

Heating and decoherence suppression using decoupling techniques

D. Vitali and P. Tombesi

Dipartimento di Matematica e Fisica and Unità INFN, Università di Camerino, via Madonna delle Carceri, I-62032 Camerino, Italy

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We study the application of decoupling techniques to the case of a damped vibrational mode of a chain of trapped ions, which can be used as a *quantum bus* in linear ion trap quantum computers. We show that vibrational heating could be efficiently suppressed using appropriate “parity kicks.” We also show that vibrational decoherence can be suppressed by this decoupling procedure, even though this is generally more difficult because the rate at which the parity kicks have to be applied increases with the effective bath temperature.

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I. INTRODUCTION

Real-world quantum systems interact with their environment to a greater or lesser extent. No matter how weak the coupling with such an environment, the evolution of an open quantum system is eventually affected by nonunitary features such as decoherence, dissipation, and heating. Decoherence, in particular, is a serious obstacle to all applications exploiting quantum coherence, such as the burgeoning field of quantum information processing.

Recently, considerable effort has been devoted to designing strategies able to counteract the undesired effects of the coupling with an external environment. Notable examples of these strategies in the field of quantum information are quantum error-correction codes [1] and error-avoiding codes [2], both based on encoding the state to be protected into carefully selected subspaces of the joint Hilbert space of the system and a number of ancillary systems. The main difference between the two encoding strategies is that error avoiding codes (also called decoherence-free subspaces) provide a passive strategy relying on the occurrence of specific symmetries in the interaction with the environment, which guarantees the existence of state space regions inaccessible to noise. Quantum error correction is instead an active strategy in which the encoding is performed in such a way that the various errors are mapped onto orthogonal subspaces so that they can be diagnosed and reversed.

A simple example of decoherence-free subspace has been recently demonstrated with two trapped ions [3], while error-correction codes for single-qubit errors has been demonstrated only in NMR quantum information processors [4]. The main limitation for the efficient implementation of these encoding strategies for combatting decoherence is the large amount of extra space resources required [5]. Correcting all the possible one-qubit errors requires at least five qubits [6] and if fault tolerant error correction is also considered, the number of ancillary qubits rapidly increases. For this reason, other alternative approaches that do not require any ancillary resources have been pursued, and that may be divided into two main categories: closed-loop (quantum feedback) [7,8], and open-loop [9–14] decoherence control strategies. In closed-loop techniques, the system to be protected is subject to appropriate measurements and the classical information obtained from this measurement is used for real-time correction of the system dynamics. This technique shares therefore

some similarities with quantum error correction, which also checks which error has taken place and eventually corrects it. However, the main limiting aspect of feedback schemes is the need of a measurement, which is always inevitably subject to the limitations due to nonunit detection efficiency. In fact, only under specific cases (see [8]), is it possible to automatically correct the error without a measurement, as in quantum error-correction codes. In open-loop control strategies instead, the system is subject to external, suitably tailored, time-dependent drivings that are independent of the system dynamics and do not require any measurement, but only a limited, *a priori*, knowledge of the system-environment dynamics. These external control Hamiltonians are chosen in order to realize an effective dynamical decoupling of the system from the environment. In this way, any undesired effect of the environment, such as dissipation, decoherence, heating, may be eliminated in principle. The essential physical idea behind these open-loop schemes comes from refocusing techniques in NMR spectroscopy, now routinely used to eliminate unwanted interactions [15]. Nonetheless, these decoupling methods have recently attracted a large interest and they have been applied in many different situations, such as the inhibition of the decay of an unstable atomic state [13], or the suppression of magnetic state decoherence [14]. The general applicability of decoupling methods has been discussed in [12], while the possibility to combine decoupling techniques together with weak-strength and slow-switching controls has been analyzed in [16], where the conditions under which noise-tolerant, universal quantum control of a system may be performed with no extra space resources, have been determined. The general algebraic structure behind decoupling strategies has been also analyzed in Ref. [17], where it is shown how decoupling can be also considered as a dynamical *symmetrization* with respect to a group. This more general algebraic framework has also provided a unifying picture for coding and decoupling noise-control strategies [18,19]. In fact, when decoupling open-loop controls are combined together with encoding into larger Hilbert spaces, fault-tolerant universal control of quantum systems becomes possible even with limited control resources. For example, it has been shown that the Heisenberg exchange interaction is sufficient to perform universal quantum computation if appropriately encoded qubits are used [20]; these encoded subspaces may actually be made deco-

herence free if appropriate decoupling controls are applied in parallel [19,21].

The main drawback of open-loop decoupling procedures is that the timing constraints are particularly stringent. In fact, the decoupling interactions have to be turned on and off at extremely short time scales, even faster than typical environmental time scales (*full-strength/fast-switching* or *quantum bang-bang* controls [12]). In fact, perfect decoupling from the environment is obtained only in the infinitely fast control limit (see Sec. II) and it is therefore important to establish in a quantitative way how effective these decoupling schemes are in a realistic situation with control pulses with finite strength and time duration. A detailed analysis of decoupling time scales has been performed only in [9] for the case of a single qubit in the presence of a purely dephasing environment, and in [11], in the case of a linearly damped vibrational degree of freedom. In this latter case, Ref. [11] proved that perfect decoupling may be achieved using extremely fast “parity kicks,” and that significant suppression of dissipation and decoherence due to the coupling with a zero-temperature bath is obtained as soon as the frequency of parity kicks becomes larger than the frequency cutoff of the environment. In the present paper, we shall reconsider the model of Ref. [11] and extend the analysis to the case of a *finite-temperature* environment. The motivation for this study is twofold. First of all, it will allow us to establish if and how thermal effects influence the decoupling strategy, that is, if temperature introduces a timescale that, together with the environmental frequency cutoff, determines the effectiveness of the parity kick decoupling strategy. Second, the damped harmonic oscillator in a finite-temperature bath studied in this paper well describes a collective vibrational mode of a chain of trapped ions, which is used as a quantum bus in linear ion trap quantum computers [22]. One of the main experimental problems for quantum information processing with linear ion traps is just the heating of these vibrational modes [23], and it is therefore extremely important to establish if the parity kick decoupling method of Ref. [11] is able to suppress heating and decoherence in this case.

