Towards a quantum Hall effect for atoms using electric fields

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An atomic analog of Landau quantization based on the Aharonov-Casher interaction is developed. The effect provides a first step towards an atomic quantum Hall system using electric fields, which may be realized in a Bose-Einstein condensate.

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I. INTRODUCTION

Paredes *et al.* [1] have recently proved the existence of anyonic excitations in rotating Bose-Einstein condensates, demonstrating a method to test exotic particle statistics. Their analysis was based on the analogy between a condensate in a rotating trap and a system of interacting electrons in a uniform static magnetic field.

Motivated by this result, we provide in this work a first step towards another atomic quantum Hall analogy that could be of interest for the physics of Bose-Einstein condensates. The idea is based on the Aharonov-Casher (AC) effect [2] (see also [3]) in which atoms may interact with an electric field via a nonvanishing magnetic moment. This interaction coincides formally in the nonrelativistic limit with that of minimal coupling, where the AC vector potential is determined by the electric field and the direction of the magnetic dipole. We demonstrate the existence of a certain field-dipole configuration in which an atomic analog of the standard Landau effect [4] occurs. This result opens up the possibility for an atomic realization of the quantum Hall effect using electric fields. It should be noted that the AC interaction in Bose-Einstein condenstates has previously been discussed [5], but in the different context related to vortex formation.

In the following section, we briefly outline the standard Landau theory for a charged particle moving in a uniform magnetic field. The precise conditions under which the AC analog of the Landau effect occurs are stated in Sec. III. Under fulfillment of these conditions the corresponding theory is developed in detail. The relation to the AC duality [2] as well as aspects of gauge and supersymmetry are delineated in Sec. IV. These aspects illuminate an additional richness in the physics of the present Landau effect compared to that of the standard one. The paper ends with concluding remarks.

II. STANDARD LANDAU THEORY

Consider a particle with charge q moving in a plane perpendicular to a uniform magnetic field $\mathbf{B}=B\mathbf{z}$, say. This system is the basic constituent of the quantum Hall effect and is described by Hamiltonian operator (SI units are used throughout this paper)

$$H=\frac{\mathbf{\Pi}^2}{2m}=\frac{1}{2m}(-i\hbar\nabla-q\mathbf{A})^2, \tag{1}$$

where m is the mass of the particle, $2\pi\hbar$ Planck's constant, and $\mathbf{A}$ is the vector potential fulfilling $\mathbf{B}=\nabla\times\mathbf{A}$. The eigenvalues of H are computed using standard ladder operator technique based on the canonical structure of the commutator

$$[\Pi_x,\Pi_y]=i\hbar m\omega=i\sigma\hbar^2/\ l^2. \tag{2}$$

Here $\omega=qB/m=\sigma|qB|/m$ is the cyclotron frequency for which the sign σ describes the revolution direction of the corresponding classical motion. The natural unit of length in the quantum Hall regime is the magnetic length $l=\sqrt{\hbar/|qB|}$. Next we introduce the annihilation and creation operators

$$a=\frac{1}{\sqrt{2m\hbar|\omega|}}(\Pi_x+i\sigma\Pi_y),$$

$$a^\dagger=\frac{1}{\sqrt{2m\hbar|\omega|}}(\Pi_x-i\sigma\Pi_y) \tag{3}$$

that fulfill the commutation relation $[a,a^\dagger]=1$. In terms of these, we may write the Hamiltonian operator as

$$H=\left(a^\dagger a+\frac{1}{2}\right)\hbar|\omega|+\frac{p_z^2}{2m}. \tag{4}$$

Thus the motion in the $x-y$ plane has been transformed into a one-dimensional harmonic oscillator accompanying free motion in the z direction. It follows that the energy eigenvalues are

$$E_{\nu,k_z}=\left(\nu+\frac{1}{2}\right)\hbar|\omega|+\frac{\hbar^2k_z^2}{2m}, \tag{5}$$

where $\nu=0,1,2,\ldots;$ and k_z is real valued. Note that these eigenvalues are independent of both the revolution direction and the orbit center of the corresponding classical motion. The latter independence is related to the fact that the above energy eigenvalues are degenerate. This degeneracy is revealed by labeling the corresponding eigenfunctions with the eigenvalue η of the quantum orbit center operator [6]. Such a degenerate set $\{\psi_{\eta,\nu,k_z}\}_\eta$ defines a Landau level [4].

