i j n/3}|j\rangle \otimes |j+m \bmod 3\rangle/\sqrt{3}, \tag{2}
$$

where $n,m,j=0,1,2$. Explicitly,

$$
\begin{aligned}
|\Psi_{00}\rangle &= (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3},\\
|\Psi_{10}\rangle &= (|00\rangle + e^{2\pi i/3}|11\rangle + e^{4\pi i/3}|22\rangle)/\sqrt{3},\\
|\Psi_{20}\rangle &= (|00\rangle + e^{4\pi i/3}|11\rangle + e^{2\pi i/3}|22\rangle)/\sqrt{3},\\
|\Psi_{01}\rangle &= (|01\rangle + |12\rangle + |20\rangle)/\sqrt{3},\\
|\Psi_{11}\rangle &= (|01\rangle + e^{2\pi i/3}|12\rangle + e^{4\pi i/3}|20\rangle)/\sqrt{3},\\
|\Psi_{21}\rangle &= (|01\rangle + e^{4\pi i/3}|12\rangle + e^{2\pi i/3}|20\rangle)/\sqrt{3},\\
|\Psi_{02}\rangle &= (|02\rangle + |10\rangle + |21\rangle)/\sqrt{3},\\
|\Psi_{12}\rangle &= (|02\rangle + e^{2\pi i/3}|10\rangle + e^{4\pi i/3}|21\rangle)/\sqrt{3},\\
|\Psi_{22}\rangle &= (|02\rangle + e^{4\pi i/3}|10\rangle + e^{2\pi i/3}|21\rangle)/\sqrt{3}.
\end{aligned}
\tag{3}
$$

Superdense coding can be done in the following way. Suppose Alice and Bob share the maximally entangled state $|\Psi_{00}\rangle$. Through simple calculation, it can be shown that the single-body operators

$$
U_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

$$
U_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{bmatrix},
$$

$$U_{20}=\begin{bmatrix}1&0&0\\0&e^{4\pi i/3}&0\\0&0&e^{2\pi i/3}\end{bmatrix},$$

$$U_{01}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},$$

$$U_{11}=\begin{bmatrix}0&0&e^{4\pi i/3}\\1&0&0\\0&e^{2\pi i/3}&0\end{bmatrix},$$

$$U_{21}=\begin{bmatrix}0&0&e^{2\pi i/3}\\1&0&0\\0&e^{4\pi i/3}&0\end{bmatrix},$$

$$U_{02}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},$$

$$U_{12}=\begin{bmatrix}0&e^{2\pi i/3}&0\\0&0&e^{4\pi i/3}\\1&0&0\end{bmatrix},$$

$$U_{22}=\begin{bmatrix}0&e^{4\pi i/3}&0\\0&0&e^{2\pi i/3}\\1&0&0\end{bmatrix} \quad (4)$$

will transform $|\Psi_{00}\rangle$ into the corresponding states in Eq. (3), respectively,

$$U_{nm}|\Psi_{00}\rangle=|\Psi_{nm}\rangle. \quad (5)$$

Bob operates on one of the above unitary transformations and sends his particle back to Alice. Alice takes only one measurement in the basis $\{|\Psi_{00}\rangle,|\Psi_{10}\rangle,\ldots,|\Psi_{22}\rangle\}$, and she will know what operation Bob has performed, that is, what the messages are that Bob has encoded in the quantum state. As a result, Alice gets $\log_2 9$ bits of information through only one measurement. It should be pointed out that Reck and Zeilinger [8] have given the method by which to realize any discrete unitary operators and the operators used here can be constructed according to their protocol.

It is straightforward to generalize the above protocol to arbitrarily high dimensions for two parties. We denote the dimension as d. The general Bell-basis states are

$$|\Psi_{nm}\rangle=\sum_j e^{2\pi ijn/d}|j\rangle\otimes|j+m \bmod d\rangle/\sqrt{d}, \quad (6)$$

where $n,m,j=0,1,\ldots,d-1$. Obviously, there exist one-body operators U_{nm} on Bob's particle that satisfy $U_{nm}|\Psi_{00}\rangle=|\Psi_{nm}\rangle$. The matrix elements of the unitary transformation U_{nm} may be explicitly written as

$$(U_{nm})_{j'j}=e^{2\pi jn/d}\delta_{j',j+m \bmod d}, \quad (7)$$

where $n,m,j=0,1,\ldots,d-1$. The procedure for realizing dense coding in this high-dimensional case is similar to that for the two-dimensional case: Alice and Bob share the state $|\Psi_{00}\rangle$, then Bob performs one of the operations in Eq. (7) on his particle and sends this particle back to Alice. Alice then performs a collective measurement in the basis states in Eq. (6) to find out what Bob has done to the particle and hence reads out the encoded message. In this case, Alice gets $\log_2 d^2$ bits of information by just making only one measurement.

