Multipartite entangled coherent states

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We propose a scheme for generating multipartite entangled coherent states via entanglement swapping, with an example of a physical realization in ion traps. Bipartite entanglement of these multipartite states is quantified by the concurrence. We also compute multipartite entanglement for certain systems. Finally we establish that these results for entanglement can be applied to more general multipartite entangled nonorthogonal states.

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I. INTRODUCTION

Quantum entanglement is at the heart of quantum information theory and plays a key role in quantum information such as quantum teleportation [1], superdense coding [2], quantum key distribution [3],and telecoloning [4]. Genuine entanglement arises when the state of a multipartite system is nonseparable. Despite extensive efforts to quantify entanglement [5,6], characterization and classification of general mixed entangled states remains an important challenge. Entangled nonorthogonal states are even more challenging than their well-studied orthogonal counterparts, yet have not received the same degree of attention despite the importance of nonorthogonality to quantum theory and the importance of entanglement of nonorthogonal states to quantum information such as quantum key distribution [7]. The entangled coherent state (ECS) [8–12] or multipartite superposition of coherent states [13,14], is the most well-known example of entangled nonorthogonal states, along with the related entangled squeezed states [11] and entangled SU(2) and SU(1,1) coherent states [15].

The bipartite ECS can exhibit various nonclassical properties such as sub-Poissonian statistics, two-mode squeezing, and violations of the Cauchy-Schwarz inequalities [13], as well as violating Bell's inequality [8,10]. Although most attention has been devoted to the bipartite ECS, the multipartite case has been studied as well [14], but not to the extent we undertake here, namely, generating such states, discussing their potential realization in ion traps, and quantifying the degree of entanglement. Such studies of the ECS are of interest beyond obtaining a fundamental understanding of nonorthogonal entangled states: such states can be employed in quantum information theory, and in quantum computing applications, in particular [16]. This particular application is relevant to our discussion of the multipartite entangled coherent state (MECS), as it considers qubits encoded as two-dimensional center-of-mass (c.m.) vibrational motion of two ions in an ion trap. These qubits are measured by swapping entanglement from the vibrational to the internal states of the ion [16]. Our analysis of the MECS could enable generalizations of such qubits to large entangled systems.

Here we discuss the MECS and propose a scheme to generate such states via entanglement swapping, and we calculate the degree of entanglement by employing concurrence [6]. We also compute [17,18] to characterize the multipartite entanglement of the tripartite and even-number MECS.

II. GENERATION OF MULTIPARTITE ENTANGLED COHERENT STATES

We consider a set of N ions in a linear trap coupled together via the Coulomb interaction. The collective motion of the ions is described by the dynamics of the normal modes, with a_i ($a_i^\dagger$) the annihilation (creation) operator for the ith mode and ν_i the angular frequency of the ith mode, $i \in \{1,2,\ldots,N\}$. The fundamental mode ($i=1$) is the c.m. mode with frequency ν_1, and the $i=2$ mode is the breathing mode with angular frequency $\nu_2=\sqrt{3}\nu_1$. Frequencies for higher-order normal modes can be calculated [19]. We propose a method for generating entangled coherent states of the normal phonon modes for the trapped ions.

The ions behave as effective two-level systems, with $|0\rangle$ and $|1\rangle$ being the lower and upper states. Unitary transformations that create superpositions of these two states are generated by the Pauli raising and lowering operators $\sigma_\pm$ and the inversion operator σ_z. In $^9Be^+$, the states $|0\rangle$ and $|1\rangle$ exist as two hyperfine sublevels with superpositions created via stimulated Raman transitions [20].

The driving frequency between states $|0\rangle$ and $|1\rangle$ can be modified to excite one of the normal modes of oscillation. The normal mode can be excited by driving any one of the atoms, and for a given atom j, an effective interaction Hamiltonian between the ith phonon mode and the jth ion is [21,22]

$$H_{ij}=g_{ij}a_i^\dagger a_i\sigma_{jz}, \tag{1}$$

where g_{ij} is the coupling constant. By choosing the duration, strength, and frequency of the Raman pulse appropriately, the coupling strength can be set to the same magnitude. Thus we can consider a fixed coupling strength g. This means that any ion can be coupled to any of the normal modes of oscillation, with the choice of mode and coupling strength determined by experimental parameters. Thus, $g_{ij}=g$ in Eq. (1). Although challenging, this coupling between ions and normal-mode oscillations is possible.