The paper is organized as follows: In Sec. II, decoupling strategies in general, and the parity kick method of Ref. [11] as a particular example, are presented. In Sec. III, the dynamics of the vibrational mode in the presence of a nonzero-temperature bath and parity kicks is analyzed in detail and in Sec. IV the numerical results for both heating and decoherence rates are presented. Section V is for concluding remarks.

II. DYNAMICAL DECOUPLING VIA PARITY KICKS

The starting point of decoupling techniques is the observation that even though one does not have access to the large number of uncontrollable degrees of freedom of the environment, it is still possible to interfere with its dynamics by inducing motions into the *system*, which are at least as fast as the environment dynamics. This indirect influence of the environment is obtained through the application of suitable time-dependent perturbations acting on the system variables only. Let us now review the main points of the decoupling

technique following the lines of Ref. [12].

We consider a quantum system S coupled to an arbitrary bath B , whose overall Hamiltonian may be written as

$$H_0 = H_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes H_B + H_{SB} = \sum_{\alpha} S_{\alpha} \otimes B_{\alpha}. \quad (1)$$

A decoupling strategy consists of trying to protect the evolution of S against the effect of the interaction H_{SB} , by seeking a perturbation $H_1(t) \otimes \mathbb{1}_B$ to be added to H_0 so that the total Hamiltonian becomes $H(t) = H_0 + H_1(t) \otimes \mathbb{1}_B$. One usually restricts to situations where the control field is *cyclic*, i.e., associated to a decoupling operator $U_1(t)$ that is periodic over some cycle time T_c

$$U_1(t) \equiv T \exp \left\{ - (i/\hbar) \int_0^t du H_1(u) \right\} = U_1(t + T_c), \quad (2)$$

where T denotes time ordering. In this case, one need only focus on the *stroboscopic* evolution at times $T_N = NT_c$, and it is possible to see that in this case, the evolution is driven by an effective *average* Hamiltonian [24]

$$U_{tot}(T_N) = e^{- (i/\hbar) \bar{H} T_N}. \quad (3)$$

The calculation of the average Hamiltonian \bar{H} is performed on the basis of a standard Magnus expansion of the time-ordered exponential defining the cycle propagator [25],

$$\begin{aligned} U_{tot}(T_c) &= \exp(-i\bar{H}T_c/\hbar) \\ &= T \exp \left\{ - (i/\hbar) \int_0^{T_c} du \tilde{H}(u) \right\} \\ &= e^{-i[\bar{H}^{(0)} + \bar{H}^{(1)} + \dots]T_c/\hbar}, \end{aligned} \quad (4)$$

where

$$\tilde{H}(t) = U_1^\dagger(t) H_0 U_1(t) = \sum_{\alpha} [U_1^\dagger(t) S_{\alpha} U_1(t)] \otimes B_{\alpha}. \quad (5)$$

The various contributions in the right-hand side of Eq. (4) collect terms of equal order in $\tilde{H}(t)$. In particular,

$$\bar{H}^{(0)} = \frac{1}{T_c} \int_0^{T_c} du \tilde{H}(u), \quad (6)$$

$$\bar{H}^{(1)} = - \frac{i}{2T_c} \int_0^{T_c} dv \int_0^v du [\tilde{H}(v), \tilde{H}(u)]. \quad (7)$$

One says that k th-order decoupling is achieved if the control field $H_1(t)$ can be devised so that contributions mixing S and B degrees of freedom are no longer present in $\bar{H}^{(0)}$ and the first nonvanishing correction arises from $\bar{H}^{(k)}$, $k \geq 1$. One then considers the infinitely fast control limit, which, for a finite evolution time T , requires considering $T_c = T/N$ in the limit $T_c \rightarrow 0$ and $N \rightarrow \infty$. In this limit, first-order decoupling is sufficient, contributions higher than zeroth-order are negligible in Eq. (4), and one can focus on the problem of design-

ing the effective Hamiltonian $\bar{H}^{(0)}$ of Eq. (6) in such a way that there is no residual system-environment coupling.

The more general way to engineer the average Hamiltonian $\bar{H}^{(0)}$ is through *symmetrization* with respect to a finite group \mathcal{G} [12,17]. In fact, if we consider a finite group of unitary operators $\mathcal{G}=\{g_j\}, j=1, \dots, |\mathcal{G}|$, symmetrization is the map (acting on system operator only)

$$\mathcal{S} \rightarrow \Pi_{\mathcal{C}}(\mathcal{S}) = \frac{1}{|\mathcal{G}|} \sum_{g_j \in \mathcal{G}} g_j^\dagger \mathcal{S} g_j, \quad (8)$$

which is also the projection on the so-called *centralizer* of \mathcal{G} , composed of operators commuting with every element g_j of the group \mathcal{G} [12,17]. The map (8) may be dynamically implemented through a simple piecewise constant decoupling operator