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III. AC ANALOG OF LANDAU THEORY

In the nonrelativistic limit one may describe the interaction between an atom with nonvanishing magnetic moment μ and an electric field $\mathbf{E}$ by the AC Hamiltonian operator [neglecting terms of $O(\mathbf{E}^2)$] [7]

$$H=\frac{\mathbf{\Pi}^2}{2m}+\frac{\mu\hbar}{2mc^2}\boldsymbol{\nabla}\cdot\mathbf{E}. \tag{6}$$

Here $\mathbf{\Pi}=-i\hbar\boldsymbol{\nabla}-\mu c^{-2}(\mathbf{n}\times\mathbf{E})$ is the kinematic momentum with $\mathbf{n}$ the direction of the magnetic dipole moment $\boldsymbol{\mu}$,m is the mass of the particle, and c is the speed of light. This defines the AC vector potential

$$\mathbf{A}_{AC}=c^{-2}(\mathbf{n}\times\mathbf{E}), \tag{7}$$

the associated field strength

$$\mathbf{B}_{AC}=c^{-2}\boldsymbol{\nabla}\times(\mathbf{n}\times\mathbf{E}), \tag{8}$$

and the coupling strength μ. It follows that

$$[\Pi_k,\Pi_l]=i\hbar\mu\epsilon_{klm}(\mathbf{B}_{AC})_m, \tag{9}$$

where ϵ_{klm} is the Levi-Civita symbol with k,l,m running over x,y,z and the summation convention is used.

The precise conditions on the field-dipole configuration under which the AC analog of the Landau effect occurs are as follows. (i) Condition for vanishing torque on the dipole, $\mathbf{n}\times(\langle\mathbf{\Pi}\rangle\times\mathbf{E})=0$, where $\langle\cdot\rangle$ denotes expectation value; (ii) Conditions for electrostatics, $\partial_t\mathbf{E}=0$ and $\boldsymbol{\nabla}\times\mathbf{E}=0$, and (iii) $\mathbf{B}_{AC}$ uniform.

Condition (i) follows from the fact that a dipole moving with velocity $\langle\mathbf{\Pi}\rangle$ sees an effective magnetic field $\mathbf{B}_{\mathrm{eff}}\propto\langle\mathbf{\Pi}\rangle\times\mathbf{E}$ in its own reference frame and that this magnetic field produces a torque $\dot{\mathbf{n}}\propto\mathbf{n}\times\mathbf{B}_{\mathrm{eff}}$.

For $\mathbf{n}=(0,0,1)$, conditions (i) and (ii) are fulfilled if $\mathbf{E}=(E_x(x,y),E_y(x,y),0)$ with $\partial_xE_y-\partial_yE_x=0$ and the atom moves in the $x-y$ plane so that $\langle\Pi_z\rangle$ vanishes. Using the vector identity $\mathbf{B}_{AC}=\mathbf{n}(\boldsymbol{\nabla}\cdot\mathbf{E})+(\mathbf{n}\cdot\boldsymbol{\nabla})\mathbf{E}$, the second term vanishes and the condition (iii) reduces to Gauss's law $\boldsymbol{\nabla}\cdot\mathbf{E}=\rho_0/\epsilon_0$, where ρ_0 is a nonvanishing uniform volume charge density and ϵ_0 is the electric vacuum permittivity.

With the Landau conditions fulfilled and the above choice of $\mathbf{n}$ and $\mathbf{E}$, we may proceed as in the preceding section to work out energy eigenstates using standard ladder operator technique. The kinetic part of the Hamiltonian operator is taken care of by noting that the only nonvanishing commutator in Eq. (9) is

$$[\Pi_x,\Pi_y]=i\hbar m\omega_{AC}=i\sigma\hbar^2/l_{AC}^2 \tag{10}$$

with the cyclotron frequency

$$\omega_{AC}=\frac{\mu\rho_0}{mc^2\epsilon_0}=\frac{\sigma|\mu\rho_0|}{mc^2\epsilon_0}. \tag{11}$$

Again $\sigma=\pm1$ labels the revolution direction of the corresponding classical motion. The natural unit of length in the AC case is $l_{AC}=\sqrt{\hbar c^2\epsilon_0/|\mu\rho_0|}$. Next we define the annihilation and creation operators

$$a_{AC}=\frac{1}{\sqrt{2m\hbar|\omega_{AC}|}}(\Pi_x+i\sigma\Pi_y),$$

$$a_{AC}^{\dagger}=\frac{1}{\sqrt{2m\hbar|\omega_{AC}|}}(\Pi_x-i\sigma\Pi_y) \tag{12}$$

that fulfill the commutation relation $[a_{AC},a_{AC}^{\dagger}]=1$. Inserting Eq. (12) into Eq. (6) we obtain

$$H=\left[a_{AC}^{\dagger}a_{AC}+\frac{1}{2}(1+\sigma)\right]\hbar|\omega_{AC}|+\frac{p_z^2}{2m}, \tag{13}$$

where we have used that

$$\frac{\mu\hbar}{2mc^2}\boldsymbol{\nabla}\cdot\mathbf{E}=\frac{1}{2}\hbar\omega_{AC}. \tag{14}$$

