Quantum revivals in periodically driven systems close to nonlinear resonances

Farhan Saif[1,2,*] and Mauro Fortunato[1,3,†]

[1]*Abteilung für Quantenphysik, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany*
[2]*Department of Electronics, Quaid-i-Azam University, 54320 Islamabad, Pakistan*
[3]*Dipartimento di Matematica e Fisica and INFM, Università di Camerino, I-62032 Camerino, Italy*

We calculate the quantum revival time for a wave packet initially well localized in a one-dimensional potential in the presence of an external periodic modulating field. The dependence of the revival time on various parameters of the driven system is shown analytically. As an example of an application of our approach, we compare the analytically obtained values of the revival time for various modulation strengths with the numerically computed ones in the case of a driven gravitational cavity. We show that they are in very good agreement.

In nature, interference phenomena lead to recurrences [1,2]. In the quantum-mechanical evolution, for instance, interference plays a crucial role and manifests itself in quantum recurrences [3]. A quantum wave packet spreads all over the available space after a few classical periods following wave mechanics and collapses. However, due to quantum dynamics, it rebuilds itself after a certain evolution time that is larger than the classical period. In one-dimensional systems, this phenomenon has been well studied and is known as quantum revivals and fractional revivals [4–9]. In higher-dimensional systems, which exhibit classical and quantum chaos, this phenomenon has been numerically observed earlier [10–14]. In particular, the quantum revivals occurring in driven gravitational cavities [15,16] have been investigated analytically and numerically.

In this paper, we provide a general analytical prescription to the revival phenomenon occurring in any periodically driven time-dependent systems close to a quantum nonlinear resonance. By using semiclassical secular theory [17–19], we derive a simple relation that allows us to calculate the quantum revival time in the presence of the external periodic modulation. For the sake of concreteness, we apply our general result to the dynamics of atoms in a modulated gravitational cavity [20], which is accessible to laboratory experiments [21]. Hence, we argue that dynamical revivals are generic to all periodically driven explicitly time-dependent systems, as far as resonances prevail.

Let us consider a one-dimensional system driven by an external periodic field. In order to calculate its quasienergy, we consider the nonlinear resonances of the time-dependent system [17,18]. We denote the eigenstates and eigenvalues of the corresponding time-independent system by $|n\rangle$ and E_n, respectively, so that $H_0|n\rangle = E_n|n\rangle$, where H_0 is the unperturbed Hamiltonian. We consider the evolution of a wave packet in an arbitrary one-dimensional potential in the presence of an external periodic field. The Hamiltonian of the driven system in dimensionless form may be expressed as

$$H = H_0 + \lambda V(x)\sin t, \tag{1}$$

where λ is the dimensionless modulation strength and $V(x)$ defines the coupling.

In order to study the quantum nonlinear resonances of the system, we make the ansatz [17,18] that the solution of the Schrödinger equation corresponding to the Hamiltonian (1) may be written in the form

$$|\psi(t)\rangle = \sum_n C_n(t)|n\rangle \exp\left\{-i\left[E_r + (n-r)\frac{k}{N}\right]\frac{t}{k}\right\}, \tag{2}$$

where k denotes the effective Planck's constant, and E_r is the average energy of the wave packet in the Nth resonance.

We substitute Eq. (2) into the Schrödinger equation and project the state vector $\langle m|$ onto the result. This leads to

$$ik\dot{C}_m = \left[E_m - E_r + (m-r)\frac{k}{N}\right]C_m(t) + \frac{\lambda}{2i}\sum_n V_{m,n}$$
$$\times [e^{i(n-m+N)t/N} - e^{i(n-m-N)t/N}]C_n(t), \tag{3}$$

where $V_{n,m} = \langle n|V(x)|m\rangle$ are the matrix elements of $V(x)$. In Eq. (3), we drop the fast oscillating terms and keep only the resonant ones, that is, $n = m \pm N$. Moreover, for large m, we may take [16,17] $V_{m,m+N} \approx V_{m,m-N} = V$. Hence, Eq. (3) reduces to

$$ik\dot{C}_m = \left[E_m - E_r + (m-r)\frac{k}{N}\right]C_m(t) + \frac{\lambda V}{2i}(C_{m+N} - C_{m-N}). \tag{4}$$