$$U_1(t) \equiv g_j, \quad j\Delta t \leq t < (j+1)\Delta t, \quad (9)$$

corresponding to a partition of the cycle time T_c into $|\mathcal{G}|$ intervals of equal length $\Delta t \equiv T_c/|\mathcal{G}|$. Then, by Eq. (5),

$$\bar{H}^{(0)} = \Pi_{\mathcal{C}}(H_0) = \sum_{\alpha} \Pi_{\mathcal{C}}(\mathcal{S}_{\alpha}) \otimes \mathcal{B}_{\alpha}, \quad (10)$$

showing that the average Hamiltonian $\bar{H}^{(0)}$, generating time evolution in the infinitely fast control limit, has been symmetrized, i.e., has become invariant with respect to the group \mathcal{G} . Perfect decoupling from the environment is achieved when

$$\Pi_{\mathcal{C}}(H_{SB}) = 0, \quad (11)$$

and in this case, the effective open-system evolution for the reduced density operator of the system ρ_S over time T is governed by

$$\lim_{N \rightarrow \infty} \rho_S(T = NT_c) = e^{-i\bar{H}_S T/\hbar} \rho_S(0) e^{+i\bar{H}_S T/\hbar}, \quad (12)$$

where $\bar{H}_S = \Pi_{\mathcal{C}}(H_S)$. This means that the system is no more interacting with the environment and the residual time evolution is driven by a projected system Hamiltonian, invariant with respect to \mathcal{G} .

A number of examples of decoupling groups \mathcal{G} has now appeared in the literature, especially for the case of many-qubits dissipative registers with various kinds of interaction with the environment [9,10,12,17]. Another important example for applications in quantum computing is the case of a linearly dissipative vibrational degree of freedom, which can be used as a *quantum bus* in linear ion trap quantum computers [22], and which has been shown in Ref. [11] to be decoupled by the group \mathcal{Z}_2 , composed by the identity and the parity operator P . In fact, it is straightforward to check that the decoupling condition (11) is equivalent to the condition of Eq. (7) of Ref. [11].

As it can be expected, implementing the above general decoupling strategy is by no means trivial. First of all, for a given H_{SB} , the identification of a minimal group \mathcal{G} able to

produce decoupling is nontrivial. Second, the decoupling prescription (9) requires the capability of instantaneously changing the evolution operator from g_j to g_{j+1} over successive subintervals. This means assuming the capability of implementing arbitrarily strong and extremely fast control operations. Such impulsive full-power control configurations correspond to so-called *quantum bang-bang controls* as introduced in [9]. As it has been already shown in [9,11], the most stringent condition is not on the strength but rather on the extremely high speed of the control operations: one has to be faster than the typical *environmental* time scale, which is usually fixed by the frequency cutoff of the bath spectrum, ω_c . However, the identification of the frequency cutoff ω_c as the only relevant parameter determining the threshold for the decoupling cycle frequency $1/T_c$ above which the decoupling procedure becomes effective, has been done in Refs. [9,11] only on the basis of two specific examples. In Ref. [11], the case of a harmonic oscillator coupled to a zero-temperature bath, able to induce only system dissipation (and the associated decoherence) has been considered. The case of nonzero temperature has been discussed in Ref. [9] but only in the particular case of a single qubit subject to a purely dephasing, energy-conserving, environment. It is therefore important to establish the effectiveness of decoupling techniques in the general case of a *nonzero-temperature, dissipative bath*. From now on we shall specialize to the case of the linearly damped harmonic oscillator of Ref. [11], which is of relevance for linear ion trap quantum computation. In fact, we shall demonstrate that decoupling techniques may be successfully used to efficiently suppress heating of the vibrational center-of-mass motion of the ion chain.

III. PARITY KICKS FOR A DAMPED HARMONIC OSCILLATOR

We choose a harmonic oscillator as system of interest

$$H_S = \hbar \omega_0 a^\dagger a, \quad (13)$$

describing a collective vibrational mode of a linear chain of trapped ions with frequency ω_0 . It has been already experimentally verified [23,26] that the nonunitary features of the vibrational dynamics (heating and decoherence) are well described by modeling the environment as a collection of independent bosonic modes [27]

$$H_B = \sum_k \hbar \omega_k b_k^\dagger b_k, \quad (14)$$

interacting with the vibrational mode via the following bilinear interaction Hamiltonian in which the ‘‘counter-rotating’’ terms are dropped

$$H_{SB} = \sum_k \hbar \gamma_k (a b_k^\dagger + a^\dagger b_k). \quad (15)$$

The symmetrization with respect to the group \mathcal{Z}_2 is performed by periodically pulsing the oscillation frequency, that is, by changing the potential so that ω_0 is changed to $\omega_0 + \delta\omega$ for a time interval τ and with a time period T_c (see

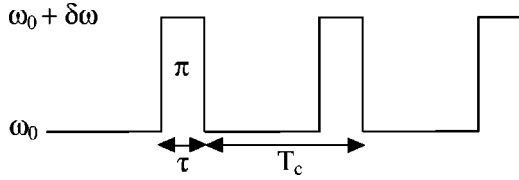


FIG. 1. Sketch of the implementation of the parity kick decoupling procedure by pulsing the oscillation frequency.

Fig. 1). The pulse realizes the “parity kick” of Ref. [11] when the condition $\delta\omega\tau = \pi$ is satisfied. In this way, the cyclic time-dependent control Hamiltonian is given by

$$H_1(t) = \hbar \delta\omega a^\dagger a \sum_{n=1}^{\infty} \theta(t - nT_c + \tau) \theta(nT_c - t), \quad (16)$$

so that the cyclic decoupling operator $U_1(t)$ is equal to $U_1(t) = \mathbb{1}_S$ for $0 < t < T_c - \tau$ and $U_1(t) = \exp\{i\pi a^\dagger a\} = P$, for $T_c - \tau < t < T_c$.