Dense coding between two parties can be generalized into *multiparties*. Bose *et al.* [7] have generalized the Bennett-Wiesner scheme of dense coding into multiparties in the qubit system. The multiparty dense coding scheme can be understood in the following way. There are $N+1$ users sharing an $(N+1)$-particle maximally entangled state, possessing one particle each. Suppose that one of them, say user 1, intends to receive messages from the N other users. The N senders mutually decide *a priori* to perform only certain unitary operations on their particles. After performing their unitary operations, each of the N senders sends his particle to user 1. User 1 then performs a collective measurement on the $N+1$ particles and identifies the state. Thus, he can learn about the operation each of the other N users has performed. That is to say that a single measurement is sufficient to reveal the messages sent by all the N users. We now discuss the high-dimension generalization of this scenario. We also begin with an example with three particles in three dimensions. We take the maximally entangled states as our basis,

$$|\Psi_{nm}^k\rangle=\sum_j e^{2\pi ijk/3}|j\rangle\otimes|j+n \bmod 3\rangle\otimes|j+m \bmod 3\rangle/\sqrt{3}, \quad (8)$$

where $n,m,k=0,1,2$. More explicitly,

$$|\Psi_{00}^0\rangle=(|000\rangle+|111\rangle+|222\rangle)/\sqrt{3},$$

$$|\Psi_{01}^0\rangle=(|001\rangle+|112\rangle+|220\rangle)/\sqrt{3},$$

$$|\Psi_{02}^0\rangle=(|002\rangle+|110\rangle+|221\rangle)/\sqrt{3},$$

$$\ldots$$

$$|\Psi_{22}^2\rangle=(|022\rangle+e^{4\pi i/3}|100\rangle+e^{2\pi i/3}|211\rangle)/\sqrt{3}. \quad (9)$$

Suppose Alice, Bob, and Claire share the maximally entangled state

$$|\Psi_{00}^0\rangle=(|000\rangle+|111\rangle+|222\rangle)/\sqrt{3}, \quad (10)$$

and Bob and Claire hold particles 2 and 3, respectively. The essential issue in dense coding is to find a limited number of one-body operations that Bob and Claire can perform so that the state $|\Psi_{00}\rangle$ is transformed into all possible states in Eq. (9). Meanwhile these operations have to be identified by Alice uniquely. If Bob and Claire are both allowed to perform any of the nine operations in Eq. (4), the total number

TABLE I. Transformation table for $U_{nm}(B)U_{n'm'}(C)\Psi_{00}$.

	$U_{n'm'}(C)$								
$U_{nm}(B)$	U_{00}	U_{10}	U_{20}	U_{01}	U_{11}	U_{21}	U_{02}	U_{12}	U_{22}
U_{00}	Ψ_{00}^{0}	Ψ_{00}^{1}	Ψ_{00}^{2}	Ψ_{01}^{0}	Ψ_{01}^{1}	Ψ_{01}^{2}	Ψ_{02}^{0}	Ψ_{02}^{1}	Ψ_{02}^{2}
U_{10}	Ψ_{00}^{1}	Ψ_{00}^{2}	Ψ_{00}^{0}	Ψ_{01}^{1}	Ψ_{01}^{2}	Ψ_{01}^{0}	Ψ_{02}^{1}	Ψ_{02}^{2}	Ψ_{02}^{0}
U_{20}	Ψ_{00}^{2}	Ψ_{00}^{0}	Ψ_{00}^{1}	Ψ_{01}^{2}	Ψ_{01}^{0}	Ψ_{01}^{1}	Ψ_{02}^{2}	Ψ_{02}^{0}	Ψ_{02}^{1}
U_{01}	Ψ_{10}^{0}	Ψ_{10}^{1}	Ψ_{10}^{2}	Ψ_{11}^{0}	Ψ_{11}^{1}	Ψ_{11}^{2}	Ψ_{12}^{0}	Ψ_{12}^{1}	Ψ_{12}^{2}
U_{11}	Ψ_{10}^{1}	Ψ_{10}^{2}	Ψ_{10}^{0}	Ψ_{11}^{1}	Ψ_{11}^{2}	Ψ_{11}^{0}	Ψ_{12}^{1}	Ψ_{12}^{2}	Ψ_{12}^{0}
U_{21}	Ψ_{10}^{2}	Ψ_{10}^{0}	Ψ_{10}^{1}	Ψ_{11}^{2}	Ψ_{11}^{0}	Ψ_{11}^{1}	Ψ_{12}^{2}	Ψ_{12}^{0}	Ψ_{12}^{1}
U_{20}	Ψ_{20}^{0}	Ψ_{20}^{1}	Ψ_{20}^{2}	Ψ_{21}^{0}	Ψ_{21}^{1}	Ψ_{21}^{2}	Ψ_{22}^{0}	Ψ_{22}^{1}	Ψ_{22}^{2}
U_{12}	Ψ_{20}^{1}	Ψ_{20}^{2}	Ψ_{20}^{0}	Ψ_{21}^{1}	Ψ_{21}^{2}	Ψ_{21}^{0}	Ψ_{22}^{1}	Ψ_{22}^{2}	Ψ_{22}^{0}
U_{22}	Ψ_{20}^{2}	Ψ_{20}^{0}	Ψ_{20}^{1}	Ψ_{21}^{2}	Ψ_{21}^{0}	Ψ_{21}^{1}	Ψ_{22}^{2}	Ψ_{22}^{0}	Ψ_{22}^{1}