A judicious choice of initial state will lead, by Hamiltonian evolution, to the MECS. We begin with the coherent state of the vibrational collective mode. If the phonon mode is initially in the ground state, the driving field is first con-

figured to impart a displacement to the chosen normal mode. We can therefore assume that the ith normal mode has been prepared in a coherent state $|\alpha\rangle_i$, and the ion i can be prepared in a superposition of $|0\rangle_i$ and $|1\rangle_i$. The initial state for the ith ion and the ith normal mode is prepared as $|\psi(0)\rangle_i = 2^{-1/2}|\alpha\rangle_i \otimes (|0\rangle_i + |1\rangle_i)$. This state undergoes the Hamiltonian evolution H_{ii}, Eq. (1), to yield the time-dependent state (in the interaction picture)

$$|\psi(\tau)\rangle_i = 2^{-1/2}(|\alpha e^{i\tau}\rangle_i \otimes |0\rangle_i + |\alpha e^{-i\tau}\rangle_i \otimes |1\rangle_i) \tag{2}$$

with $\tau = gt$ a normalized unit of time, scaled to the coupling strength g.

State (2) involves just one degree of freedom for the vibrational mode. We can consider two identical ions, ions 1 and 2, each prepared in the superposition state $|0\rangle$ and $|1\rangle$. Let us also assume that the c.m. mode and the breathing mode have been prepared in identical coherent states, that is, with the same amplitude and phase; the two-mode coherent state for these two normal modes of oscillation is $|\alpha\rangle_1 \otimes |\alpha\rangle_2$. We then couple ion 1 to the fundamental mode and ion 2 to the breathing mode by adjusting the appropriate parameters of the two Raman beams, with beam 1 directed at ion 1 and beam 2 directed at ion 2. Moreover, the coupling strength g between ion 1 and the c.m. mode is equal to the coupling strength between ion 2 and the breathing mode, with both interactions occurring simultaneously, i.e., the interaction Hamiltonian is $H = g(a_1^\dagger a_1 \sigma_{1z} + a_2^\dagger a_2 \sigma_{2z})$. Then the Hamiltonian (1) applies to the coupling between ion 1 and normal mode 1 and the coupling between ion 2 and normal mode 2 simultaneously. After a time $\tau = gt$, the product state

$$\begin{aligned} |\psi(\tau)\rangle_1 \otimes |\psi(\tau)\rangle_2 = 2^{-1}(&|\alpha e^{i\tau}\rangle_1 \otimes |\alpha e^{i\tau}\rangle_2 \otimes |0\rangle_1 \otimes |0\rangle_2 \\ &+ |\alpha e^{-i\tau}\rangle_1 \otimes |\alpha e^{-i\tau}\rangle_2 \otimes |1\rangle_1 \otimes |1\rangle_2 \\ &+ |\alpha e^{i\tau}\rangle_1 \otimes |\alpha e^{-i\tau}\rangle_2 \otimes |0\rangle_1 \otimes |1\rangle_2 \\ &+ |\alpha e^{-i\tau}\rangle_1 \otimes |\alpha e^{i\tau}\rangle_2 \otimes |1\rangle_1 \otimes |0\rangle_2) \end{aligned} \tag{3}$$

eventuates and is factorizable because two independent states have each undergone independent evolutions.

The factorizable state (3) may be transformed to an entangled state via a suitable measurement process on the electronic states of the two ions. Direct measurements of the electronic states will not suffice to entangle the vibrational states, but Bell-state measurements of the electronic states will work. This technique of Bell-state measurements to entangle other states is the hallmark of entanglement swapping [23], and we apply this method to entanglement swapping for nonorthogonal states.