To determine the effects of a nonzero-temperature bath on the efficiency of the decoupling scheme, we shall consider an initially factorized state in which the vibrational mode is prepared in a given pure state $|\psi(0)\rangle$ and the environment is at the thermal equilibrium state at temperature T , ρ_B^T . To be more specific, we want to establish if decoupling via parity kicks is able to suppress efficiently both heating (which is important for quantum information processing) and quantum decoherence of the vibrational mode. To study heating, we shall assume that the collective vibrational mode has been initially cooled to its ground state [28], that is, $|\psi(0)\rangle = |0\rangle$. The study of decoherence instead will be performed, as in Ref. [11], by considering an initial linear superposition of two coherent states with opposite phases, that is, the well-known Schrödinger cat state

$$|\psi(0)\rangle = |\psi_\varphi\rangle = N_\varphi (|\alpha(0)\rangle + e^{i\varphi} |-\alpha(0)\rangle), \quad (17)$$

where $N_\varphi = (2 + 2e^{-2|\alpha(0)|^2} \cos \varphi)^{-1/2}$. The dynamics of the system in the presence of parity kicks will be exactly solved for both initial conditions of the vibrational mode, by exploiting the fact that a tensor product of coherent states retains its form at all times when the evolution is generated by the Hamiltonian of Eqs. (13), (14), and (15) [29], that is

$$|\alpha(0)\rangle \otimes \prod_k |\beta_k(0)\rangle \rightarrow |\alpha(t)\rangle \otimes \prod_k |\beta_k(t)\rangle, \quad (18)$$

where the time-dependent coherent state amplitudes are a linear combination of the initial amplitudes

$$\alpha(t) = L_{00}(t)\alpha(0) + \sum_k L_{0k}(t)\beta_k(0), \quad (19)$$

$$\beta_k(t) = L_{k0}(t)\alpha(0) + \sum_{k'} L_{kk'}(t)\beta_{k'}(0). \quad (20)$$

Equation (18) is useful also in the nonzero-temperature case. In fact, using the expression of the thermal state ρ_B^T in the Glauber-Sudarshan P -representation [30]

$$\rho_B^T = \prod_k \int \frac{d^2\beta_k}{\pi N_k} \exp\left\{-\frac{|\beta_k|^2}{N_k}\right\} |\beta_k\rangle\langle\beta_k|, \quad (21)$$

where $d^2\beta_k = d\text{Re}\beta_k d\text{Im}\beta_k$ and $N_k = (\exp\{\hbar\omega_k/k_B T\} - 1)^{-1}$ is the mean thermal excitation number of the k th bath mode, one has always to evaluate the time evolution of terms such as $|\alpha\rangle\langle\alpha'| | \beta_k\rangle\langle\beta_k|$, and then perform the average over the thermal Gaussian weight $\exp\{-|\beta_k|^2/N_k\}/\pi N_k$. Therefore, the essential dynamics is contained in the expression of the unitary matrix $L_{ij}(t)$ of Eqs. (19) and (20), which has the same structure both with and without parity kicks, because the two situations differ only by the value of the oscillation frequency. The matrix element $L_{00}(t)$ is given in terms of its Laplace transform, and, in the interaction picture with respect to H_S of Eq. (13), one has

$$L_{00}(t, \delta\omega) = \mathcal{L}^{-1}\left[\frac{1}{z + K(z, \delta\omega)}\right], \quad (22)$$

where

$$K(z, \delta\omega) = \sum_k \frac{\gamma_k^2}{z + i(\omega_k - \omega_0) - i\delta\omega}. \quad (23)$$

This expression refers to the evolution during the parity kicks, that is, for $nT_c - \tau < t < nT_c$, $n \geq 1$. The evolution in the absence of kicks is simply obtained putting $\delta\omega = 0$ in Eqs. (22) and (23). All the other matrix elements may be expressed in terms of the matrix element $L_{00}(t, \delta\omega)$ in the following way:

$$L_{0k}(t, \delta\omega) = L_{k0}(t, \delta\omega) = -i\gamma_k \int_0^t ds e^{-i(\omega_k - \omega_0 - \delta\omega)s} \times L_{00}(t-s, \delta\omega), \quad (24)$$

$$L_{kk'}(t, \delta\omega) = \delta_{kk'} e^{-i(\omega_k - \omega_0)t} - \gamma_k \gamma_{k'} \int_0^t ds e^{-i(\omega_k - \omega_0 - \delta\omega)(t-s)} \times \int_0^s ds' e^{-i(\omega_{k'} - \omega_0 - \delta\omega)(s-s')} L_{00}(s', \delta\omega). \quad (25)$$

It is evident that a decoupling cycle of duration T_c will be described by the product of unitary matrices $L(\tau, \delta\omega) \cdot L(T_c - \tau, 0)$ applied to the vector formed by the coherent amplitudes $[\alpha(t), \dots, \beta_k(t), \dots]$. As a consequence, the stroboscopic dynamics of the whole system during the decoupling procedure, in the case of an initial tensor product of coherent states, may be described exactly as

$$\begin{pmatrix} \alpha(NT_c) \\ \vdots \\ \beta_k(NT_c) \\ \vdots \end{pmatrix} = [L(\tau, \delta\omega)L(T_c - \tau, 0)]^N \begin{pmatrix} \alpha(0) \\ \vdots \\ \beta_k(0) \\ \vdots \end{pmatrix}. \quad (26)$$

