Quantum anticentrifugal force

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In a two-dimensional world, a free quantum particle of vanishing angular momentum experiences an attractive force. This force originates from a modification of the classical centrifugal force due to the wave nature of the particle. For positive energies the quantum anticentrifugal force manifests itself in a bunching of the nodes of the energy wave functions towards the origin. For negative energies this force is sufficient to create a bound state in a two-dimensional δ-function potential. In a counterintuitive way, the attractive force pushes the particle away from the location of the δ-function potential. As a consequence, the particle is localized in a band-shaped domain around the origin.

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I. INTRODUCTION

"If this is the best of all possible worlds, what are the others like?" exclaims Candide [1] in Voltaire's philosophical novel when he sees the devastating results of the earthquake in Lisbon. Almost 150 years after Voltaire, Einstein pondered the question "How much freedom had God when he created the world?" In the same spirit P. Ehrenfest [2,3] raised the problem "Why is the space we live in three-dimensional?" Since then many phenomena where dimensionality of space plays a crucial role have been discovered. They manifest themselves in quantum dots and wires in solid-state physics, phase transitions in statistical physics or in the Kaluza-Klein or string theories of particle physics. In the present paper, we point out a wave effect that is unique to two-space dimensions and that can, in principle, be observed in the recent two-dimensional trapping experiments using wires [4]: A point particle subjected to a potential that is solely confined to the coordinate origin binds locally in one and three dimensions but in two dimensions binds in a domain like a hollow pipe. The deeper reason for this surprising effect lies in the quantum anticentrifugal potential: In two dimensions the centrifugal potential corresponding to vanishing angular momentum is attractive rather than repulsive.

In the present paper, we focus on the manifestations of the quantum anticentrifugal potential in the energy eigenstates of a free particle in two dimensions. The problem of time-dependent phenomena originating from this potential will be addressed in future publications.

The paper is organized as follows: In Sec. II, we observe that a localized wave function satisfies the time-independent Schrödinger equation of a free particle. The reason for the localization stands out most clearly in the Schrödinger equation for the radial wave function, discussed in Sec. III. Indeed, for vanishing angular momentum an attractive potential arises from the wave nature of the particle and determines the decay of the radial wave function. In Sec. IV, we identify the origin of the corresponding attractive force as interference of waves. Moreover, we show that the attraction or repulsion of the potentials corresponding to vanishing or one unit of angular momentum manifests itself in the bunching or antibunching of the nodes of the radial wave function. This phenomenon of attraction is unique to two dimensions. In Sec. V, we address the question of a bound state of a "free" particle. Indeed, the attraction due to the quantum anticentrifugal force is not enough to create a bound state. An additional weakly binding potential, such as a δ-function potential, is necessary. We conclude in Sec. VI by presenting some ideas for experimental realizations of these considerations.

II. AN UNUSUAL BOUND STATE

Our analysis rests on the observation that the function [5,6]

$$\Phi^{(2)}(x,y) \equiv \frac{1}{\sqrt{\pi}} k\, K_0(k\sqrt{x^2+y^2}) \tag{1}$$

defined in terms of the zeroth modified Bessel function K_0 and the wave number k satisfies the Helmholtz equation

$$[\Delta^{(2)} - k^2]\Phi^{(2)}(x,y) = 0 \tag{2}$$

everywhere except at $x=y=0$. Here, $\Delta^{(2)}$ denotes the Laplacian in two dimensions.

When we recall the dispersion relation

$$E = -|E| = -\frac{(\hbar k)^2}{2M} \tag{3}$$

of a free particle with mass M and negative energy E, the Helmholtz equation is equivalent to the corresponding time-independent Schrödinger equation.