In view of the slow dependence of the energy E_m on the quantum number m around the initially excited level r in a nonlinear resonance, we expand the energy E_m up to second order. Hence, the equation of motion for the probability amplitude C_m takes the form

$$ik\dot{C}_m = (m-r)\left(E_r' - \frac{k}{N}\right)C_m + \frac{1}{2}(m-r)^2 E_r'' C_m$$
$$+ \frac{\lambda V}{2i}(C_{m+N} - C_{m-N}). \tag{5}$$

*Email: address: farhan@qau.edu.pk
†Email address: mauro@camcat.unicam.it

For the exact resonance case $E_r'=k/N$, and as a result, the first term on the right-hand side vanishes. We may write this equation in angle representation by introducing the Fourier representation of C_m as

$$C_m = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) e^{-i(m-r)\varphi} d\varphi$$

$$= \frac{1}{2N\pi} \int_0^{2N\pi} g(\theta) e^{-i(m-r)\theta/N} d\theta. \tag{6}$$

This particular choice of the Fourier representation of C_m yields a Schrödinger-like equation, $i\hbar \dot{g}(\theta, t) = \mathcal{H} g(\theta, t)$, where the Hamiltonian $\mathcal{H}(\theta)$ reads

$$\mathcal{H}(\theta) = -\frac{N^2 E_r''}{2} \frac{\partial^2}{\partial \theta^2} + \lambda V \sin(\theta), \tag{7}$$

and is independent of time. To obtain this equation, we have assumed that the function $g(\theta, t)$ has a $2N\pi$ periodicity in θ coordinate.

Due to the time-independent behavior of $\mathcal{H}(\theta)$, we can write the time evolution of g as $g(\theta, t) = \tilde{g}(\theta) e^{-i\mathcal{E}t/\hbar}$. On substituting this representation in the equation of motion for $g(\theta, t)$, we are left with rather simple equation,

$$\left[-\frac{N^2 E_r''}{2} \frac{\partial^2}{\partial \theta^2} - \mathcal{E} + \lambda V \sin(\theta) \right] \tilde{g}(\theta) = 0. \tag{8}$$

With the change of variable $\theta = 2z + \pi/2$, we may reduce it to the standard Mathieu equation

$$\left[\frac{\partial^2}{\partial z^2} + a - 2q \cos(2z) \right] \tilde{g}(z) = 0. \tag{9}$$

Here, $a = 8\mathcal{E}/N^2 E_r''$, $q = 4\lambda V/N^2 E_r''$, and $\tilde{g}(\varphi)$ is a π-periodic function.

Through this procedure, we have mapped our initially unsolvable time-dependent Schrödinger equation onto the known Mathieu equation [17,18] under the assumption of exact resonance. The Floquet solution of the Mathieu equation [19] may be written as $\tilde{g}_\nu(z) = e^{iz\nu} P_\nu(z)$, where P_ν are the Mathieu functions [22]. Since we require a π periodicity of $\tilde{g}$, we are forced to take only the even values of the index, i.e., $\nu = 2k/N$.

Hence, the Floquet quasienergy eigenvectors [19] may be written as

$$|\psi_k(t)\rangle = e^{i\mathcal{E}_k t/\hbar} |u_k(t)\rangle, \tag{10}$$

where $\mathcal{E}_k$ and $|u_k(t)\rangle$ are defined as

$$\mathcal{E}_k \equiv \frac{N^2 E_r''}{8} a_\nu(q), \tag{11}$$

$$|u_k(t)\rangle \equiv \frac{1}{2\pi} \sum_n e^{int/N} \int_0^{2\pi} d\varphi e^{i\nu N\varphi/2} e^{-i(n-r)\varphi/2} P_{\nu(k)} |n\rangle. \tag{12}$$

In this way, we have obtained an approximate solution for a nonlinear resonance of our explicitly time-dependent system.

