Quantum cloning machines for equatorial qubits

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Quantum cloning machines for equatorial qubits are studied. For the case of a one to two phase-covariant quantum cloning machine, we present the networks consisting of quantum gates to realize the quantum cloning transformations. The copied equatorial qubits are shown to be separable by using Peres-Horodecki criterion. The optimal one to M phase-covariant quantum cloning transformations are given.

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I. INTRODUCTION

Quantum computing and quantum information have been attracting a great deal of interest. They differ in many aspects from the classical theories. One of the most fundamental differences between classical and quantum information is the no-cloning theorem [1]. It tells us that arbitrary quantum information cannot be copied exactly. The no-cloning theorem for pure states is also extended to the case that a general mixed state cannot be broadcast [2]. However, the no-cloning theorem does not forbid imperfect cloning, and several kinds of quantum cloning machines (QCM) are proposed, the optimal fidelity and transformations of QCM's are found in [3–9].

In the proof of the no-cloning theorem, Wootters and Zurek introduced a QCM that has the property that the quality of the copy it makes depends on the input-states [1]. To diminish or cancel this disadvantage, Bužek and Hillery proposed a universal quantum cloning machine (UQCM) for an arbitrary pure state where the copying process is input-state independent. They use Hilbert-Schmidt norm to quantify distances between the input density operator and the output density operators. Bruß *et al.* [4] discussed the performance of a UQCM by analyzing the role of the symmetry and isotropy conditions imposed on the system and found the optimal UQCM and the optimal state-dependent quantum cloning. Optimal fidelity and optimal quantum cloning transformations of general N to M ($M>N$) case are presented in Refs. [6–9]. The relation between quantum cloning and superluminal signaling is proposed and discussed in Refs. [10,11]. It was also shown that the UQCM can be realized by a network consisting of quantum gates [12].

In the case of UQCM, the input states are arbitrary pure states. In this paper, we study the QCM for a restricted set of pure input states. The Bloch vector is restricted to the intersection of x-z (x-y and y-z) plane with the Bloch sphere. These kind of qubits are the so-called equatorial qubits [13] and the corresponding QCM is called phase-covariant quantum cloning. The one to two phase-covariant quantum cloning was first studied by Bruß *et al.*, [13] who studied the optimal quantum cloning for $x-z$ equatorial qubits by taking $BB84$ states as input. The fidelity of quantum cloning for the equatorial qubits is higher than the original Bužek and Hillery UQCM [3]. This is expected, as the more information about the input is given, the better one can clone each of its states.

In this paper, using the approach presented in Ref. [12], we show that the one to two optimal phase-covariant quantum cloning machines can be realized by networks consisting of quantum rotation gates and controlled NOT gates. The copied equatorial qubits are shown to be separable by using Peres-Horodecki criterion [14,15]. We then present the one to M phase-covariant quantum cloning transformations and prove that the fidelity is optimal. The general N to M ($M>N$) optimal phase-covariant quantum cloning machines are finally proposed.

The paper is organized as follows: In Sec. II, we introduce the cloning transformations for equators in x-z and x-y planes. In Sec. III, phase-covariant quantum cloning can be realized by networks consisting of quantum gates. In Sec. IV, the copied qubits are shown to be separable and quantum triplicators are studied. In Sec. V, optimal one to M phase-covariant quantum cloning machines are presented and proved. In Sec. VI, N to M ($M>N$) phase-covariant QCM is proposed. Section VII includes a brief summary.

II. ONE TO TWO PHASE-COVARIANT QUANTUM CLONING

Instead of arbitrary input states, we consider the input state that we intend to clone to be a restricted set of states. It is a pure superposition state

$$|\Psi\rangle=\alpha|0\rangle+\beta|1\rangle \tag{1}$$

with $\alpha^2+\beta^2=1$. Here, we use an assumption that α and β are real in contrast to complex when we consider the case of UQCM. That means the y component of the Bloch vector of the input qubit is zero. Because there is just one unknown parameter in the input state under consideration, we expect that we can achieve a better quality in quantum cloning if we can find an appropriate phase-covariant QCM.