The wave function $\Phi^{(2)}$ shown in Fig. 1 enjoys some rather unusual properties: Due to the modified Bessel function K_0, it diverges [7] logarithmically at the origin whereas at large distances it decreases exponentially. Despite this divergence, the wave function is still square integrable,

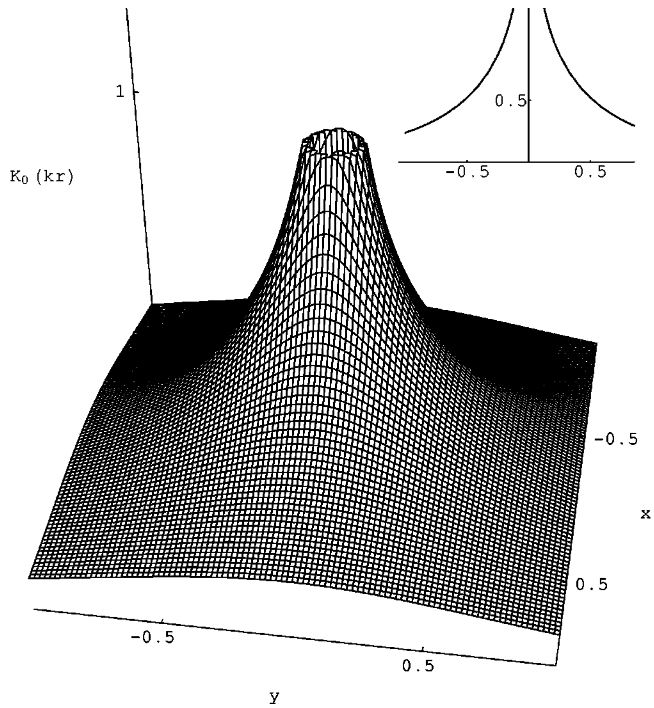

FIG. 1. The wave function $\Phi^{(2)}(r)$ represented in two-dimensional space is logarithmically divergent at the origin but decays exponentially for positions away from the origin. In the inset we show a cut along the x axis that brings out the logarithmic divergence of $\Phi^{(2)}(r)$ at the origin.

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\Phi^{(2)}(x,y)|^2 = \int_0^{\infty} dr \int_0^{2\pi} r\, d\phi \frac{k^2}{\pi} K_0^2(kr)$$
$$= 2\int_0^{\infty} d\xi\, \xi K_0^2(\xi) = 1.$$

Indeed, the area element $dx\, dy = r dr\, d\phi$ brings in an additional power of $r \equiv (x^2+y^2)^{1/2}$ and regularizes the logarithmic divergence at the origin.

For the same reason, the probability

$$W^{(2)}(r) dr \equiv 2k^2 K_0^2(kr) r\, dr$$

to find the particle between r and $r+dr$ vanishes at the origin, as shown in Fig. 2. Moreover, since the modified Bessel function K_0 decays for large distances, the radial probability displays a maximum close to the origin.

III. QUANTUM ANTICENTRIFUGAL POTENTIAL

What is the deeper reason for this localization [8,9] of a free particle? No classical potential prevents the particle from diffusing away. One part of the answer to this apparently paradoxical situation, a bound state of a free particle, lies in the Schrödinger equation

$$\left\{ \frac{d^2}{dr^2} + \frac{2M}{\hbar^2}[E - V_m^{(2)}(r)] \right\} u_m^{(2)}(r) = 0$$