IV. NUMERICAL RESULTS

In the standard description of dissipation, one always considers a continuum distribution of oscillator frequencies in order to obtain an irreversible transfer of energy from the system of interest into the reservoir. Moreover, most often, also the Markovian assumption is made, which means assuming an infinitely fast bath with an infinite frequency cutoff ω_c . This case of a standard vacuum bath in the Markovian limit is characterized by an infinite, continuous, and flat distribution of couplings [30],

$$\gamma(\omega)^2 = \frac{\gamma}{2\pi} \quad \forall \omega, \quad (27)$$

where γ is the energy damping rate. As shown in Refs. [9,11], decoupling strategies become efficient when the external controls are characterized by time scales faster than those of the environment. It is therefore evident that, in the presence of parity kicks, we cannot make any Markovian approximation. We have to solve numerically the problem, by simulating the continuous distribution of bath oscillators with a large but finite number of oscillators with closely spaced frequencies. As in Ref. [11], we have considered a bath of 201 oscillators, with equally spaced frequencies, symmetrically distributed around the resonance frequency ω_0 , i.e.,

$$\omega_k = \omega_0 + k\Delta, \quad (28)$$

$$\Delta = \frac{\omega_0}{100}, \quad (29)$$

$$k_{max} = \frac{\omega_0}{\Delta} = 100 \Rightarrow \omega_k^{max} = 2\omega_0, \quad (30)$$

$$k_{min} = -k_{max} = -100 \Rightarrow \omega_k^{min} = 0, \quad (31)$$

and we have considered a constant distribution of couplings similar to that associated with the Markovian limit

$$\gamma_k^2 = \frac{\gamma\Delta}{2\pi} \quad \forall k. \quad (32)$$

Approximating a continuous Markovian bath with a finite number of bath oscillators has two main effects. First of all, the discrete frequency distribution with a fixed spacing Δ makes all the dynamical quantities periodic with period $T_{rev} = 2\pi/\Delta$ [31]. Therefore, our numerical solution will correctly describe the interaction with the environment provided that we consider not too large times, say $t \leq \pi/\Delta$. Second, the introduction of a finite cutoff ($\omega_c = 2\omega_0$ in our case) implies a modification of the coupling spectrum $\gamma(\omega)$ at

very high frequency with respect to the infinitely flat distribution of the Markovian treatment [see Eq. (27)]. This fact manifests itself in a slight modification of the dynamics at very short times ($t \approx \omega_c^{-1}$) [31]. We have verified both short- and long-time deviations from the standard Markovian bath dynamics in our numerical calculations. However, we have checked that our model environment with a finite number of oscillators faithfully reproduces the standard Markovian bath dynamics within the time interval of interest, $0.1/\gamma < t < 3/\gamma$, say.

A. Effect of parity kicks on heating

To check if decoupling via parity kicks is able to suppress the heating of the vibrational mode, we consider the following initial state for the whole system:

$$|0\rangle\langle 0| \otimes \prod_k \int \frac{d^2\beta_k(0)}{\pi N_k} \exp\left\{-\frac{|\beta_k(0)|^2}{N_k}\right\} |\beta_k(0)\rangle\langle\beta_k(0)|, \quad (33)$$

where $|0\rangle$ is the ground state of the collective vibrational mode [28].

Using the property (18), and tracing over the environment, the evolved state of the vibrational mode after N decoupling cycles may be written as

$$\begin{aligned} \rho_S(NT_c) &= \int \prod_k \frac{d^2\beta_k(0)}{\pi N_k} \exp\left\{-\frac{|\beta_k(0)|^2}{N_k}\right\} \\ &\quad \times |\alpha(NT_c)\rangle\langle\alpha(NT_c)|, \end{aligned} \quad (34)$$

where the coherent state amplitude $\alpha(NT_c)$ is the following linear combination of the complex variables $\beta_k(0)$:

$$\alpha(NT_c) = \sum_k C_{0k}(NT_c) \beta_k(0), \quad (35)$$

where we have defined $C_{0k}(NT_c) = \{[L(\tau, \delta\omega)L(T_c - \tau, 0)]^N\}_{0k}$ [see Eqs. (19) and (26)]. The Gaussian average of Eq. (34) may be performed by first considering the normally ordered characteristic function $\chi(\lambda, NT_c)$ [30] of the state, and then performing the integration. One gets

$$\chi(\lambda, NT_c) = \exp\left\{-|\lambda|^2 \sum_k N_k |C_{0k}(NT_c)|^2\right\}, \quad (36)$$

showing that, in the presence of parity kicks, the vibrational state is a *thermal state*, with mean vibrational number $\nu(NT_c)$,

$$\nu(NT_c) = \sum_k N_k |C_{0k}(NT_c)|^2. \quad (37)$$

The stroboscopic time evolution of this mean vibrational number is plotted in Fig. 2 both in the presence (full circles) and in the absence (crosses) of parity kicks. The capability of the parity kick decoupling strategy to avoid vibrational heating is clearly visible in this figure. In Fig. 2, and in the rest of the paper, we consider a vibrational mode with frequency

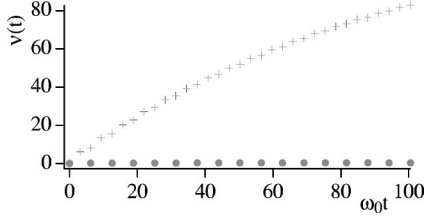


FIG. 2. Time evolution of the mean vibrational number of Eq. (37) with (full circles) and without (crosses) parity kicks. The capability of parity kicks to suppress heating is clearly visible. Parameters are: $\omega_0=10$ MHz, $\gamma=0.1$ MHz, $\omega_c=20$ MHz, effective reservoir temperature $T=10$ mK [corresponding to $N(\omega_0)\approx 130$], $T_c=157$ ns, parity kick duration $\tau=T_c/7\approx 22.4$ ns, implying $\delta\omega=140$ MHz.