The case of one to two phase-covariant quantum cloning transformation has already been found by Bruß *et al.* in [13]. They proposed the following cloning transformation for the input (1),

$$|0\rangle_{a_1}|Q\rangle_{a_2a_3}\to\left[\left(\frac{1}{2}+\sqrt{\frac{1}{8}}\right)|00\rangle_{a_1a_2}+\left(\frac{1}{2}-\sqrt{\frac{1}{8}}\right)|11\rangle_{a_1a_2}\right]|\uparrow\rangle_{a_3}+\frac{1}{2}|+\rangle_{a_1a_2}|\downarrow\rangle_{a_3}, \tag{2}$$

$$|1\rangle_{a_1}|Q\rangle_{a_2a_3}\to\left[\left(\frac{1}{2}+\sqrt{\frac{1}{8}}\right)|11\rangle_{a_1a_2}+\left(\frac{1}{2}-\sqrt{\frac{1}{8}}\right)|00\rangle_{a_1a_2}\right]|\downarrow\rangle_{a_3}+\frac{1}{2}|+\rangle_{a_1a_2}|\uparrow\rangle_{a_3}, \tag{3}$$

where the following notations are introduced

$$|+\rangle=\frac{1}{\sqrt{2}}(|10\rangle+|01\rangle),\quad |-\rangle=\frac{1}{\sqrt{2}}(|10\rangle-|01\rangle). \tag{4}$$

The fidelity of the phase-covariant cloning transformation is $F=1/2+\sqrt{1/8}$, which is larger than $F=5/6$, the fidelity of one to two UQCM [3]. Also, this fidelity was proved to be optimal for phase-covariant cloning machine [13]. Actually, because we assume α and β are real, only a single unknown parameter is copied instead of two unknown parameters for the case of a general pure state. Thus, a higher fidelity of quantum cloning can be achieved. The case of spin flip has a similar phenomenon [16,12,17]. Here, the fidelity is defined in the standard form as $F=\langle\Psi|\rho|\Psi\rangle$, ρ is the output reduced density operator at a single qubit.

For convenience, we present the following cloning transformation for pure input state (1):

$$|0\rangle_{a_1}|Q\rangle_{a_2a_3}\to(|00\rangle_{a_1a_2}+\lambda|11\rangle_{a_1a_2})q|\uparrow\rangle_{a_3}+(|10\rangle_{a_1a_2}+|01\rangle_{a_1a_2})y|\downarrow\rangle_{a_3},$$

$$|1\rangle_{a_1}|Q\rangle_{a_2a_3}\to(|11\rangle_{a_1a_2}+\lambda|00\rangle_{a_1a_2})q|\downarrow\rangle_{a_3}+(|10\rangle_{a_1a_2}+|01\rangle_{a_1a_2})y|\downarrow\rangle_{a_3}, \tag{5}$$

where we assume λ is real and $\lambda\neq\pm1$, we also use notations

$$q\equiv\sqrt{\frac{2}{3-2\lambda+3\lambda^2}},\quad y\equiv\frac{1-\lambda}{\sqrt{6-4\lambda+6\lambda^2}}. \tag{6}$$

The qubit in a_1 is the input state, the output copies appear in a_1,a_2 qubits, and a_3 is the ancilla state. In case $\lambda=0$, the cloning transformation reduces to the UQCM proposed in [3]. When $\lambda=3-2\sqrt{2}$, we obtain the optimal phase-covariant quantum cloning transformation presented in [13] for *x-z* equator. Actually, we may use both Bures fidelity and Hilbert-Schmidt norm to quantify the quality of the copies [18]. Both of them show that transformation (3) is the optimal cloning machine for input state (1).