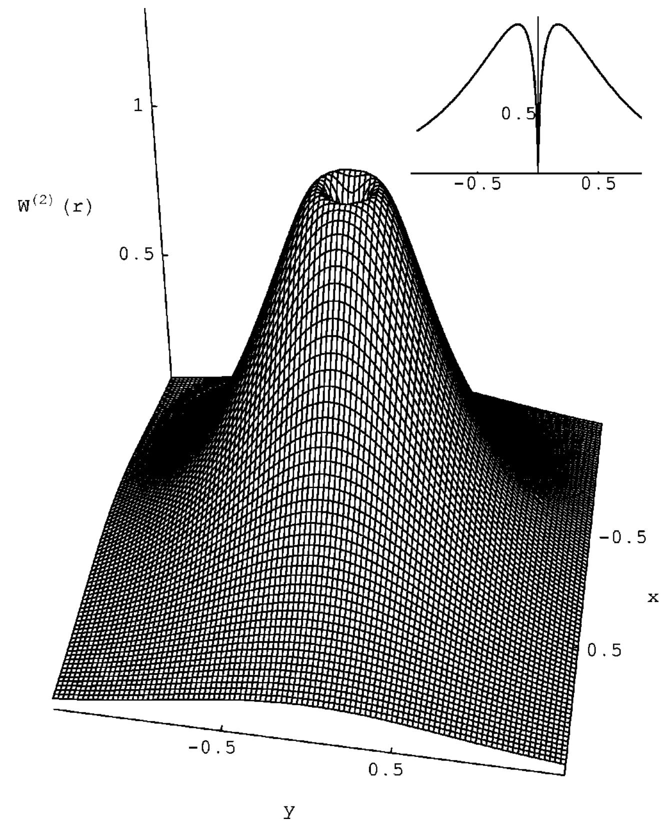

FIG. 2. The radial probability $W^{(2)}(r)$ vanishes at the origin and decays for large distances, with a maximum close to the origin. In the inset we show a cut along the x axis that brings out the cusp of $W^{(2)}(r)$ at the origin.

for the radial wave function

$$u_m^{(2)}(r) \equiv \sqrt{r} e^{-im\phi} \Phi^{(2)}(r\cos\phi, r\sin\phi). \tag{4}$$

Here we have introduced the effective potential

$$V_m^{(2)}(r) \equiv \frac{\hbar^2}{2M} \frac{m^2 - 1/4}{r^2}$$

in two dimensions. The radial wave equation Eq. (4) follows from the Helmholtz equation Eq. (2) with the help of the dispersion relation Eq. (3).

The first term in $V_m^{(2)}$, proportional to m^2, is the potential that describes the familiar centrifugal force. Less familiar is the negative correction term $-1/4$ that comes from the reduction of space from three to two dimensions. It gives rise to a centripetal force, which from this point on we shall call a quantum anticentrifugal force to emphasize that its binding power arises from quantum mechanics. Indeed, for particles with nonvanishing angular momentum ($m \neq 0$) the potential is repulsive, as shown in the bottom inset of Fig. 3. However, the repulsiveness associated with the classical centrifugal force, that is the m^2 term, is softened by the correction term $-1/4$.

The effect of this contribution stands out most clearly for a particle with zero angular momentum, that is, $m=0$. Here

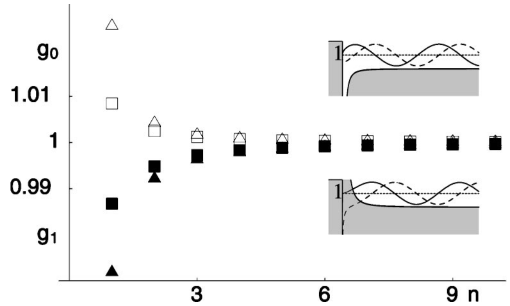

FIG. 3. Node bunching and antibunching of energy eigenfunctions of a free particle in a two-dimensional space. The centrifugal potential corresponding to a nonvanishing angular momentum is repulsive (bottom inset) and the two linearly independent eigenfunctions are determined by the Bessel function J_1 (solid line) and the Neumann function Y_1 (dotted line). In contrast, the potential corresponding to a vanishing angular momentum is attractive (top inset) and the two eigenfunctions are proportional to J_0 (solid line) and Y_0 (dotted line). The repulsive and attractive potentials give rise to an antibunching and bunching of the nodes of the energy eigenfunction, respectively. As a measure $g_m(n)\equiv\pi/\Delta_m(n)$ of bunching or antibunching, we use the inverse of the difference $\Delta_m(n)$ of neighboring zeros of the mth Bessel function J_m or Neumann function Y_m in units of the free-space separation π. Filled squares or triangles represent $g_1(n)$ for J_1 or Y_1 in the repulsive centrifugal potential. Open squares or triangles represent $g_0(n)$ for J_0 or Y_0 in the attractive potential. The zeros of Y_0 and Y_1 lie closer to the origin than those of J_0 and J_1. Consequently, the bunching or antibunching effect is more evident in the Neumann function than in the Bessel function. The physics of the nonrelativistic free particle does not contain an intrinsic unit of length. When we define a dimensionless length $\rho\equiv kr$, where k is the wave number, the dimensionless energy eigenvalue is unity.

the effective potential shown in the top inset of Fig. 3 becomes attractive. Hence, this quantum anticentrifugal potential

$$V_Q(r)\equiv V_0^{(2)}(r)=-\frac{\hbar^2}{2M}\frac{1}{4r^2}$$

is the reason for the decay of the wave function in Eq. (1) at large distances.