$\omega_0=10$ MHz, damping rate $\gamma=0.1$ MHz, and environmental frequency cutoff $\omega_c=20$ MHz. The curve referring to the situation without parity kicks in Fig. 2 well reproduces the standard Markovian result [30] $\nu(t)=N(\omega_0)(1-e^{-\gamma t})$, where $N(\omega_0)=(\exp\{\hbar\omega_0/k_B T\}-1)^{-1}$ is the mean vibrational number of the oscillator at thermal equilibrium in the usual Born-Markov approximation. Figure 2 refers to an effective reservoir temperature $T=10$ mK [corresponding to $N(\omega_0)\approx 130$], and to the following decoupling cycle parameters: $T_c=157$ ns, parity kick duration $\tau=T_c/7\approx 22.4$ ns, implying $\delta\omega=140$ MHz.

The influence of the environmental temperature T on heating suppression is analyzed in Fig. 3, where the mean vibrational number after one relaxation time $t=1/\gamma$, $\nu(1/\gamma)$, is plotted as a function of the rescaled decoupling cycle time $\omega_c T_c/2\pi$ for three different bath temperatures, $T=10$ mK (a), $T=100$ mK (b), and $T=1$ K (c). For each value of T_c , we have always chosen the kick duration $\tau=T_c/7$, as in Fig. 2, and the frequency shift $\delta\omega$ is always correspondingly adjusted so that $\delta\omega=\pi/\tau$. We can see that a well-visible threshold for the decoupling cycle time T_c exists and that as soon as the parity kicks are sufficiently fast, $T_c < 2\pi/\omega_c$, heating suppression becomes significant. This is a phase transitionlike behavior analogous to that found for decoherence suppression in the zero-temperature case [11]. What is more important is that *bath temperature has no effect* on the effectiveness of the decoupling scheme: the results are essentially identical for the three different temperatures studied, and this means that, at least for what concerns heating, the only relevant environmental timescale is given by the frequency cutoff ω_c . This result is particularly important for the application of the parity kick strategy to suppress heating in linear ion trap quantum computers, where heating is due to some technical imperfections originating from fluctuating patch fields [23]. Determining the effective temperature T of the thermal bath modeling these fluctuating fields is generally very difficult, but our results shows that this is not relevant, and that parity kick decoupling is very promising for eliminating vibrational heating.

B. Effect of parity kicks on decoherence

Let us now consider the possibility of suppressing decoherence. We assume an initially prepared Schrödinger cat

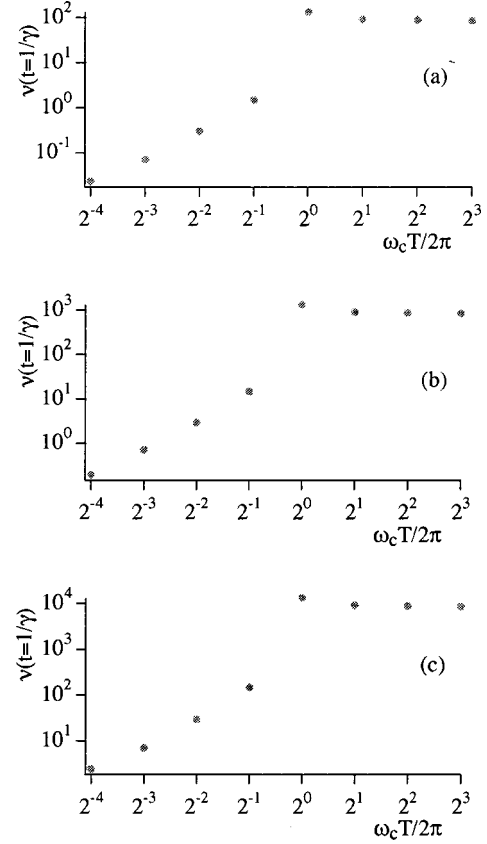


FIG. 3. Mean vibrational number after one relaxation-time $t=1/\gamma$, $\nu(1/\gamma)$, as a function of the rescaled decoupling cycle time $\omega_c T_c/2\pi$, for three different bath temperatures: $T=10$ mK [corresponding to $N(\omega_0)\approx 130$] (a), $T=100$ mK [corresponding to $N(\omega_0)\approx 1302$] (b), and $T=1$ K [corresponding to $N(\omega_0)\approx 13144$] (c). For each value of T_c , we have always chosen $\tau=T_c/7$, and, correspondingly, $\delta\omega=\pi/\tau$. The other parameters are as in Fig. 2

state of the vibrational mode, and therefore, the following initial state for the whole system:

$$|\psi_\varphi\rangle\langle\psi_\varphi|\otimes\prod_k\int\frac{d^2\beta_k(0)}{\pi N_k}\exp\left\{-\frac{|\beta_k(0)|^2}{N_k}\right\}\times|\beta_k(0)\rangle\langle\beta_k(0)|, \quad (38)$$

where $|\psi_\varphi\rangle$ is given by Eq. (17).