Sometimes, we study *x-y* equator instead of *x-z* equator so that some results may be obtained easier, and the two cases are connected by a transformation. We consider the input state as

$$|\Psi\rangle=\frac{1}{\sqrt{2}}[|0\rangle+e^{i\phi}|1\rangle], \tag{7}$$

where $\phi\in[0,2\pi)$. One can check that the y component of the Bloch vector of this state is zero. The cloning transformation takes the form,

$$|0\rangle_{a_1}|00\rangle_{a_2a_3}\to\frac{2(1-\lambda)}{\sqrt{6-4\lambda+6\lambda^2}}|00\rangle_{a_1a_2}|0\rangle_{a_3}+\frac{1+\lambda}{\sqrt{6-4\lambda+6\lambda^2}}(|01\rangle_{a_1a_2}+|10\rangle_{a_1a_2})|1\rangle_{a_3},$$

$$|1\rangle_{a_1}|00\rangle_{a_2a_3}\to\frac{2(1-\lambda)}{\sqrt{6-4\lambda+6\lambda^2}}|11\rangle_{a_1a_2}|1\rangle_{a_3}+\frac{1+\lambda}{\sqrt{6-4\lambda+6\lambda^2}}(|01\rangle_{a_1a_2}+|10\rangle_{a_1a_2})|0\rangle_{a_3}. \tag{8}$$

As the case of *x-y* equator, $\lambda=0$ corresponds to UQCM, and the case $\lambda=3-2\sqrt{2}$ is the optimal phase-covariant quantum cloning for input (7), which takes the following form:

$$|0\rangle_{a_1}|00\rangle_{a_2a_3}\to\frac{1}{\sqrt{2}}|00\rangle_{a_1a_2}|0\rangle_{a_3}+\frac{1}{2}(|01\rangle_{a_1a_2}+|10\rangle_{a_1a_2})|1\rangle_{a_3},$$

$$|1\rangle_{a_1}|00\rangle_{a_2a_3}\to\frac{1}{\sqrt{2}}|11\rangle_{a_1a_2}|1\rangle_{a_3}+\frac{1}{2}(|01\rangle_{a_1a_2}+|10\rangle_{a_1a_2})|0\rangle_{a_3}. \tag{9}$$

III. QUANTUM CLONING NETWORKS FOR EQUATORIAL QUBITS

In this section, following the method proposed by Bužek *et al.* [12], we show that the quantum cloning transformations for equatorial qubits can be realized by networks consisting of quantum logic gates. Let us first introduce the method proposed by Bužek *et al.* [12], and then analyze the case of phase-covariant cloning. The network is constructed by one- and two-qubit gates. The one-qubit gate is a single qubit rotation operator $\hat{R}_j(\vartheta)$, defined as

$$\hat{R}_j(\vartheta)|0\rangle_j=\cos\vartheta|0\rangle_j+\sin\vartheta|1\rangle_j,$$

$$\hat{R}_j(\vartheta)|1\rangle_j=-\sin\vartheta|0\rangle_j+\cos\vartheta|1\rangle_j. \tag{10}$$

The two-qubit gate is the controlled NOT gate represented by the unitary matrix

$$\hat{P}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}. \tag{11}$$

Explicitly, the controlled NOT gate $\hat{P}_{kl}$ acts on the basis vectors of the two qubits as follows:

$$\hat{P}_{kl}|0\rangle_k|0\rangle_l=|0\rangle_k|0\rangle_l,\quad \hat{P}_{kl}|0\rangle_k|1\rangle_l=|0\rangle_k|1\rangle_l,$$

$$\hat{P}_{kl}|1\rangle_k|0\rangle_l=|1\rangle_k|1\rangle_l,\quad \hat{P}_{kl}|1\rangle_k|1\rangle_l=|1\rangle_k|0\rangle_l. \tag{12}$$

Due to Bužek *et al.*, the action of the copier is expressed as a sequence of two unitary transformations,

$$|\Psi^{(in)}_{a_1}|0\rangle_{a_2}|0\rangle_{a_3}\rightarrow|\Psi\rangle^{(in)}_{a_1}|\Psi\rangle^{(prep)}_{a_1a_2}\rightarrow|\Psi\rangle^{(out)}_{a_1a_2a_3}. \tag{13}$$

This network may be described by a figure in Ref. [12]. The preparation state is constructed as

$$|\Psi\rangle^{(prep)}_{a_2a_3}=\hat{R}_2(\vartheta_3)\hat{P}_{32}\hat{R}_3(\vartheta_2)\hat{P}_{23}\hat{R}_2(\vartheta_1)|0\rangle_{a_2}|0\rangle_{a_3}. \tag{14}$$