We have chosen this name for the potential to bring out in the most striking way the counterintuitive nature of this attraction. However, we emphasize that, despite the name, the attraction is not related to the angular but to the radial motion.

To illustrate this statement we compare the effective potential $V_m^{(2)}$ in two dimensions to the effective potential

$$V_l^{(3)}(r)\equiv\frac{\hbar^2}{2M}\frac{l(l+1)}{r^2} \tag{5}$$

in three dimensions. Here, l denotes the quantum number of angular momentum.

Both potentials seem to be quantum translations of the classical centrifugal potential

$$V_{\rm cl}(r)\equiv\frac{\vec{L}^2}{2Mr^2}, \tag{6}$$

where $\vec{L}$ is the angular momentum vector. Indeed, in three dimensions the ‘‘quantum square’’ of angular momentum reads $\vec{L}^2=\hbar^2 l(l+1)$. In two dimensions, it seems to take the less familiar form $\vec{L}^2=\hbar^2(m^2-1/4)=\hbar^2(m-1/2)(m+1/2)$.

However, this picture is misleading. Whereas the quantum square $l(l+1)$ is solely a consequence of the angular momentum algebra, the correction term $-1/4$ in two dimensions does not result from the angular motion, but from the radial motion. It can be traced back to the radial derivatives in the Laplacian

$$\Delta^{(2)}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}, \tag{7}$$

expressed in polar coordinates.

This feature suggests that the quantum anticentrifugal force is a metric force. It originates from the use of curvilinear coordinates, that is, the description of the wave in cylindrical coordinates.

IV. NODE BUNCHING AND ANTIBUNCHING

How can we gain some insight into the physical origin of the quantum anticentrifugal potential V_Q? One strategy is to first consider the familiar case of a free particle of positive energy and compare and contrast the wave functions of an attractive potential $(m=0)$ and a repulsive potential $(m>0)$. Then we extend these considerations to negative energies and emphasize the uniqueness of two dimensions.

A. Positive energy

For $E>0$ the two linear independent solutions of the two-dimensional Helmholtz equation are the ordinary Bessel functions J_m and the Neumann functions Y_m. For the attractive potential V_Q the independent solutions are proportional to J_0 or Y_0, whereas for the repulsive potential $V_1^{(2)}$, corresponding to $m=1$, we find J_1 and Y_1. The different nature of the potentials—attractive vs repulsive—manifests itself in the wave functions through the distribution of nodes determined by the zeros $j_{m,n}$ or $y_{m,n}$ of the Bessel function J_m or the Neumann function Y_m. A measure for the distribution of nodes is the normalized density

$$g_m(n)\equiv\frac{\pi}{\Delta_m(n)} \tag{8}$$

of the zeros of the Bessel functions. Here,

$$\Delta_m(n)\equiv j_{m,n+1}-j_{m,n} \tag{9}$$

denotes the separation of neighboring zeros of J_m and

$$\Delta_m(n)\equiv y_{m,n+1}-y_{m,n} \tag{10}$$

denotes the same quantity for Y_m. We have normalized the separation to the free-space separation π of the zeros.

In J_0 and Y_0 the separation $\Delta_0(n)$ between neighboring zeros decreases for decreasing n, in agreement with the intuitive picture that the particle accelerates towards the origin. In Fig. 3, we represent by open squares and triangles the normalized density $g_0(n)$ of the zeros of J_0 and Y_0, respectively, clearly demonstrating node bunching.

In the language of cold atoms the energy wave function $u_0^{(2)}$ has a negative scattering length, indicating an attractive potential. In the case of cold atoms the origin of this attraction is a physical interaction. In contrast, the attractive quantum anticentrifugal potential is not due to a classical interaction but arises from the wave equation.