In order to check the correctness of our result, we first consider the case of zero modulation strength, that is $\lambda = 0$, for which $q = 0$. The corresponding value for the Mathieu characteristic parameter becomes $a_{\nu(n)}(q=0) = 4(n-r)^2/N^2$. This reduces the quasienergy given in Eq. (11) to

$$\mathcal{E}_k(q=0) = \frac{E_r''}{2} k^2, \tag{13}$$

which is indeed the energy of the unmodulated system expanded around $n=r$ for the exact resonance case. This is obtained by considering $k=n-r$, and therefore we may express $\nu_n(q=0) = 2(n-r)/N$.

The eigenfunctions given in Eq. (10) form a complete set of basis vectors. Therefore, we may write the propagated wave packet at any later time in terms of the quasienergy eigenstates, such that

$$|\psi(t)\rangle = \sum_n \xi_n e^{-i\mathcal{E}_n t/\hbar} |u_n(t)\rangle, \tag{14}$$

where ξ_n describes the probability amplitude in the nth state.

In order to investigate the dynamical revivals, we calculate the autocorrelation function between the initially excited wave packet and the wave packet $|\psi(t)\rangle$ after a certain evolution time t, i.e.,

$$\langle \psi(0)|\psi(t)\rangle = \sum_n |\xi_n|^2 e^{-i\mathcal{E}_n t/\hbar}. \tag{15}$$

We take into account that the wave packet is initially centered around $n=r$ in the energy representation. Therefore, by expanding the quasienergy of the system around the resonant level r, we may write

$$\langle \psi(0)|\psi(t)\rangle = \sum_n |\xi_n|^2 \exp\{-i[\omega^{(0)} + (n-r)\omega^{(1)}$$

$$+ (n-r)^2 \omega^{(2)} + \cdots] t\}, \tag{16}$$

where $\omega^{(j)} \equiv (j!\hbar)^{-1} \mathcal{E}_n^{(j)}|_{n=r}$ and $\mathcal{E}_n^{(j)}|_{n=r}$ denotes the jth derivative of $\mathcal{E}_n$ with respect to n, calculated at $n=r$.

We then compare the autocorrelation function for the driven system, as given in Eq. (16), with the autocorrelation function for the corresponding one-dimensional unperturbed system [5,15]. This comparison helps us to identify the classical period for the driven system as

$$T_{cl} \equiv \frac{2\pi}{\omega^{(1)}} = \frac{2\pi k}{\mathcal{E}_n^{(1)}|_{n=r}}, \tag{17}$$

and the quantum revival time as

$$T_\lambda \equiv \frac{2\pi}{\omega^{(2)}} = \frac{4\pi k}{\mathcal{E}_n^{(2)}|_{n=r}}. \tag{18}$$

Let us now consider the case of small perturbations, i.e., $\lambda \ll 1$. Under this condition, we may use the expansion for the Mathieu characteristic parameter a_ν [22] only up to second order in q. Equation (11) then becomes

$$\mathcal{E}_n = \frac{E_r''}{8}\left(\nu_n^2 + \frac{q^2}{2}\frac{1}{\nu_n^2 - 1}\right) + O(q^4), \qquad (19)$$

and remove an expression for the revival time in the driven system may be derived from Eqs. (18) and (19).