The quantum copying is performed by

$$|\Psi\rangle^{(out)}_{a_1a_2a_2}=\hat{P}_{a_3a_1}\hat{P}_{a_2a_1}\hat{P}_{a_1a_3}\hat{P}_{a_1a_2}|\Psi\rangle^{(in)}_{a_1}|\Psi\rangle^{(prep)}_{a_2a_3}. \tag{15}$$

Note that the output copies appear in the a_2,a_3 qubits instead of a_1,a_2 qubits. For UQCM, we should choose [12]

$$\vartheta_1=\vartheta_3=\frac{\pi}{8},\quad \vartheta_2=-\arcsin\left(\frac{1}{2}-\frac{\sqrt{2}}{3}\right)^{1/2}. \tag{16}$$

We now consider the cloning transformations for equatorial qubits. The network proposed by Bužek *et al.* is rather general. We only need to take a different angles ϑ_j, $j=1$, 2, 3 to realize the phase-covariant cloning. In the case of cloning transformation for x-y equator (8), the preparation state takes the form

$$|\Psi\rangle^{(perp)}_{a_2a_3}=\frac{2(1-\lambda)}{\sqrt{6-4\lambda+6\lambda^2}}|00\rangle_{a_2a_3}+\frac{1+\lambda}{\sqrt{6-4\lambda+6\lambda^2}}(|01\rangle_{a_1a_2}+|10\rangle_{a_2a_3}). \tag{17}$$

The preparation state corresponding to cloning transformation (5) for x-z equator may be written as

$$|\Psi\rangle^{(perp)}_{a_2a_3}=q|00\rangle_{a_2a_3}+q\lambda|11\rangle_{a_2a_3}+y|10\rangle_{a_2a_3}+y|01\rangle_{a_2a_3}. \tag{18}$$

We can check that for some angles ϑ_j, $j=1$, 2, 3, the above preparation states can be realized. Actually we have several choices. When $\lambda=0$, we obtain the result for UQCM. Here, we present the result for the optimal case, i.e., $\lambda=3-2\sqrt{2}$.

For x-y equator, let

$$\vartheta_1=\vartheta_3=\arcsin\left(\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)^{1/2},$$

$$\vartheta_2=-\arcsin\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)^{1/2}. \tag{19}$$

Then, the preparation state has the form

$$|\Psi\rangle^{(perp)}_{a_2a_3}=\frac{1}{\sqrt{2}}|00\rangle_{a_2a_3}+\frac{1}{2}(|01\rangle_{a_2a_3}+|10\rangle_{a_2a_3}). \tag{20}$$

For x-z equator, let

$$\vartheta_1=\vartheta_3=\arcsin\left(\frac{1}{2}-\sqrt{\frac{1}{8}}\right)^{1/2},\quad \vartheta_2=0. \tag{21}$$

The preparation state is

$$|\Psi\rangle^{(perp)}_{a_2a_3}=\left(\frac{1}{2}+\sqrt{\frac{1}{8}}\right)|00\rangle_{a_2a_3}+\frac{1}{2\sqrt{2}}(|01\rangle_{a_2a_3}+|10\rangle_{a_2a_3})+\left(\frac{1}{2}-\sqrt{\frac{1}{8}}\right)|11\rangle_{a_2a_3}. \tag{22}$$

After the preparation stage, perform the copying procedure (15), we obtain the output state, and the output copies appear in the a_2 and a_3 qubits. The optimal quantum cloning transformations for equatorial qubits may achieve the highest fidelity $1/2+\sqrt{1/8}$. The reduced density operator of both copies at the output in a_2 and a_3 qubits may be expressed as

$$\rho^{(out)}=\left(\frac{1}{2}+\sqrt{\frac{1}{8}}\right)|\Psi\rangle\langle\Psi|+\left(\frac{1}{2}-\sqrt{\frac{1}{8}}\right)|\Psi_\perp\rangle\langle\Psi_\perp|. \tag{23}$$

IV. SEPARABILITY OF COPIED QUBITS AND QUANTUM TRIPLICATORS

A. Separability

For the UQCM, the density matrix for the two copies $\rho^{(out)}_{a_2a_3}$ is shown to be inseparable by use of Peres-Horodecki criterion [14,15]. That means it cannot be written as the convex sum,

$$\rho^{(out)}_{a_2a_3}=\sum_m w^{(m)}\rho^{(m)}_{a_2}\otimes\rho^{(m)}_{a_3}, \tag{24}$$

where the positive weights $w^{(m)}$ satisfy $\Sigma_m w^{(m)}=1$. There are correlations between the copies, i.e., the two qubits at the output of the quantum copier are nonclassically entangled

[12]. We shall show in this section that, different from the UQCM, the copied qubits are separable for the case of optimal phase-covariant quantum cloning by Peres-Horodecki criterion.