of operations is $9\times 9=81$, which is greater than the total number of the basis states in Eq. (9). Alice cannot identify the operations of Bob and Claire uniquely. Thus not all the operations are allowed, and some restriction has to be made. In the qubit system, this problem is solved by allowing Bob to perform all four possible unitary operations I, σ_x, $i\sigma_y$, and σ_z, and Claire only performs any two of these operations. However, direct generalization of this rule needs caution. By a direct calculation, we find that allowing Bob to perform all nine operations and restricting Claire to performing any three of the nine operations will not always work. Look at Table I, where we have given the results of the operations $U_{nm}(B)U_{n'm'}(C)$ on $|\Psi_{00}\rangle$. Now allow Bob to perform all nine unitary operations. Then from Table I we see that the product of the nine operations of Bob with each operation in the subset $\{U_{00},U_{10},U_{20}\}$ of Claire can only give nine of the 27 basis states in Eq. (9). This is evident from Table I: the first three columns are just a rearrangement of the same nine basis states. The same is true for the other two subsets of operations $\{U_{01},U_{11},U_{21}\}$ and $\{U_{02},U_{12},U_{22}\}$. Thus, Claire's three operations can be chosen one from each of the three subsets arbitrarily. One such set is the three-operation U_{nm} with $n=0$: U_{00}, U_{01}, and U_{02}. Other combinations are also possible. For convenience, we can simply choose this subset: $\{U_{nm}$ with $n=0\}$ as the allowed operations of Claire.

With the identification of the allowed operations, say, Bob is allowed to perform all nine unitary transformations and Claire is allowed to perform the three operations U_{0m}, $m=0$, 1, and 2, the three-party dense coding can be done easily. Bob and Claire perform their operations on their respective particles and return the particles to Alice. When Alice receives the operated on particles from Bob and Claire, she can find the messages Bob and Claire have encoded by just one measurement in the basis of Eq. (9). That is, Alice gets $\log_2 27$ bits of information with a single measurement.

From the above example, we can generalize the scheme into multiparty superdense coding in high dimensions. Suppose the dimension is d. We construct the following basis:

$$|\Psi^{n}_{i_1,i_2,\ldots,i_N}\rangle=\sum_j e^{2\pi ijn/d}|j\rangle\otimes|j+i_1 \bmod d\rangle \otimes\cdots\otimes/|j+i_N \bmod d\rangle\sqrt{d}, \qquad (11)$$

where $n,j,i_1,i_2,\ldots,i_N=0,1,\ldots,d-1$. The d^2 one-body unitary operations can be written explicitly as

$$(U_{nm})_{j'j}=e^{2\pi jn/dN}\delta_{j',j+m \bmod d}, \qquad (12)$$

where $j=0,1,\ldots,d-1$. Thus, if the N senders mutually decide *a priori* that user 2 can perform all the d^2 one-body operations in Eq. (12), user 3 through user $N+1$ can only perform the d one-body operations, U_{0m}, $m=0,1,\ldots,d-1$. Then the N users can encode their messages by performing their allowed operations on the particles at their disposal and return the particles to user 1. After user 1 receives all the particles, he can then perform a single collective measurement of all his $N+1$ particles so that he can read out the messages the N users have encoded. In this way, the N users can send out $\log_2 d^{N+1}$-bit messages and the receiver, user 1, receives and identifies them with only one measurement. This is because the N senders are allowed to perform $d^2\times d\times\cdots\times d=d^{N+1}$ different combinations of the one-body unitary operations on the initial $(N+1)$-particle maximally entangled state

$$|\Psi_{00\ldots0}\rangle=(|00\ldots0\rangle+|11\ldots1\rangle +\cdots+|d-1,d-1\ldots d-1\rangle)/\sqrt{d}, \qquad (13)$$

while there are d^{N+1} orthogonal states in the Hilbert space of $N+1$ particles.

In summary, we have given general schemes for multiparty high-dimensional dense coding. Explicit expressions for the measuring basis and the forms of the one-body unitary transformation operators have been constructed. In particular, the operations allowed for users 3 through N must be chosen carefully.

Note added in proof. Recently, it was brought to our attention that the capacity of multiparticle qubit channel dense coding has been studied in Ref. [9].

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