In contrast, in J_1 and Y_1 the separation $\Delta_1(n)$ of neighboring zeros increases as n decreases, corresponding to a deceleration of the particle running up the potential well. Again, in the language of cold atoms this case corresponds to a positive scattering length. The filled squares and triangles of Fig. 3, corresponding to the normalized density $g_1(n)$ of zeros of J_1 and Y_1, respectively, reflect the phenomenon of node antibunching.

Where is the attraction coming from? The answer is: Interference of waves. When we interfere infinitely many plane waves of identical amplitudes and wave numbers, and allow all propagation directions with equal weight, the interference pattern is that of the Bessel function J_0. This surprising feature is just the physical interpretation of the Sommerfeld integral representation

$$J_0(kr)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, e^{ikr\sin\theta}$$

of the Bessel function.

The particle represented by the wave function containing J_0 feels the quantum anticentrifugal force. Each plane wave contributing to the Bessel-interference pattern does not feel any force. The interference of all plane waves acts as an effective force. Attraction from interference.

B. Negative energy

So far, we have focused on the case of positive energies. An interesting selection of solutions occurs when we make the transition from positive to negative energies. Due to the sign change of the energy and the quadratic dispersion relation of the free particle, the wave number becomes purely imaginary. Consequently, the ordinary Bessel functions J_m and Y_m turn into the modified Bessel functions I_m and K_m. However, based on physical arguments, no solutions of negative energy exist for $m\geqslant 1$. Nevertheless, for $m=0$, we have the two solutions I_0 and K_0. Since the modified Bessel function I_0 increases exponentially for large distances, whereas K_0 decreases, the boundary conditions imposed by the quantum anticentrifugal potential select K_0 and thus the solution Eq. (1).

The reduction from two equally contributing waves to a single one as we cross the zero-energy line is reminiscent of the behavior of the Airy function when we cross [10] the Stokes line going from negative to positive arguments. Indeed, for large negative values we can approximate the Airy function by two counterpropagating waves, whereas for large positive values we only find a single decaying exponential.

C. Higher dimensions

This phenomenon of attraction is unique to two dimensions [11]. Indeed, for a free particle of vanishing angular momentum the N-dimensional, (hyper)spherical Schrödinger equation [12]

$$\left\{\frac{d^2}{dr^2}+\frac{2M}{\hbar^2}[E-V_0^{(N)}]\right\}u^{(N)}(r)=0$$

for the radial variable $r\equiv(x_1^2+\cdots+x_N^2)^{1/2}$ contains the quantum potential [13]

$$V_0^{(N)}(r)\equiv\frac{\hbar^2}{2M}\frac{(N-1)(N-3)}{4r^2}.$$

For $N=1$ and $N=3$ the quantum potential vanishes. For higher dimensions $N\geqslant 3$ it is repulsive. Only for $N=2$ this potential becomes attractive. Therefore, the anticentrifugal force effect is a consequence of the dimensionality of space.

V. BOUND STATE OF A "FREE" PARTICLE

These considerations suggest that in two dimensions there exists a bound state of a free particle with the wave function given by Eq. (1). However, we emphasize that the wave number k and thus the energy E are free parameters. There is no length scale in the problem. What fixes the energy of this bound state? The logarithmic singularity of $\Phi^{(2)}$ at the origin indicates that there the wave function does not satisfy the time-independent Schrödinger equation. Indeed, the wave function (1) satisfies the equation [14],

$$[\Delta^{(2)}-k^2]\Phi^{(2)}(\vec{r})=U_0\delta^{(2)}(\vec{r})$$

with an additional δ-function potential [15] of strength U_0. A nonlinear relation between k and U_0 determines [16] the eigenenergy of the bound state.

Hence, we are not really dealing with a free particle, but with a particle in the presence of a δ-function potential. Notwithstanding the problems [16,17] associated with the definition of a δ-function potential in two and higher dimensions, it is well known that under appropriate conditions such potentials entertain bound states [16,17]. Indeed, in one dimension the strength U_0 of the potential has to be negative and the corresponding probability distribution

$$W^{(1)}(x)dx\equiv|\Phi^{(1)}(x)|^2dx=(\sqrt{k}e^{-k|x|})^2dx=ke^{-2k|x|}dx$$

displays a maximum at the location of the potential.

