"Single-sided" focusing of the time-dependent Schrödinger equation

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(Received 1 December 2000; revised manuscript received 4 April 2001; published 13 December 2001)

The problem of focusing is considered for the one-dimensional time-dependent Schrödinger equation with a local potential. Focusing is also considered for the closely related plasma wave equation. It is supposed that an experimenter can send in an incident wave *from one side of the potential*. Then to what degree can the wave be focused? It will be shown that an incident wave can be found so that the *real part of the wave* collapses to a delta function $\delta(x-x_o)$ when $t=0$. The equation that governs this previously unsolved "single-sided" focusing problem is derived and shown to be Marchenko's equation—the canonical equation of one-dimensional inverse-scattering theory.

DOI: 10.1103/PhysRevA.65.012707 PACS number(s): 03.65.Nk

I. INTRODUCTION

Suppose a distorting layer separates an experimenter from half of space. Furthermore, suppose that the experimenter can probe this distorting layer by sending in arbitrary time-domain waves from one side and by measuring the resulting reflected waves. Finally, suppose that x denotes the spatial coordinate and x_o an arbitrary focal point, which may be inside the support of the distorting layer. Then the following question can be asked. "To what degree can the experimenter find a wave, incident from one side, such that the wave collapses to $\delta(x-x_o)$ at a prespecified time, i.e., 'single-sided' focusing of the wave at x_o?"

An exact theory of such focusing does not currently exist. Here, this gap is partly closed and an exact one-dimensional (1D) theory is presented for both the time-dependent Schrödinger equation (TDSE) and the plasma wave equation (PWE). Marchenko's equation [1–6]—the canonical equation of 1D inverse-scattering theory—is shown to govern single-sided focusing for both wave equations. The real part of the wave can be focused for the TDSE (generally, the imaginary part remains unfocused). For the closely related PWE, the total wave can be focused (generally, the time derivative of the wave remains unfocused). Mathematical proofs and analytic examples for single-sided focusing of the PWE will be published separately [7].

This work was motivated by the fact that many imaging methods rely explicitly on the ability to focus a wave of one kind or another. Furthermore, imaging is plausibly an inverse-scattering method since it remotely determines the properties of an object from scattered or reflected waves. The conjunction of these two facts suggests that a connection might exist between exact inverse-scattering theory and an exact theory of focusing. This possibility motivated the development of a theory of focusing for the 1D Schrödinger equation where the problem of inverse scattering is well understood. A paper by Dyson [8] provided additional motivation. He considered adaptive focusing through a turbulent atmosphere and showed that the "least-squares" optimal focus is obtained by solving a multidimensional Marchenko-like equation.

This paper is organized as follows. Section II introduces the needed wave equations, reviews scattering theory, and defines notation. More importantly, it shows that single-sided focusing for the TDSE is equivalent to single-sided focusing for the PWE. Section III derives the equation that governs single-sided focusing for the PWE and thus for the TDSE. Section IV shows by numerical example that Marchenko's equation can, in fact, be used to focus the wave. Finally, the paper is concluded with a discussion and summary.

II. EQUIVALENCE OF "SINGLE-SIDED" FOCUSING FOR THE PWE AND THE TDSE

This section reviews the formalism needed to solve the single-sided focusing problem. Section II A introduces the time-independent Schrödinger equation and the PWE. Section II B shows that the PWE and TDSE are equivalent with respect to single-sided focusing.

"Time" for the TDSE does not have the same nature as the "time" for the PWE. For the TDSE time is the variable conjugate to the energy. For the PWE time is the variable conjugate to the wave number. This distinction is maintained by using "t" for the TDSE time and "τ" for the PWE time. These different definitions of time lead to a crucial difference between the TDSE and PWE. Namely, the speed of propagation is unbounded for the (parabolic) TDSE but is finite and bounded by the value 1 for the (hyperbolic) PWE.

The repeated use of the phrase single-sided focusing is cumbersome. From this point onward, the word focusing will replace the phrase single-sided focusing except where emphasis is needed.