In general, the initially excited wave packet is away from the center of the resonance and the condition $E_r' = k/N$ is not satisfied. For the sake of clarity, in the above discussion, we have only considered the case of the exact resonance. However, taking into account the nonresonant situation, the revival time for a time-dependent system may be calculated in the general case and reads [15]

$$T_\lambda = T_0\left[1 - \frac{1}{2}\left(\frac{\lambda V}{E_r''}\right)^2\left(\frac{2}{N}\right)^4\frac{3\nu_r^2 + 1}{(\nu_r^2 - 1)^3}\right]. \qquad (20)$$

Here, $T_0 = 4\pi k/E_r''$ denotes the revival time of the initial wave packet in the undriven potential and the index ν_n is defined as

$$\nu_n = \frac{2}{N}(n - r) + \frac{2(E_r' - k/N)}{NE_r''}, \qquad (21)$$

where the second term on the right-hand side is due to the nonresonant condition. Substituting in Eq. (20) the value of ν_n calculated at $n = r$ from Eq. (21), we obtain the general expression for the quantum revival time in a periodically driven system as

$$T_\lambda = T_0\left[1 - \frac{1}{2}\left(\frac{\lambda V}{E_r''}\right)^2\frac{3\mu^2 + N^2/4}{(\mu^2 - N^2/4)^3}\right], \qquad (22)$$

where $\mu = (E_r' - k/N)/E_r''$. Equation (22) constitutes our main result: It expresses the modification of the quantum revival time in the presence of an external periodic driving field as a function of the modulation strength λ and of the other characteristic parameters of the system.

The present approach is valid for small modulation strengths. In the weak binding potentials, for which the level spacing decreases as we increase the quantum number, the nonlinear resonances disappear after small modulations, as observed numerically for a gravitational cavity [23,24] and also for the hydrogen atom. As a consequence, this technique is reasonably good for these kind of potentials. For tight binding potentials, for which the level spacing increases as we go up in the energy, we find nonlinear resonances for higher modulation as well. In this kind of potential we need to consider higher-order terms in the expansion of the Mathieu characteristic parameter in order to correctly calculate the recurrence time.

As an application of our method, we consider the dynamics of atoms in a gravitational cavity [20] in the presence of an external periodic field [25,26]. In this case, the atoms move in a potential $x + V_0 e^{-\kappa x}$, where the linear term is due to the gravitational potential and the exponential term comes from the evanescent-wave field that may be obtained by total internal reflection of a laser light field on the surface of a glass prism [20,21,27–30]. In the approximation of a triangular well potential, the coupling constant for the Nth resonance is given by

$$V \equiv \langle m|x|m \pm N\rangle = -\frac{2E_m}{N^2\pi^2[1 \mp N/6m + O(m^{-2})]^2}, \qquad (23)$$

which in the limit of large m becomes $V \cong -2E_N/N^2\pi^2$. Substituting this expression into Eq. (22), it is possible to calculate the time of revival for a wave packet initially excited with average energy E_r around the Nth resonance, which reads

$$T_\lambda = T_0\left\{1 - \frac{1}{8}\left(\frac{\lambda}{E_r}\right)^2\frac{3(1 - \alpha)^2 + \beta^2}{[(1 - \alpha)^2 - \beta^2]^3}\right\}, \qquad (24)$$

where $\alpha \equiv (E_N/E_r)^{1/2}$ and $\beta \equiv \alpha^2 k/4E_r$. If the initial energy is large enough, that is $E_r \gg 1$, we may neglect β^2 with respect to $(1 - \alpha)^2$ in Eq. (24), obtaining the simpler formula

$$T_\lambda = T_0\left[1 - \frac{3}{8}\left(\frac{\lambda}{E_r}\right)^2\frac{1}{(1 - \alpha)^4}\right]. \qquad (25)$$