In three dimensions the parameter U_0 has to be positive in order for the δ-function potential to support a bound state. As in one dimension, the probability distribution

$$W^{(3)}(r)dr \equiv |\Phi^{(3)}(r)|^2 4\pi r^2 dr = \left(\sqrt{\frac{k}{2\pi}}\frac{1}{r}e^{-kr}\right)^2 4\pi r^2 dr$$
$$= 2k\, e^{-2kr} dr$$

is an exponential and exhibits a maximum at the origin.

The reason for this common feature is quite intriguing. In one dimension it is simply due to the fact that the wave function $\Phi^{(1)}(x)$ has a maximum at $x=0$. In three dimensions the situation is more subtle. Here the radial wave function $\Phi^{(3)}(r)$ contains a $1/r$ singularity, creating a $1/r^2$ singularity in the probability density. However, the volume element $4\pi r^2 dr$ of a spherical shell in three dimensions cancels the singularity in the probability and only the exponential at the origin survives.

In two dimensions the situation is drastically different. Independent of the sign of U_0 there always exists a single bound state, with wave function $\Phi^{(2)}$, Eq. (1). Moreover, the area element $2\pi r\, dr$ of a ring prevails over the logarithmic singularity contained in K_0. This creates a node at the origin. As a consequence, the maximum of the probability distribution gets pushed away from the center of attraction.

In this sense, the intuitive picture of a repulsive centrifugal force reappears: The maximum of the probability is not at the origin, but in a ring surrounding it. The quantum anticentrifugal potential keeps the packet together.

This behavior is reminiscent of the probability distribution of the electron in the hydrogen atom, in a s state. Here, the wave function is an exponential and displays a maximum at the origin. The volume element $4\pi r^2 dr$ of a three-dimensional spherical shell creates a node at the origin and thus a maximum at the Bohr radius. However, there is a fundamental difference to our situation: The exponential decay of the wave function in the atom is enforced by a classical potential, namely, the Coulomb potential. In contrast, for the free particle in two dimensions it is the quantum anticentrifugal potential that demands the decay.

VI. CONCLUSIONS

There is an interesting connection between the energy eigenstates of a free particle in two dimensions and diffraction-free beams [18], that is Bessel beams [19] in classical optics. Here, the ordinary Bessel function J_0 describes the wave field with a purely real wave number corresponding to positive energy. However, the present effect corresponds to negative energies and relies on purely imaginary wave numbers giving rise to modified Bessel functions. This is analogous to axicons used in classical optics.

This phenomenon of binding a particle with the help of the quantum anticentrifugal force could have interesting applications in the context of waveguides. Needless to say, all the conclusions hold for electromagnetic fields when we can ignore polarization. Here the maximum of the intensity does not lie in the waveguide, that is, the δ-function potential, but rather outside.

The newly emerging field of cold atoms offers interesting possibilities for experimentally verifying the existence of the quantum anticentrifugal force. Here we do not go into the details of such an experiment, but only give an idea. The interaction between two cold atoms is usually modeled by a δ function. We can use this feature to create the δ-function potential necessary for the wave function $\Phi^{(2)}$ defined in Eq. (1) to be an eigenstate of the self-adjoint extension of the kinetic-energy operator. The cylindrical symmetry we achieve by using a dilute atomic beam guided by a laser beam. In order not to affect the atom to be trapped, we have to work with two different atomic elements. In the sense of a Born-Oppenheimer approximation the atom feels a time-averaged δ-function potential.

We conclude by emphasizing that this phenomenon of attraction in a free particle crucially depends on the fact that we have restricted the space to two dimensions. For positive energies the special case of vanishing angular momentum selects the origin as a special point of the two-dimensional plane. In the case of negative energies with a δ-function potential the origin becomes a singular point, much in the spirit of the singularity provided by the magnetic-flux line in the Aharonov-Bohm effect. These facts demonstrate that in two dimensions a single point matters: It changes the topology. In contrast, in three dimensions a single point is less important.

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