A. The wave equations and notation

The time-independent Schrödinger equation and two equivalent time-domain wave equations are introduced. The time-independent Schrödinger equation

$$\left(-\frac{d^2}{dx^2}+\nu(x)\right)\psi(k,x)=E\psi(k,x) \tag{1a}$$

is chosen to describe wave propagation on the line (for the quantum scattering problem insert $\hbar^2/2m$ before the second derivative). For simplicity, the potential $\nu(x)$ is chosen to be a positive, finite, and real function with compact support to the right of the origin of coordinates, $x=0$. Bound states are excluded by construction since the potential is everywhere positive. These conditions on the potential can all be significantly relaxed [7]. E denotes the energy and k the wave number. For the class of potentials mentioned above, the energy is related to the wave number by $E=k^2$ and the wave equation becomes

$$\left(-\frac{d^2}{dx^2}+\nu(x)\right)\psi(k,x)=k^2\psi(k,x). \tag{1b}$$

Scattering solutions of Eqs. (1) are generated by an incident right-going plane wave. They are defined by the following solutions of the Lippmann-Schwinger equation:

$$\psi^+(k,1,x)=e^{ikx}+\int_{-\infty}^{\infty}dx'\,\hat{G}_o^+(k,|x-x'|)\nu(x')\psi^+(k,1,x'). \tag{2}$$

Here, the argument "1" indicates waves incident from the left. The 1D noninteracting causal Green's function is

$$\hat{G}_o^+(k,x,x')\equiv\frac{e^{ik|x-x'|}}{2ik}. \tag{3}$$

Note that if k is a real number,

$$\psi^+(k,1,x)=\psi^{+*}(-k,1,x), \tag{4}$$

since both are solutions of the same Lippmann-Schwinger equation. This fact is needed to connect the time-independent Schrödinger equation and the PWE. The data for the focusing problem are the reflection coefficient for all real k, where $\hat{R}(k)$ is defined by

$$\psi^+(k,1,x)\equiv e^{ikx}+\hat{R}(k)e^{-ikx} \tag{5}$$

for $x<0$.

The plasma wave equation is

$$\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial x^2}+\nu(x)\right)U(\tau,x)=0. \tag{6}$$

The PWE follows from Fourier transforming the time-independent Schrödinger equation, Eq. (1b), with respect to the wave number k. Only real solutions U will be considered.

The scattering solutions for the PWE follow by Fourier transforming ψ^+ and are

$$u^+(\tau,1,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,e^{-ik\tau}\psi^+(k,1,x). \tag{7}$$

Here, the integration has been extended to negative k by using the reality condition, Eq. (4). The Lippmann-Schwinger equation becomes

$$u^+(\tau,1,x)=\delta(\tau-x)+\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}d\tau'\times G_o^+(\tau-\tau',|x-x'|)\nu(x')u^+(\tau',1,x'). \tag{8}$$

The total wave due to an arbitrary incident wave $U_{\text{in}}(\tau-x)$ is denoted by

$$U(\tau,x)=\int_{-\infty}^{\infty}d\tau'\,u^+(\tau-\tau',1,x)U_{\text{in}}(\tau'). \tag{9}$$

Finally, we define *the impulse response function R* to be the Fourier transform of the reflection coefficient with respect to k,

$$R(\tau)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\hat{R}(k)e^{-ik\tau}. \tag{10}$$

The impulse response is just the reflected wave (adjusted for the travel time) due to a delta-function incident wave $\delta(\tau-x)$. It is the data needed to solve the focusing problem.

The TDSE follows from the time-independent Schrödinger equation in the standard way,

$$\left(i\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}-\nu(x)\right)\Psi(t,x)=0. \tag{11}$$

The scattering solutions are denoted by $\Psi^+(t,1,x)$ and are given by

$$\Psi^+(t,1,x)=\frac{1}{\pi}\int_0^{\infty}dk\,\psi^+(k,1,x)e^{-ik^2t} \tag{12}$$

The total wave due to an arbitrary wave incident from the left is

$$\Phi^+(t,1,x)=\frac{1}{\pi}\int_0^{\infty}dk\,\hat{U}_{\text{in}}(k)\psi^+(k,1,x)e^{-ik^2t}. \tag{13}$$

B. Focusing the PWE is equivalent to focusing the TDSE

This section shows that focusing the PWE implies focusing the TDSE and *vice versa*. Once this equivalence is shown, one can infer the focusing properties of the TDSE by establishing the focusing properties of the PWE.

The PWE is said to be focused at x_o if some real right-propagating incident wave, $U_{\text{in}}(\tau-x;x_o)$, causes the total wave to collapse to a δ function at x_o at $\tau=0$, i.e.,

$$U(\tau=0,1,x)=\int_{-\infty}^{\infty}d\tau'\,U_{\text{in}}(-\tau';x_o)u^+(\tau',1,x)=\delta(x-x_o). \tag{14}$$

See Eq. (9). The second argument for U_{in} indicates that this particular incident wave focuses the total wave to x_o.