Using the property (18), and tracing over the environment, the evolved state of the vibrational mode after N decoupling cycles may be written as

$$\rho_S(NT_c)=N_\varphi^2\int\prod_k\frac{d^2\beta_k(0)}{\pi N_k}\exp\left\{-\frac{|\beta_k(0)|^2}{N_k}\right\}\{\alpha_+(NT_c)\}\times\langle\alpha_+(NT_c)|+\alpha_-(NT_c)\rangle\langle\alpha_-(NT_c)|+e^{i\varphi}\langle\beta_k^+(NT_c)|\beta_k^-(NT_c)\rangle\langle\alpha_-(NT_c)\rangle\times\langle\alpha_+(NT_c)|+e^{-i\varphi}\langle\beta_k^-(NT_c)|\beta_k^+(NT_c)\rangle\times|\alpha_+(NT_c)\rangle\langle\alpha_-(NT_c)|\}. \quad (39)$$

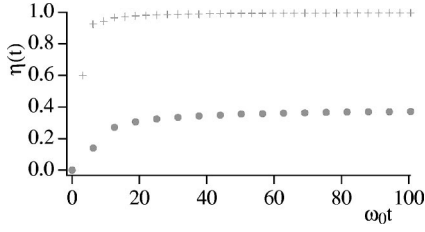


FIG. 4. Time evolution of the fringe visibility function of Eq. (43) with (full circles) and without (crosses) parity kicks. The capability of parity kicks to suppress decoherence is clearly visible. Parameters are the same as in Fig. 2, except that the decoupling cycle parameters now are: $T_c = 78.5$ ns, parity kick duration $\tau = T_c/7 \approx 11.2$ ns, implying $\delta\omega = 280$ MHz.

The coherent-state amplitudes $\alpha_{\pm}(NT_c)$ and $\beta_k^{\pm}(NT_c)$ are now given by the following linear combinations of the initial amplitudes [see Eqs. (19)–(20)]:

$$\alpha_{\pm}(NT_c) = \pm \alpha_0 C_{00}(NT_c) + \sum_k C_{0k}(NT_c) \beta_k(0), \quad (40)$$

$$\beta_k^{\pm}(NT_c) = \pm \alpha_0 C_{k0}(NT_c) + \sum_{k'} C_{kk'}(NT_c) \beta_{k'}(0). \quad (41)$$

The Gaussian average of Eq. (39) may be performed, as in the preceding subsection, by first considering the normally ordered characteristic function of the state and then performing the integration. The integration is straightforward but lengthy, and the resulting reduced vibrational state may be better expressed in terms of its Wigner function $W_S(\alpha, NT_c)$,

$$\begin{aligned} W_S(\alpha, NT_c) = & \frac{2N_{\varphi}^2}{\pi[1+2\nu(NT_c)]} \\ & \times \left\{ \exp\left[-\frac{2|\alpha - \alpha_0 C_{00}(NT_c)|^2}{1+2\nu(NT_c)}\right] \right. \\ & + \exp\left[-\frac{2|\alpha + \alpha_0 C_{00}(NT_c)|^2}{1+2\nu(NT_c)}\right] \\ & + 2 \exp\{-2|\alpha_0|^2 \eta(NT_c)\} \\ & \times \exp\left[-\frac{2|\alpha|^2}{1+2\nu(NT_c)}\right] \\ & \left. \times \cos\left[\varphi + \frac{4 \operatorname{Im}[\alpha \alpha_0 C_{00}(NT_c)]}{1+2\nu(NT_c)}\right] \right\}, \quad (42) \end{aligned}$$

where $\nu(NT_c)$ is again the mean vibrational number of the cat state of Eq. (37), the matrix element $C_{00}(NT_c)$ describes the amplitude decay, and $\eta(NT_c)$ is the *fringe visibility function* [32], determining the relative strength of the quantum interference term in the cat state, and which may be expressed as

$$\eta(NT_c) = 1 - \frac{|C_{00}(NT_c)|^2}{1+2\nu(NT_c)}. \quad (43)$$

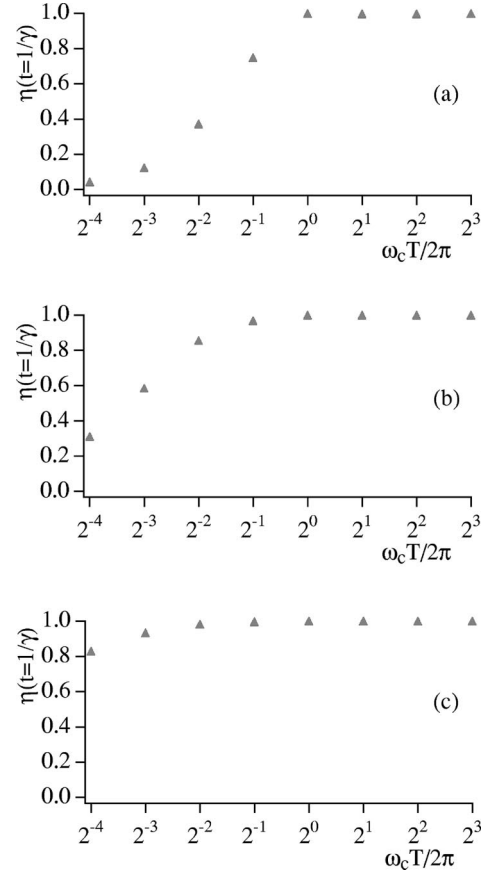


FIG. 5. Fringe visibility function after one relaxation time $t = 1/\gamma$, $\eta(1/\gamma)$, as a function of the rescaled decoupling cycle time $\omega_c T_c / 2\pi$, for three different bath temperatures: $T = 10$ mK [corresponding to $N(\omega_0) \approx 130$] (a), $T = 100$ mK [corresponding to $N(\omega_0) \approx 1302$] (b), and $T = 1$ K [corresponding to $N(\omega_0) \approx 13144$] (c). For each value of T_c , we have always chosen $\tau = T_c/7$, and, correspondingly, $\delta\omega = \pi/\tau$. The other parameters are as in Fig. 2. The quality of decoherence suppression degrades with increasing temperature.