This expression allows us to compare the analytically calculated revival time with the numerically computed ones.
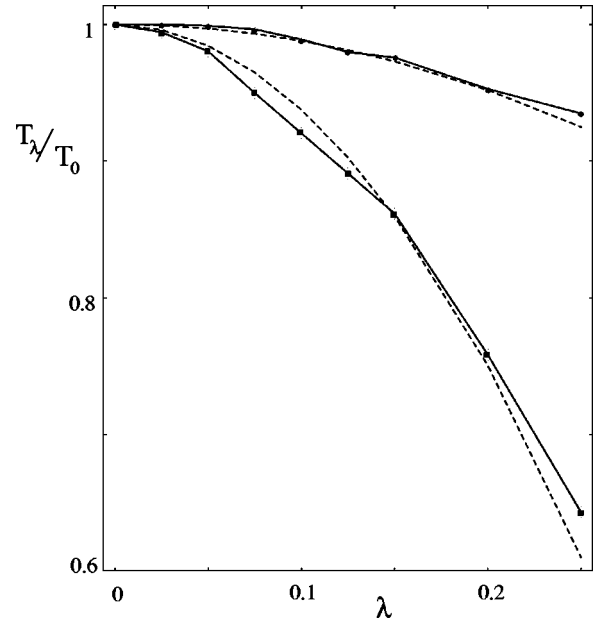

FIG. 1. Comparison between the numerically computed ratio T_λ/T_0 for two different initial conditions, $E_r = 104.1$ (solid line with circles) and $E_r = 70.28$ (solid line with squares), and the corresponding [Eq. (25)] analytical results (dashed lines) in the case of an atom bouncing in a modulated gravitational cavity. The other parameters are $k = 1$, $V_0 = 1$, and $\kappa = 1$.

In the example considered above, one may prepare a wave packet by trapping and cooling cesium atoms in a magneto-optical trap (MOT) [19,20,26]. The wave packet may be placed at (or close to) the Nth resonance by accurately controlling the position of the center of the MOT where the atom is localized and giving the atom some velocity using another laser. In laboratory experiments, the driving frequency may be changed from 0 to 2 MHz [21]. Hence, if we select a frequency $\omega = 2\pi \times 0.93$ kHz out of this domain, for cesium atoms of mass $M = 2.2 \times 10^{-25}$ kg, the dimensionless Plank's constant is $k = 1$. Moreover, with an effective Rabi frequency $\Omega_{\text{eff}} = 2\pi \times 3.72$ kHz and a potential steepness equal to 0.57 μm^{-1}, we have $V = 1$ and $\kappa = 1$. In our numerical calculations we use these values of the dimensionless parameters. In order to estimate the quantum recurrence time, in the numerical simulations we have placed the atomic wave packet above the atomic mirror at $x_0 = 29.8 \, \mu\text{m}$, for which $E_r = 104.1$, and $x_0 = 20.1 \, \mu\text{m}$, for which $E_r = 70.28$.

Our numerical results show a complete qualitative and quantitative agreement with the analytical results. In fact, from Eq. (25) we learn that (i) the revival time changes quadratically as a function of the strength λ of the external modulation field, and (ii) the revival time depends inversely on the square of the initial average energy of the wave packet. Our numerical investigation confirms both these analytical results.

In Fig. 1, we compare the quantum revival times calculated from Eq. (25) (dashed lines) with those obtained numerically (solid lines) for an atomic wave packet bouncing in a modulated gravitational cavity. We see from the numerical data that the revival time displays a quadratic dependence on the strength λ of the external modulation. Moreover, the change in the revival time is just about 8% for λ ranging from 0 to 0.25 when the initial average energy $E_r = 104.1$, (solid line with circles), whereas it increases to almost 40% for the same range of λ when the initial average energy $E_r = 70.28$ (solid line with squares), in agreement with the inverse square dependence of T_λ on E_r.

In summary, we have presented a general approach to the investigation of the phenomenon of quantum revivals in a certain class of periodically driven systems. We have derived a simple relation that provides the quantum revival time as a function of the modulation strength. We have finally applied our theory to the case of the revival time arising in the dynamics of atoms bouncing in a modulated gravitational cavity, obtaining an excellent agreement with the exact numerical data. For the set of parameters that we have used in our calculations, a modulated gravitational cavity may be realized within the framework of presently available technology [15]. We strongly suggest an experiment, designed with the help of these parameters, to test our prediction of the quantum recurrences in the driven gravitational cavity.