The TDSE is said to be focused if some $\hat{U}_{\text{in}}$, incident from the left, causes the real part of the total wave to collapse to a δ function at x_o at $t=0$, i.e.,

$$\mathrm{Re}\,\Phi(t=0,1,x)=\mathrm{Re}\frac{1}{\pi}\int_0^\infty dk\,\hat{U}_{\mathrm{in}}(k;x_o)\psi^+(k,1,x)$$
$$=\delta(x-x_o). \quad (15)$$

See Eq. (13).

I will first show that if the PWE is focused—i.e., if Eq. (14) is true—then the TDSE is also focused—i.e., Eq. (15) is true. Start by supposing that Eq. (14) is true. Fourier transform the integral on the right-hand side to the k domain. Note that $\hat{U}_{\mathrm{in}}(-k;x_o)=\hat{U}^*_{\mathrm{in}}(k;x_o)$ since $U_{\mathrm{in}}(\tau;x_o)$ is real by construction. Next, use the reality condition, Eq. (4), to rewrite the integral over positive k. The result is Eq. (15). Thus, the TDSE is focused if the PWE is focused.

Next, I show that if the TDSE is focused—i.e., if Eq. (15) is true—then the PWE is also focused—i.e., Eq. (14) is true. At $t=0$, the real part of Φ [defined by Eq. (13)] can be written as

$$\mathrm{Re}\,\Phi(t=0,1,x)=\frac{1}{2\pi}\int_0^\infty dk\,(\hat{U}_{\mathrm{in}}(k;x_o)\psi^+(k,1,x)$$
$$+\hat{U}_{\mathrm{in}}{*}(k;x_o)\psi^+{*}(k,1,x))$$
$$=\delta(x-x_o). \quad (16)$$

Equation (16) becomes, after using the reality conditions for $\hat{U}_{\mathrm{in}}$ and ψ^+,

$$\mathrm{Re}\,\Phi(t=0,1,x)=\frac{1}{2\pi}\int_{-\infty}^\infty dk\,\hat{U}_{\mathrm{in}}(k;x_o)\psi^+(k,1,x)$$
$$=\delta(x-x_o). \quad (17)$$

Finally, Fourier transform the integrand on the right-hand side of Eq. (17) from k to τ and obtain the integral found on the right-hand side of Eq. (14). Thus, if the TDSE is focused, then so is the PWE.

The equivalence of focusing for the TDSE and PWE is easily verified for the following trivial example—focusing in free space, i.e., $v(x)=0\ \forall x$. The required incident wave that focuses the TDSE and PWE is

$$\hat{U}_{\mathrm{in}}(k;x_o)=e^{-ikx_o}, \quad (18)$$

while $\psi^+(k,1,x)=e^{ikx}$ is the total wave. The required delta function for the TDSE follows when Eq. (18) is substituted into Eq. (17). The δ function for the PWE follows when $\hat{U}_{\mathrm{in}}$ and ψ^+ are Fourier transformed to the τ domain and then substituted in Eq. (14).

III. SINGLE-SIDED FOCUSING OF THE PWE

The single-sided focusing problem for the PWE is solved in this section and the governing role of Marchenko's equation is clarified. Section III A sets up the focusing problem. Next, a key time antisymmetry of the wave is derived in Sec. III B. In Sec. III C the equation that governs focusing is derived and found to be Marchenko's equation.

The main points of the derivation are especially well represented visually. Figure 1 shows the focusing of the PWE to a point behind a square well. Somewhat inconsistently time is labeled by t rather than τ in Figs. 1 and 2. This schematic drawing (actually a numerical result, see Sec. IV) shows the propagation of the total wave through the square well and its collapse to a delta function $\delta(x-x_o)$ at $\tau=0$. The incident wave, shown by the curve labeled -4, consists of a sharp Gaussian peak (the “delta function”) followed by a long “tail.” Below, I will show that one computes just this “tail” when one solves the Marchenko equation. The curves labeled -2.2, -1.8, and 0.0 show the evolution of the wave as it propagates through the square well and collapses to a δ function at $x_o=4.0$.