With the constraint $\langle\Pi_z\rangle=\langle p_z\rangle=0$ imposed by the condition (i) we obtain the energy eigenvalues

$$E_{\nu}^{(\sigma)}=\left[\nu+\frac{1}{2}(1+\sigma)\right]\hbar|\omega_{AC}|,$$

$$\nu=0,1,2,\ldots\ . \tag{15}$$

These energies are independent of the classical orbit center but they depend on the revolution direction. The σ-dependent set of degenerate states $\{\psi_{\eta,\nu}^{(\sigma)}\}_{\eta}$ defines the AC analog of a Landau level (η is again the eigenvalue of the quantum orbit center operator). Note that the condition for vanishing torque on the dipole puts an additional constraint on the stationary states $\psi_{\eta,\nu}^{(\sigma)}$, viz. that they must fulfill $\partial_z\psi_{\eta,\nu}^{(\sigma)}=0$.

IV. PHYSICAL INTERPRETATION OF THE AC ANALOG

A. Relation to the AC duality

The physical origin of the AC analog may be understood from the standard Landau effect using the AC duality [2]

$$q\Phi\leftrightarrow\frac{\mu\lambda}{c^2\epsilon_0}, \tag{16}$$

where Φ is a magnetic flux and λ is a uniform linear charge density in the direction of the magnetic dipole. Consider the separation of the energies in Eq. (5) for fixed k_z

$$\Delta E=\hbar\frac{|qB|}{m}. \tag{17}$$

The magnetic field can be expressed in terms of its flux Φ as $B=\Phi/S$,S being the area (perpendicular to the magnetic field) through which the flux is measured. This leads to

$$\Delta E = \hbar \frac{|q\Phi/S|}{m}. \tag{18}$$

Thus, according to Eq. (16) the AC dual to ΔE is

$$\Delta E_{AC} = \hbar \frac{|\mu\lambda/S|}{mc^2\epsilon_0} = \hbar \frac{|\mu\rho_0|}{mc^2\epsilon_0}, \tag{19}$$

which is the Landau level separation in the AC case for fixed σ. We have used that $\rho_0 = \lambda/S$ is a uniform volume charge density with the direction of λ perpendicular to S.

B. Gauge symmetries

In the case of a charged particle interacting with an electromagnetic field, two distinct experimental setups cannot be related by a gauge transformation. This is not so for the AC interaction, where there exist different choices of physical setups associated with the same physical effect. The basic reason for this fact is that the AC vector potential is directly linked to physical quantities, viz. the electric field $\mathbf{E}$ and the direction of the magnetic dipole $\mathbf{n}$, in such a way that two different pairs $(\mathbf{n},\mathbf{E})$ and $(\mathbf{n}',\mathbf{E}')$ may yield the same $\mathbf{B}_{AC}$.

To explore this additional gauge symmetry in more detail, let us consider the AC vector potential $\mathbf{A}_{AC} = c^{-2}(-E_y, E_x, 0)$ for $\mathbf{n} = (0,0,1)$ and $\mathbf{E} = (E_x(x,y), E_y(x,y), 0)$. Any electric field $\mathbf{E}'$ related to this $\mathbf{E}$ by

$$E'_x = E_x + \partial_y \chi$$

$$E'_y = E_y - \partial_x \chi \tag{20}$$

defines the AC vector potential

$$\mathbf{A}'_{AC} = \mathbf{A}_{AC} + \boldsymbol{\nabla}\chi \tag{21}$$

so that $\mathbf{B}'_{AC} = \mathbf{B}_{AC}$. In particular, with uniform $\mathbf{B}_{AC}$ the Landau conditions are fulfilled for the atom moving in the $x-y$ plane also for $\mathbf{E}'$ in Eq. (20), if $E'_z = 0, \chi = \chi(x,y)$, and $\nabla^2\chi = 0$.

In the Landau case with $\mathbf{n} = (0,0,1)$, it is in this context instructive to consider a uniform volume charge density ρ_0 confined to regions with differently shaped x-y cross sections. Solving for a cylindrical shape yields the electric field $\mathbf{E} = [\rho_0/(2\epsilon_0)](x,y,0)$ in the interior of the cylinder. Here x and y are measured relative to the symmetry axis of the cylinder, which we take to be the z axis. The corresponding vector potential becomes $\mathbf{A}_{AC} = [\rho_0/(2c^2\epsilon_0)](-y,x,0)$. This is the AC analog of symmetric gauge. On the other hand, within a uniformly charged plate of finite width in the x direction, but infinite extension in the y and z directions, the electric field takes the form $\mathbf{E} = (\rho_0/\epsilon_0)(x,0,0)$ with x measured relative to one of the surfaces of the plate. The corresponding vector potential is $\mathbf{A}_{AC} = [\rho_0/(c^2\epsilon_0)](0,x,0)$. This is the AC analog of Landau gauge. These two configurations yield identical AC Landau level energies and are related by the gauge function $\chi = [\rho_0/(2\epsilon_0 c^2)]xy$. Other choices of gauge may be obtained by further changes of the shape of the x-y cross section.