---

[1] W. H. F. Talbot, Philos. Mag. **4**, 401 (1836).

[2] M. Born and E. Wolf, *Principles of Optics*, 5th ed. (Pergamon Press, New York, 1975).

[3] M. Berry, I. Marzoli, and W. Schleich, Phys. World **14(6)**, 39 (2001).

[4] J. Parker and C. R. Stroud, Jr., Phys. Rev. Lett. **56**, 716 (1986).

[5] I. Sh. Averbukh and N. F. Perel'man, Phys. Rev. A **139**, 449 (1989).

[6] C. Leichtle, I. Sh. Averbukh, and W. P. Schleich, Phys. Rev. Lett. **77**, 3999 (1996); Phys. Rev. A **54**, 5299 (1996).

[7] P. A. Braun and V. I. Savichev, J. Phys. B **29**, L329 (1996).

[8] Wen-Yu Chen and G. J. Milburn, Phys. Rev. A **51**, 2328 (1995).

[9] D. L. Aronstein and C. R. Stroud, Jr., Phys. Rev. A **62**, 022102 (2000).

[10] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 1992).

[11] T. Hogg and B. A. Huberman, Phys. Rev. Lett. **48**, 711 (1982).

[12] J. K. Breslin, C. A. Holmes, and G. J. Milburn, Phys. Rev. A **56**, 3022 (1997).

[13] S. Tomsovic and J. Lefebvre, Phys. Rev. Lett. **79**, 3629 (1997).

[14] F. Saif, Phys. Rev. E **62**, 6308 (2000).

[15] F. Saif, Ph.D. thesis, University of Ulm, 1998; Hyperthesis **1**, 2 (1999).

[16] F. Saif, G. Alber, V. Savichev, and W. P. Schleich, J. Opt. B: Quantum Semiclassical Opt. **2**, 668 (2000); F. Saif, J. Phys. Soc. Jpn. **69**(8), 2363L (2000).

[17] G. P. Berman and G. M. Zaslavsky, Phys. Lett. A **61**, 295 (1977).

[18] M. E. Flatté and M. Holthaus, Ann. Phys. (N.Y.) **245**, 113 (1996).

[19] L. E. Reichl, *The Transition to Chaos in Conservative Classical Systems: Quantum Manifestations* (Springer-Verlag, Berlin, 1992).

[20] C. G. Aminoff, A. M. Steane, P. Bouyer, P. Desbiolles, J. Dalibard, and C. Cohen-Tannoudji, Phys. Rev. Lett. **71**, 3083 (1993).

[21] A. Steane, P. Szriftgiser, P. Desbiolles, and J. Dalibard, Phys. Rev. Lett. **74**, 4972 (1995).

[22] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1992).

[23] H. Wallis, J. Dalibard, and C. Cohen-Tannoudji, Appl. Phys. B: Lasers Opt. **54**, 407 (1992).

[24] J. P. Dowling and J. Gea-Banacloche, Adv. At., Mol., Opt. Phys. **37**, 1 (1996).

[25] F. Saif, I. Bialynicki-Birula, M. Fortunato, and W. P. Schleich, Phys. Rev. A **58**, 4779 (1998).

[26] F. Saif, Phys. Lett. A **274**, 98 (2000).

[27] R. J. Cook and R. K. Hill, Opt. Commun. **43**, 258 (1982).

[28] V. I. Balykin, V. S. Letokhov, Yu. B. Ovchinnikov, and A. I. Sidorov, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 282 (1987) [JETP Lett. **45**, 353 (1987)]; Phys. Rev. Lett. **60**, 2137 (1988).

[29] M. A. Kasevich, D. S. Weiss, and S. Chu, Opt. Lett. **15**, 607 (1990).

[30] Yu. B. Ovchinnikov, I. Manek, and R. Grimm, Phys. Rev. Lett. **79**, 2225 (1997).