Peres-Horodecki's positive partial transposition criterion states that the positivity of the partial transposition of a state is both necessary and sufficient condition for its separability [14,15]. For x-z equator where the input state is $\alpha|0\rangle+\beta|1\rangle$, with $\alpha=\cos\theta,\beta=\sin\theta$, the partially transposed output density operator at a_2,a_3 qubits is expressed by a matrix,

$$[\rho_{a_2a_3}^{(out)}]^{T_2}=\frac{1}{3-2\lambda+3\lambda^2}\begin{pmatrix} 2(\alpha^2+\lambda^2\beta^2) & \alpha\beta(1-\lambda^2) & \alpha\beta(1-\lambda^2) & \frac{1}{2}(1-\lambda)^2 \\ \alpha\beta(1-\lambda^2) & \frac{1}{2}(1-\lambda)^2 & 2\lambda & \alpha\beta(1-\lambda^2) \\ \alpha\beta(1-\lambda^2) & 2\lambda & \frac{1}{2}(1-\lambda)^2 & \alpha\beta(1-\lambda^2) \\ \frac{1}{2}(1-\lambda)^2 & \alpha\beta(1-\lambda^2) & \alpha\beta(1-\lambda^2) & 2(\beta^2+\alpha^2\lambda^2) \end{pmatrix}. \tag{25}$$

Here, the cloning transformation corresponds to Eq. (5). Note that the output of copies appear in a_2,a_3 qubits. We have the following four eigenvalues:

$$\frac{1}{3-2\lambda+3\lambda^2}\left\{\frac{1}{2}(1-6\lambda+\lambda^2),\ \frac{1}{2}(1+2\lambda+\lambda^2),\right.$$
$$1+\lambda^2+\frac{1}{2}(1-\lambda)\sqrt{5+6\lambda+5\lambda^2},$$
$$\left.1+\lambda^2-\frac{1}{2}(1-\lambda)\sqrt{5+6\lambda+5\lambda^2}\right\}. \tag{26}$$

For optimal phase-covariant quantum cloning, $\lambda=3-2\sqrt{2}$, the four eigenvalues are

$$\left\{0,0,\frac{1}{4},\frac{3}{4}\right\}. \tag{27}$$

We see that none of the four eigenvalues is negative. This is different from the UQCM, where one negative eigenvalue exists for $\lambda=0$. According to Peres-Horodecki criterion, the copied qubits in phase-covariant quantum cloning are separable. Analyzing the four eigenvalues (26), we find that the optimal point $\lambda=3-2\sqrt{2}$ is the only separable point for the copied qubits. If we analyze the x-y equator, we obtain the same result.

B. Optimal quantum triplicators

The networks for equatorial qubits can realize the quantum copying. The copies at the output appear in a_2 and a_3 qubits. And the output reduced density operator is written as

$$\rho^{(out)}=\frac{2(1-\lambda^2)}{3-2\lambda+3\lambda^2}\rho^{(in)}+\frac{1-2\lambda+5\lambda^2}{6-4\lambda+6\lambda^2}\times 1. \tag{28}$$