This fringe visibility is always contained in the interval $[0, 1]$ and provides a good quantitative description of dynamical decoherence processes. For this reason, we shall study the stroboscopic evolution of this quantity, as in Ref. [11], to quantify the eventual decoherence suppression caused by the decoupling.

The time evolution of the fringe visibility is plotted in Fig. 4, both with (full circles) and without (crosses) parity kicks. The possibility to suppress decoherence using parity kicks is clearly demonstrated in this figure. Parameters are the same as in Fig. 2, except that the decoupling cycle parameters now are: $T_c = 78.5$ ns, parity kick duration $\tau = T_c/7 \approx 11.2$ ns, implying $\delta\omega = 280$ MHz. The curve referring to the situation without parity kicks in Fig. 4 (crosses) well reproduces the Markovian result [32]

$$\eta(t) = 1 - \frac{e^{-\gamma t}}{1 + 2N(\omega_0)(1 - e^{-\gamma t})}. \quad (44)$$

The influence of temperature on decoherence suppression is studied in Fig. 5, where the fringe visibility function after

one relaxation-time $t = 1/\gamma$, $\eta(1/\gamma)$, is plotted as a function of the rescaled decoupling cycle time $\omega_c T_c/2\pi$ for three different bath temperatures, $T=10$ mK (a), $T=100$ mK (b), and $T=1$ K (c). For each value of T_c , we have always chosen the kick duration $\tau = T_c/7$, as in Figs. 2 and 3, and the frequency shift $\delta\omega$ is always correspondingly adjusted so that $\delta\omega = \pi/\tau$.

We can see from Fig. 5 that the situation is rather different from that with heating suppression. In fact, decoherence suppression by parity kicks *strongly depends on the bath temperature*, and it is significant only in the lower-temperature case [Fig. 5(a)], which is the only case in which a threshold for the decoupling cycle time T_c at about $T_c \approx 2\pi/\omega_c$, as in the zero-temperature case [11], is visible. In the other cases, decoherence suppression worsens for increasing bath temperature. This result shows that eliminating decoherence via decoupling techniques is generally more difficult than eliminating heating. This may be easily explained in terms of the so-called thermal acceleration of decoherence [32,33], that is, the fact that in the case of a thermal bath at temperature T , the decoherence process is accelerated roughly by a factor $[1 + 2N(\omega_0)]$ with respect to the zero-temperature case. This thermal effect on the decoherence rate may be also easily checked from the Markovian limit expression of Eq. (44). In fact, the fringe visibility function $\eta(t)$ reaches its asymptotic value in a time of the order of $t_{dec} \approx \{\gamma[1 + 2N(\omega_0)]\}^{-1}$, and it is evident that decoherence suppression with parity kicks is possible only if the cycle time T_c is smaller than this decoherence time t_{dec} , and not only smaller than $2\pi/\omega_c$, as in the zero-temperature case. This means that, in a nonzero-temperature bath, one has a new, *temperature-dependent*, threshold for decoherence suppression, given by

$$T_c < \min\{2\pi/\omega_c, \{\gamma[1 + 2N(\omega_0)]\}^{-1}\}, \quad (45)$$

and this generalized expression easily explains the results of Fig. 5.

V. CONCLUSIONS

We have studied the application of open-loop decoupling schemes in an experimentally realistic scenario. In fact, decoupling strategies have been proven to provide perfect iso-

lation of a system from its environment in the infinitely fast control limit, i.e., in the case of very intense and very fast control pulses [12]. The efficiency of decoupling strategies in concrete situations involving finite strength and finite duration control pulses has been analyzed only in the specific cases of a single qubit in a nondissipative environment in [9], and for a damped harmonic oscillator in a zero-temperature bath in [11]. Here, we have extended these studies to the case of a dissipative and nonzero-temperature reservoir. We have specialized to the case of a collective vibrational mode of a linear ion chain, which is used as a quantum bus in linear ion trap quantum computers [22]. We have shown that the parity kick decoupling strategy introduced in [11] may be successfully applied to suppress vibrational heating, which is one important limitation for quantum information processing in linear ion traps [23]. In fact, heating is suppressed as soon as the decoupling cycle time T_c becomes smaller than $2\pi/\omega_c$, where ω_c is the bath frequency cutoff, and more importantly, the efficiency of this suppression is not affected by the temperature of the bath. The parity kick method may be applied using present technologies and its experimental implementation in the case of trapped ions would be the first example of the application of decoupling techniques outside the field of NMR, where the so-called “refocusing” techniques [15] are easier to use because the involved magnetic environment is usually very slow (see, however, Ref. [34] for a proof-of-principle demonstration of quantum bang-bang control in a photon polarization qubit).

We have also shown that, different from heating, the suppression of vibrational decoherence is more difficult, because in a nonzero-temperature bath, the threshold for the decoupling cycle frequency is determined not only by the bath frequency cutoff, but also by the decoherence rate, which increases for increasing temperatures. The parity kick cycle frequency has to be larger than *both* rates and this makes suppression of vibrational decoherence more difficult for higher temperatures.

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