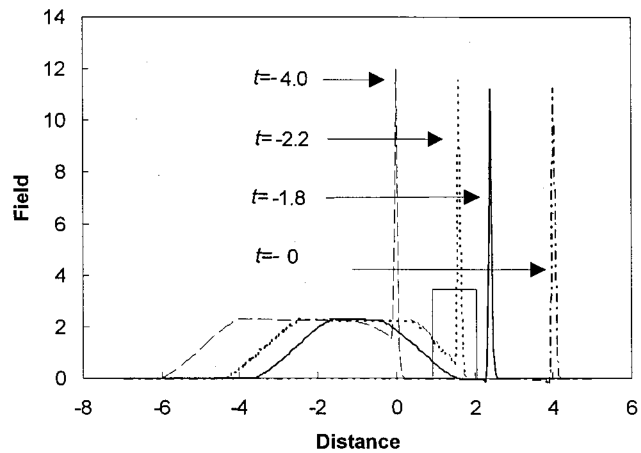

FIG. 1. The leftmost dashed curve shows the incident wave that focuses to the dash-dotted curve at $x=4.0$ and $t=0.0$. The dash-dotted line shows the focused wave, while the solid and dotted lines show the wave at intermediate times. The rectangle represents a repulsive square-well potential with height 3.0 and width 1.0. For the quantum scattering problem the unit for distance is meters while the unit for the field is $\mathrm{sec}^{-1}\,\mathrm{m}^{-1/2}$.

Figure 2 shows that, if the wave is focused, then the left-going reflected wave at $\tau=x_o$ is the negative of the right-going incident wave at $\tau=-x_o$ for all points $-2x_o<x<0$. That is, the curve labeled -4.0 is opposite and equal to the curve labeled 4.0 over this range. Thus, this part of the reflected wave can be immediately inferred from the incident wave. This illustrates an important result that will be used in the derivation. To wit, focusing at any $x_o>0$ causes the total wave to be antisymmetric in time—for points to the left of the focal point and during a certain time interval. In particular, I will show that at $x=0$, $U_{\mathrm{tot}}(\tau,x=0)=-U_{\mathrm{tot}}(-\tau,x=0)$ for $-x_o<\tau<x_o$.

A. Preliminaries for the derivation

The equivalence of the PWE and the Schrödinger equation for potential-scattering problems is notable and widely used in inverse scattering theory [1,2]. The PWE is a dispersive hyperbolic wave equation that propagates a sharp wave front at velocity 1. The PWE simplifies to the (Helmholtz) wave equation if the potential is everywhere zero (i.e., in free space).

The focusing problem for the PWE is as follows. Suppose that one has a physical system governed by the PWE and that

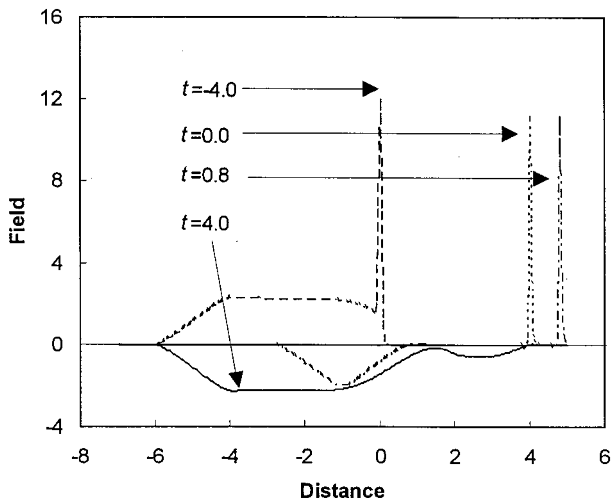

FIG. 2. The tail of the right-going incident wave at $t=-x_o=-4.0$ and that of the left-going reflected wave for $t=x_o=4.0$ are opposite and equal for $x<0.0$. The dotted curve shows the focused wave at $x=4.0$. The dashed-dotted curve represents the wave after it has passed through the focal point; note that it suddenly becomes nonzero over the range $t=-3.0<x<1.0$. For the quantum-scattering problem the unit for distance is meters while the unit for the field is $\sec^{-1}\,\mathrm{m}^{-1/2}$.

one knows the impulse response function. Determine the right-going wave, incident from $-\infty$, that results in the total wave collapsing to $\delta(x-x_o)$ at $\tau=0$. Here, x_o is inside or to the right of the support of the potential.