Here, we are also interested in the output state in a_1 qubit. According to the cloning transformations or cloning networks for equatorial qubits, we find that the reduced density operator of the output state in a_1 qubit may be written as

$$\rho_{a_1}^{(out)}=\frac{(1+\lambda)^2}{3-2\lambda+3\lambda^2}[\rho^{(in)}]^T+\frac{(1-\lambda)^2}{3-2\lambda+3\lambda^2}\times 1, \tag{29}$$

where the superscript T means transposition. For x-z equator, the output reduced density operator is invariant under the action of transposition. Comparing the output reduced density operators in a_2 and a_3 qubits (28) and a_1 qubit (29), in case $\lambda=1/3$, we have a triplicator,

$$\rho_{a_1}^{(out)}=\rho_{a_2}^{(out)}=\rho_{a_3}^{(out)}=\frac{2}{3}\rho^{(in)}+\frac{1}{6}\times 1, \tag{30}$$

with fidelity 5/6 [12]. Explicitly, the triplicator cloning transformation for x-z equator has the form,

$$|0\rangle_{a_1}|00\rangle_{a_2a_3}\to\frac{1}{\sqrt{12}}[3|000\rangle_{a_1a_2a_3}+|011\rangle_{a_1a_2a_3}+|101\rangle_{a_1a_2a_3}+|110\rangle_{a_1a_2a_3}],$$

$$|1\rangle_{a_1}|00\rangle_{a_2a_3}\to\frac{1}{\sqrt{12}}[3|111\rangle_{a_1a_2a_3}+|100\rangle_{a_1a_2a_3}+|001\rangle_{a_1a_2a_3}+|010\rangle_{a_1a_2a_3}]. \tag{31}$$

For x-y equator, by applying a transformation $|0\rangle\leftrightarrow|1\rangle$ in a_1 qubit, and still let $\lambda=1/3$, we find the output density operator in a_1 (29) equals to that of a_2 and a_3 (28). And the triplicator cloning for x-y equator takes the form,

$$|0\rangle_{a_1}|00\rangle_{a_2a_3}\rightarrow\frac{1}{\sqrt{3}}[|001\rangle_{a_1a_2a_3}+|100\rangle_{a_1a_2a_3}+|010\rangle_{a_1a_2a_3}],$$

$$|1\rangle_{a_1}|00\rangle_{a_2a_3}\rightarrow\frac{1}{\sqrt{3}}[|110\rangle_{a_1a_2a_3}+|011\rangle_{a_1a_2a_3}+|101\rangle_{a_1a_2a_3}]. \tag{32}$$

The fidelity for quantum triplicator is 5/6. Actually, we can find the fidelity takes the same value 5/6 when $\lambda=0$ and $\lambda=1/3$ corresponding to UQCM and quantum triplicator, respectively. D'Ariano and Presti [19] proved that the optimal fidelity for one to three phase-covariant quantum cloning is 5/6, and presented the cloning transformation. The quantum triplicators presented above achieve the bound of the fidelity and agree with the results in Ref. [19].

V. OPTIMAL 1 TO *M* PHASE-COVARIANT QUANTUM CLONING MACHINES

We have investigated the $1\rightarrow 2$ and $1\rightarrow 3$ optimal quantum cloning for equatorial qubits. In what follows, we shall study the general N to M ($M>N$) phase-covariant quantum cloning.

We first discuss $1\rightarrow M$ phase-covariant quantum cloning. We start from the cloning transformations similar to the UQCM [6], then determine the parameters to give the highest fidelity, and finally prove that the determined cloning transformation is the optimal QCM for equatorial qubits. For x-y equator $|\Psi\rangle=(|\uparrow\rangle+e^{i\phi}|\downarrow\rangle)/\sqrt{2}$, we suppose the cloning transformations take the following form:

$$U_{1,M}|\uparrow\rangle\otimes R=\sum_{j=0}^{M-1}\alpha_j|(M-j)\uparrow,j\downarrow\rangle\otimes R_j,$$

$$U_{1,M}|\downarrow\rangle\otimes R=\sum_{j=0}^{M-1}\alpha_{M-1-j}|(M-1-j)\uparrow,(j+1)\downarrow\rangle\otimes R_j, \tag{33}$$