C. Supersymmetry

The dependence of the Landau energies on the revolution direction in the AC case, divides the set of Landau levels into two classes, each labeled by the value of σ. As we show now, this may be understood in terms of supersymmetry [8].

We start with reinterpreting σ as the eigenvalue of an operator τ and introduce the supercharge

$$Q = a_{AC} f^\dagger, \tag{22}$$

where $[f,f^\dagger] = -\tau, ff^\dagger + f^\dagger f = 1$, and $ff = f^\dagger f^\dagger = 0$. This yields

$$H/(\hbar\omega_{AC}) = QQ^\dagger + Q^\dagger Q = a^\dagger_{AC} a_{AC} + \frac{1}{2}(1 - [f,f^\dagger]). \tag{23}$$

Next we introduce the boson and fermion number operators

$$N_B = a^\dagger_{AC} a_{AC}$$

$$N_F = \frac{1}{2}(1+\tau) = \frac{1}{2}(1 - [f,f^\dagger]), \tag{24}$$

and the Fock space $\{|n_{AC}, n_F\rangle\}$ defined by the action of the operators a_{AC}, $a^\dagger_{AC}$, f, and $f^\dagger$,

$$a_{AC}|n_B, n_F\rangle = \sqrt{n_B}|n_B - 1, n_F\rangle,$$

$$a^\dagger_{AC}|n_B, n_F\rangle = \sqrt{n_B + 1}|n_B + 1, n_F\rangle,$$

$$f|n_B, n_F\rangle = |n_B, n_F - 1\rangle,$$

$$f^\dagger|n_B, n_F\rangle = |n_B, n_F + 1\rangle. \tag{25}$$

With the identifications $n_B = \nu$ and $n_F = \frac{1}{2}(1+\sigma)$, a Landau level may be defined as a set $\{|\eta, n_B, n_F\rangle\}_\eta$ of extended Fock states. Thus the lowest Landau level corresponds to the set $\{|\eta,0,0\rangle\}_\eta$, in which the number of bosons and fermions both vanish. The states in this set are annihilated by the supercharge Q and its adjoint $Q^\dagger$ (unbroken supersymmetry), and are therefore associated with zero energy. Higher levels $n_B > 0$ are obtained by repeated action of $a^\dagger_{AC}$. The supersymmetric partners correspond to the sets of Fock states $\{\{|\eta, n_B, 1\rangle\}_\eta\}_{n_B}$ that contain one fermion. Each such set is obtained from the set $\{|\eta, n_B + 1, 0\rangle\}_\eta$ under action of Q.

V. CONCLUSIONS

An atomic analog of the Landau quantization based on the Aharonov-Casher (AC) effect has been discussed. The effect is intimately related to the AC duality between the charge and magnetic moment. We have shown how symmetric gauge and Landau gauge can be realized using two differently shaped distributions of charge. Finally, we have argued that supersymmetry plays a role in that it makes the zero-point energy to vanish in the AC analog of the Landau effect. Further extension of the Landau quantization to other multipole moments should be of interest, the most important of which would be that of an electric dipole moving in a magnetic field.

This work could be of interest for creating an atomic quantum Hall system that may be realized in a Bose-Einstein condensate. In the strong-interaction regime of such a system, the present result may provide a realization of the fractional quantum Hall effect using electric fields.

It should be kept in mind that a significant AC Landau quantization requires extreme conditions. In order to achieve reasonable separation of Landau energies and sufficiently small magnetic length l_{AC} a dense charge distribution is needed. For example, taking the energy level separation $\Delta E_{AC} \sim 10^{-6}$ eV, we need $\rho_0 \sim 10^{11}$ C/m^3 that implies $l_{AC} \sim 50$ Å. In the case of a charged plate this corresponds to a voltage of 10^{10} V for a plate of thickness 1 μm. Clearly, these experimental parameters are very hard to achieve within present day technology, but may be realizable upon improving the energy resolution. Moreover, if we succeed in creating appropriate conditions it will still be challenging to distinguish the present effect from those associated with the charge-induced polarization of the atoms. Technical difficulties of this kind must be overcome before the effect can be studied in the laboratory.

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