A hypothetical experiment proceeds as follows. The experimenter is assumed to have access to the left half-line, $x\leqslant 0$, where the potential is zero. Since the accessible region is free space, it is assumed that the experimenter can distinguish left- and right- going waves, and can also prepare arbitrary right-going incident waves $U_{\rm in}(\tau-x)$ and measure arbitrary left-going reflected waves $U_{\rm out}(\tau+x)$. The total wave is denoted by $U_{\rm tot}(\tau,x)$ and is defined for all x and τ. The initial condition for the wave $U_{\rm tot}(\tau,x)$ is that at some sufficiently early time $U_{\rm tot}$ has support confined to $x<0$ and is identical to the incident wave $U_{\rm in}(\tau-x)$. As time progresses, $U_{\rm tot}$ spreads onto the right half-line, interacts strongly with the potential, and partially backscatters—yielding a left-going part that becomes identical to the reflected wave $U_{\rm out}(\tau+x)$ at late times. For $x<0$, the total wave can be written as the sum of the right-going incident and the left-going reflected waves. That is,

$$U_{\rm tot}(\tau,x)=U_{\rm in}(\tau-x)+U_{\rm out}(\tau+x). \tag{19}$$

Now let us turn to the incident wave for which the total wave is focused to x_o at $\tau=0$. It is hypothesized (and later verified) to have the form

$$U_{\rm in}(\tau-x;x_o)=\delta(\tau-x+x_o)+\Omega(\tau-x;x_o). \tag{20}$$

Physically, the incident wave consists of a δ-function wave front followed by a long tail; see the curve marked -4.0 in Fig. 2. Ω is a to-be-determined function supported to the left of the delta function wave front, i.e., $\Omega(\tau;x_o)=0$ for $\tau<-x_o$.

The impulse response function $R(\tau)$ is the input data for the focusing equation and is defined by Eq. (10). In the time-domain picture R is the reflected wave measured at $x=0$ that evolves from an incident delta-function wave $\delta(\tau-x)$. More precisely define the scattered wave $U_{\rm tot}^{+\rm sc}$ to be the difference between this total wave and the incident δ function. The impulse response function is

$$R(\tau)=U_{\rm tot}^{+\rm sc}(\tau,0). \tag{21}$$

The solution of the single-sided focusing problem follows from considerations of how the reflected wave $U_{\rm out}$ is determined from the incident wave $U_{\rm in}$. As discussed in the next section, two independent linear equations determine $U_{\rm out}$ from $U_{\rm in}$ if the total wave is focused. The equation that governs focusing follows when these two equations are combined and $U_{\rm out}$ is eliminated.

The first equation that determines $U_{\rm out}$ from $U_{\rm in}$ follows from supposing that the total wave has a single-sided focus at some point x_o. I will show that such a total wave must be antisymmetric in time over the interval $-x_o<\tau<x_o$ at the observation point $x=0$. The antisymmetry determines the reflected wave (measured at late times) from the incident wave (measured at early times). The second equation that determines $U_{\rm out}$ from $U_{\rm in}$ is true whether the wave is focused or not. It follows straightforwardly from the linearity of the PWE, causality, and the knowledge of the impulse response function. To wit, the reflected wave is determined by convolving the incident wave and the impulse response function, i.e., ordinary linear scattering theory. To sum up, *for focused waves*, there are two independent ways of finding the reflected wave, given the incident wave: (1) antisymmetry in time and (2) linearity and the impulse response function.

B. Temporal antisymmetry for focused waves

I will show that if the total wave focuses to some $x_o>0$, then $U_{\rm tot}(\tau,x=0)$ (the total wave evaluated at $x=0$) is antisymmetric in τ during the time interval $-x_o<\tau<x_o$. This antisymmetry follows from the definition of focusing and two key properties of the PWE. First, the wave front propagates with a velocity of 1. Second, the total wave at any point x and time τ is determined by certain initial-value data. These data are the wave at $\tau=0$ and the time derivative of the wave at $\tau=0$ both evaluated over a sufficiently large region about x. What are the data required to determine the total wave at $x=0$ for times $-x_o<\tau<x_o$? Since the wave front velocity is 1, causality implies that at $\tau=0$ only waves and their derivatives in the range $-x_o<x<x_o$ contribute.