where we use the same notations as those of Ref. [6], R denotes the initial state of the copy machine and $M-1$ blank copies, R_j are orthogonal normalized states of ancilla, and $|(M-j)\psi,j)\psi_\perp\rangle$ denotes the symmetric and normalized state with $M-j$ qubits in state ψ and j qubits in state $\psi_\perp$. For arbitrary input state, the case $\alpha_j=\sqrt{2(M-j)/M(M+1)}$ is the optimal $1\rightarrow M$ quantum cloning [6]. Here, we consider the case of x-y equator instead of the arbitrary input state. The quantum cloning transformations should satisfy the property of orientation invariance of the Bloch vector and that we have identical copies. The cloning transformation (33) already ensure that we have M identical copies. The unitarity of the cloning transformation demands the relation $\Sigma_{j=0}^{M-1}\alpha_j^2=1$. Under this condition, we can check that the cloning transformation has the property of orientation invariance of the Bloch vector. Thus, the relation (33) is the quantum cloning transformation for x-y equator. The fidelity of the cloning transformation (33), takes the form

$$F=\frac{1}{2}[1+\eta(1,M)], \tag{34}$$

where

$$\eta(1,M)=\sum_{j=0}^{M-1}\alpha_j\alpha_{M-1-j}\frac{C_{M-1}^j}{\sqrt{C_M^jC_M^{j+1}}}. \tag{35}$$

We examine the cases of $M=2,3$. For $M=2$, we have $\alpha_0^2+\alpha_1^2=1$ and $\eta(1,M)=\sqrt{2}\alpha_0\alpha_1$. In case $\alpha_0=\alpha_1=1/\sqrt{2}$, we have the optimal fidelity and recover the previous result (9). For $M=3$, we have $\alpha_0^2+\alpha_1^2+\alpha_2^2=1$, and

$$\eta(1,3)=\frac{2}{3}\alpha_1^2+\frac{2}{\sqrt{3}}\alpha_0\alpha_2. \tag{36}$$

For $\alpha_0=\alpha_2=0,\alpha_1=1$, we have $\eta(1,3)=2/3$, which reproduces the case of quantum triplicator for x-y equator (32).

We present the result of the one to M phase-covariant quantum cloning transformations. When M is even, we have $\alpha_j=\sqrt{2}/2, j=M/2-1,M/2$ and $\alpha_j=0$, otherwise. When M is odd, we have $\alpha_j=1, j=(M-1)/2$ and $\alpha_j=0$, otherwise. The fidelity are $F=1/2+\sqrt{M(M+2)}/4M$ for M is even, and $F=1/2+(M+1)/4M$ for M is odd. The explicit cloning transformations have already been presented in Eq. (33).

Though the fidelity for $M=2,3$ are optimal, we need to prove that for general M, the fidelity achieve the bound as well. We apply the same method introduced by Gisin and Massar in Ref. [6]. In order to use some results later, we consider the general N to M cloning transformation. Generally, we write the N identical input state for equatorial qubits as

$$|\Psi\rangle^{\otimes N}=\frac{1}{2^{N/2}}\sum_{j=0}^{N}e^{ij\phi}\sqrt{C_N^j}|(N-j)\uparrow,j\downarrow\rangle. \tag{37}$$

The most general N to M QCM for equatorial qubits is expressed as

$$|(N-j)\uparrow,j\downarrow\rangle\otimes R\rightarrow\sum_{k=0}^{M}|(M-k)\uparrow,k\downarrow\rangle\otimes|R_{jk}\rangle, \tag{38}$$

where R still denotes the $M-N$ blank copies and the initial state of the QCM, and $|R_{jk}\rangle$ are unnormalized final states of the ancilla. The unitarity relation is written as,

$$\sum_{k=0}^{M}\langle R_{j'k}|R_{jk}\rangle=\delta_{jj'}. \tag{39}$$

The fidelity of the QCM takes the form

$$F=\langle\Psi|\rho^{out}|\Psi\rangle=\sum_{j',k',j,k}\langle R_{j'k'}|R_{jk}\rangle A_{j'k'jk}, \tag{40}$$

where ρ^{out} is the reduced density operator of each output qubit by taking partial trace over all M but one output qubits.