All the elements are now in place to show the antisymmetric nature of the total wave. The data required to find $U_{\rm tot}(\tau,x=0)$ for $-x_o<\tau<x_o$ are (1) $U_{\rm tot}(\tau=0,x)$ and (2) its time derivative, $\partial U_{\rm tot}(\tau,x)/\partial\tau|_{\tau=0}$, both on the spatial interval $-x_o<x<x_o$. But the total wave $U_{\rm tot}(\tau=0,x)$ is zero on this interval since by hypothesis the total wave is focused to $\delta(x-x_o)$ at $\tau=0$. Thus, for times $-x_o<\tau<x_o$,

$U_{tot}(\tau,x=0)$ depends only on the time derivative of the total wave at $\tau=0$, i.e., $\partial U_{tot}(\tau,x)/\partial\tau|_{\tau=0}$. Antisymmetry arises (as argued in the next paragraph) because the data, $\partial U_{tot}(\tau,x)/\partial\tau|_{\tau=0}$, changes sign under time reversal, $\tau\to-\tau$.

Temporal antisymmetry in time is shown as follows. First, define $U_{tot}(\tau,x=0)$ by propagating the data towards the future from $\tau=0$. Second, obtain $-U_{tot}(-\tau,x=0)$ by propagating the same data towards the past. Propagating the wave to the past can be broken up into the following three steps: (a) time reverse the data ($\tau\to-\tau$), (b) propagate this new problem to its future to obtain $-U_{tot}(\tau,x=0)$, and (c) time reverse the solution ($-\tau\to\tau$) to get $-U_{tot}(-\tau,x=0)$. Thus,

$$U_{tot}(\tau,x=0)=-U_{tot}(-\tau,x=0) \tag{22}$$

for $-x_o<\tau<x_o$.

The antisymmetry of the total focused wave at $x=0$ implies that the incident and reflected waves are also related by antisymmetry. Combine Eqs. (19) and (22) to obtain

$$U_{in}(\tau;x_o)+U_{out}(\tau;x_o)=-U_{in}(-\tau;x_o)-U_{out}(-\tau;x_o) \tag{23a}$$

for $-x_o<\tau<x_o$. Finally, equate the left-going waves on both sides of Eq. (23a) to find

$$U_{out}(\tau;x_o)=-U_{in}(-\tau;x_o) \tag{23b}$$

for $-x_o<\tau<x_o$.

C. Marchenko's equation governs single-sided focusing

The equation that governs focusing is derived and shown to be Marchenko's equation. First, the linearity of the PWE ensures that the reflected wave can be found by convolving the incident wave with the impulse response function. One uses the definition of the impulse response function, causality, and the superposition principle to obtain

$$U_{out}(\tau;x_o)=\int_{-\infty}^{\infty}d\tau'\,R(\tau-\tau')U_{in}(\tau';x_o). \tag{24}$$

Second, the reflected wave is determined from the incident wave by antisymmetry as expressed by Eq. (23b).

The focusing equation follows since we have two equations [Eqs. (23b) and (24)] and two unknowns ($U_{in}=\delta+\Omega$ and U_{out}). We solve for Ω using Eq. (20) and obtain

$$\Omega(-\tau;x_o)+R(\tau+x_o)+\int_{-\infty}^{\infty}d\tau'\,R(\tau+\tau')\Omega(-\tau';x_o)=0 \tag{25}$$

for $\tau<x_o$ and $\Omega(-\tau)=0$ for $\tau>x_o$. Equation (25), if it can be solved, yields Ω, the trailing part of the incident wave—again, see the curve labeled -4.0 in Fig. 1. This, in turn, determines the incident wave needed to focus the total wave at x_o via Eq. (20).

Inspection shows that the focusing equation (25) is identical to Marchenko's equation after the simple change of notation,

$$A_l(x,y)=\Omega(-y;x). \tag{26}$$

Here, A_l denotes the solution of Marchenko's equation in the notation of Ref. [1]. Finally, note that the single-sided focusing problem is uniquely soluble since Marchenko's equation has a unique solution [1–6].

IV. NUMERICAL SIMULATION OF SINGLE-SIDED FOCUSING

Numerical solutions for several square-well examples were used to test the utility of Marchenko's equation for focusing the PWE. Since focusing the PWE and TDSE are equivalent, these results also test the use of Marchenko's equation to focus the TDSE. Numerical solutions to Marchenko's equation were obtained for a variety of attractive and repulsive square wells. These numerical solutions were then used to obtain focused waves. In particular, the incident waves $U_{in}(\tau;x_o)$ were found from Eqs. (20) and (25). Finally, these incident waves were propagated through the potential and the results graphed.

A finite difference algorithm implemented in FORTRAN with double-precision arithmetic was used to simulate wave propagation. A spatial region of length 12.0 was discretized in 6000 equal intervals and the travel time from the origin to focal point x_o was discretized in 30 000 equal intervals. Besides discretization and finite-precision arithmetic, the most important approximation was to replace the δ function by a narrow Gaussian wave with an integrated strength of 1. The half-width of the Gaussian was chosen to be 0.05. The potentials were chosen so that the origin (the observation point) was always to the left of the potential's support. The solution to Marchenko's equation was obtained by iteration.

Figures 1 and 2 show finite-difference calculations for a repulsive (positive) square well that had strength $V_o=3.0$ and extended from 1.0 to 2.0. The focal point was chosen to lie to the right of the potential at $x_o=4.0$. Figure 1 shows the evolution of the wave for four different times up to and including the time of focus at $t=0.0$. The incident wave is shown by the curve labeled -4.0. The curve labeled -2.2 shows the peak traversing the potential. As it does so, the tail begins to decrease. The third peak, labeled -1.8, shows that the tail is further reduced as the peak passes through the potential. Finally, focus is achieved and the tail disappears completely at $\tau=0.0$. Figure 2 shows the wave evolving at later times. The antisymmetry of the scattering process for $x<0.0$ is evident from the curves labeled -4.0 and 4.0.

Additional calculations were carried out for focusing in and through a variety of attractive and repulsive square wells. Single-sided focuses were obtained in all cases examined including attractive square-well potentials with bound states.

V. DISCUSSION AND SUMMARY

The work was motivated by the desire to explore the connection, if any, between focusing and inverse scattering. I have shown that the same fundamental equation—

Marchenko's equation—governs both problems for the 1D Schrödinger equation.

Single-sided focusing of the TDSE concentrates the real part of the total wave to $\delta(x-x_o)$ at $t=0$. The imaginary part of the wave is not concentrated at x_o and consequently single-sided focusing does not concentrate the probability density to x_o at $t=0$. However, the probability current density

$$j(t,x)\equiv i\left(\Phi(t,x)\frac{\partial}{\partial x}\Phi^*(t,x)-\Phi^*(t,x)\frac{\partial}{\partial x}\Phi(t,x)\right) \tag{27}$$

is concentrated to x_o at $t=0$. The basic result of the paper is that upon focusing, $\mathrm{Re}\,\Phi(t=0,x)=\delta(x-x_o)$. Since the focused wave $\Phi(t=0,x)$ is purely imaginary for $x\neq x_o$ it follows by direct evaluation of Eq. (27) that $j(t=0,x)=0$ for all $x\neq x_o$. Similarly, although single-sided focusing of the PWE concentrates the total wave to x_o at $\tau=0$, it does not concentrate the time derivative of the total wave to that point. Consequently, the energy density, which depends in part on the time derivative of the wave, is generally nonzero for $x\neq x_o$.

In summary, this paper has sought the equation that determines the incident wave that focuses the real part of the wave to $\delta(x-x_o)$ at time $t=0$ for the TDSE. The result was Marchenko's equation. The fact that Marchenko's equation governs single-sided focusing was tested numerically for the PWE and verified for square-well potentials.

ACKNOWLEDGMENTS

Thanks are due to Dr. Paul Sacks, Dr. Jeorg Schmalian, and Dr. Ronald Roberts for criticizing preliminary versions of the manuscript. The Ames Laboratory is operated for the U.S. DOE by Iowa State University under Contract No. W-7405-ENG-82. This work is supported, in part, by the Director, Office of Basic Energy Sciences.

[1] G. L. Lamb, Jr., *Elements of Soliton Theory* (Wiley, New York, 1980). See Eq. (2.10.18) on p. 66 for Marchenko's equation. The relevant impulse response function is given on p. 53.
[2] R. Burridge, Wave Motion **2**, 305 (1980).
[3] V. A. Marchenko, Dokl. Akad. Nauk SSSR **104**, 695 (1955).
[4] Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach, New York, 1963).
[5] L. D. Fadeev, Trans. Am. Math. Soc. **65**, 139 (1967).
[6] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer-Verlag, New York, 1989).
[7] T. Aktosun and J. H. Rose (unpublished).
[8] F. J. Dyson, J. Opt. Soc. Am. **65**, 551 (1975).