We impose the condition that the output density operator has the property of Bloch vector invariance, and find the following for $N=1$,

$$A_{j'k'jk}=\frac{1}{4}\left\{\delta_{j'j}\delta_{k'k}+(1-\delta_{j'j})\left[\delta_{k',(k+1)}\frac{\sqrt{(M-k)(k+1)}}{M}+\delta_{k,(k'+1)}\frac{\sqrt{(M-k')(k'+1)}}{M}\right]\right\}, \tag{41}$$

where $j,j'=0,1$ for case $N=1$. The optimal fidelity of the QCM for equatorial qubits is related to the maximal eigenvalue λ_{max} of matrix A by $F=2\lambda_{max}$ [6]. The matrix A (41) is a block diagonal matrix with block B given by

$$B=\frac{1}{4}\begin{pmatrix} 1 & \frac{\sqrt{(M-k)(k+1)}}{M} \\ \frac{\sqrt{(M-k)(k+1)}}{M} & 1 \end{pmatrix}. \tag{42}$$

Thus, we have proved that the optimal fidelty of one to M QCM for equatorial qubits takes the form

$$F=2\lambda_{max}=\begin{cases} \frac{1}{2}+\frac{\sqrt{M(M+2)}}{4M}, \text{M is even,} \\ \frac{1}{2}+\frac{(M+1)}{4M}, \text{M is odd.} \end{cases} \tag{43}$$

We thus find in this section, the optimal one to M phase-covariant quantum cloning transformation. This is the main result of this paper.

VI. N TO M PHASE-COVARIANT QCM

We conjecture that the optimal N to M phase-covariant QCM for x-y equator take the following form:

Case A, when $M=N+2L$, the cloning transformation is

$$U_{N,N+2L}|(N-j)\uparrow,j\downarrow\rangle\otimes R=|(N-j+L)\uparrow,(j+L)\downarrow\rangle\otimes R_L, \tag{44}$$

which implies that we just need one ancilla state and can omit it in cloning relation. The corresponding fidelity is

$$F=\frac{1}{2}+\frac{1}{2^N}\sum_{j=0}^{N-1}\sqrt{C_N^jC_N^{j+1}}\,\frac{\sqrt{(L+j+1)(N+L-j)}}{N+2L}. \tag{45}$$

Case B, when $M=N+2L+1$, the cloning transformation is

$$U_{N,N+2L+1}|(N-j)\uparrow,j\downarrow\rangle\otimes R=\frac{1}{\sqrt{2}}|(N-j+L+1)\uparrow,(j+L)\downarrow\rangle\otimes R_L \tag{46}$$

$$+\frac{1}{\sqrt{2}}|(N-j+L)\uparrow,(j+L+1)\downarrow\rangle\otimes R_{L+1}. \tag{47}$$

The corresponding fidelity is

$$F=\frac{1}{2}+\frac{1}{2^{N+1}}\sum_{j=0}^{N-1}\sqrt{C_N^jC_N^{j+1}}\times\frac{1}{N+2L+1}[\sqrt{(L+j+1)(N+L-j+1)}+\sqrt{(L+j+2)(N+L-j)}]. \tag{48}$$

When $N=1$, the cloning transformations and the fidelity reduce to the previous results given in the last section. For case $N>1$, the upper bound on the fidelity obtained by the method introduced in Ref. [6] is too conservative because that it is sometimes greater than unity.

It is proved that the optimal fidelity of $N\to\infty$ quantum cloning equals to the corresponding optimal fidelity of quantum estimation [7,20,13]. In the limit $L\to\infty$, the fidelity for N to $N+2L$, $N+2L+1$ quantum cloning becomes

$$F=\frac{1}{2}+\frac{1}{2^{N+1}}\sum_{j=0}^{N-1}\sqrt{C_N^jC_N^{j+1}}, \tag{49}$$

which is equal to the optimal fidelity of the quantum phase-estimation presented in Ref. [20]. This confirms that the optimal fidelity (45),(48) in the limit $L\to\infty$ gives a correct result. However, we still need a rigorous proof for the case of general N to M phase-covariant quantum cloning.

VII. SUMMARY

In this paper, the networks consisting of quantum gates for phase-covariant quantum cloning have been studied. The copied qubits of phase-covariant cloning machines are showed to be separable. We have given explicitly the $1\to M$ cloning transformations for x-y equator. And the optimal fidelity has been proved by using the method by Gisin and Massar [6]. The general $N\to M$ phase-covariant quantum clonings are conjectured.

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