Measurements must be performed with respect to the Bell-state bases for joint electronic states of both ions. The Bell states are defined to be

$$|\Phi^\pm\rangle_{12} = 2^{-1/2}(|0\rangle_1 \otimes |0\rangle_2 \pm |1\rangle_1 \otimes |1\rangle_2), \tag{4}$$

$$|\Psi^\pm\rangle_{12} = 2^{-1/2}(|0\rangle_1 \otimes |1\rangle_2 \pm |1\rangle_1 \otimes |0\rangle_2). \tag{5}$$

These Bell-state measurements on the state (3) yield the normalized ECSs

$$\frac{|\alpha e^{i\tau}\rangle_1 \otimes |\alpha e^{i\tau}\rangle_2 \pm |\alpha e^{-i\tau}\rangle_1 \otimes |\alpha e^{-i\tau}\rangle_2}{\sqrt{2 \pm 2\exp(-4|\alpha|^2 \sin^2\tau)\cos[2|\alpha|^2 \sin(2\tau)]}} \tag{6}$$

and

$$\frac{|\alpha e^{i\tau}\rangle_1 \otimes |\alpha e^{-i\tau}\rangle_2 \pm |\alpha e^{-i\tau}\rangle_1 \otimes |\alpha e^{i\tau}\rangle_2}{\sqrt{2 \pm 2\exp(-4|\alpha|^2 \sin^2\tau)}}, \tag{7}$$

respectively. For the specific case that $\tau = \pi/2$, Eqs. (6) and (7) reduce to the even and odd ECS [13]

$$\frac{|i\alpha\rangle_1 \otimes |i\alpha\rangle_2 \pm |-i\alpha\rangle_1 \otimes |-i\alpha\rangle_2}{\sqrt{2 \pm 2\exp(-4|\alpha|^2)}}. \tag{8}$$

Thus far we have considered only the single-particle and bipartite cases. As the purpose of this paper is to develop and study the MECS, beyond the bipartite case, we wish to generalize the above analysis from $N = 2$ particles, or ions, to arbitrary N. We therefore consider N systems, each prepared in the identical state $|\psi(\tau)\rangle_i$, i.e.,

$$|\psi(\tau)\rangle_1 \otimes |\psi(\tau)\rangle_2 \otimes \cdots \otimes |\psi(\tau)\rangle_N. \tag{9}$$

By analogy with the Bell-state measurements, in the multiparticle case, measurements are performed with respect to the maximally entangled multipartite electronic states

$$2^{-1/2}(|i_1\rangle_1 \otimes |i_2\rangle_2 \otimes \cdots \otimes |i_N\rangle_N \pm |\overline{i_1}\rangle_1 \otimes |\overline{i_2}\rangle_2 \otimes \cdots \otimes |\overline{i_N}\rangle_N), \tag{10}$$

for $i_k \in \{0,1\}, k \in \{1,2,\ldots,N\}$, and $\overline{i_k} \equiv 1 - i_k$. The result of this measurement on the above state collapses the vibrational state to the MECS

$$\begin{aligned} &|\alpha e^{i\tau_1}\rangle_1 \otimes |\alpha e^{i\tau_2}\rangle_2 \otimes \cdots \otimes |\alpha e^{i\tau_N}\rangle_N \pm |\alpha e^{-i\tau_1}\rangle_1 \otimes |\alpha e^{-i\tau_2}\rangle_2 \\ &\otimes \cdots \otimes |\alpha e^{-i\tau_N}\rangle_N \end{aligned} \tag{11}$$

up to a normalization constant, where $\tau_k \equiv \tau(-1)^{i_k}$.

Specifically, for all $i_k = 0$ and $\tau = \pi/2$, the above state reduces to the MECS [14]

$$\begin{aligned} &[2 \pm 2e^{-2N|\alpha|^2}]^{-1/2}(|i\alpha\rangle_1 \otimes |i\alpha\rangle_2 \otimes \cdots \otimes |i\alpha\rangle_N \pm |-i\alpha\rangle_1 \\ &\otimes |-i\alpha\rangle_2 \otimes \cdots \otimes |-i\alpha\rangle_N). \end{aligned} \tag{12}$$

Therefore, by measuring the combined electronic states of the ions via a generalized Bell-state measurement, the outcome of the electronic-state measurements is a MECS. Preparing the MECS follows a natural generalization to the entanglement-swapping method for preparing the bipartite ECS.

In order to generate the MECS, we need to perform a(challenging) measurement on certain maximally entanged electronic states. Here we provide a way to realize the measurement using controlled-NOT (denoted by C) gates [24]. A

series of $\mathcal{C}$ gates is applied to the electronic state (10) followed by a Hadamard (H) gate,

$$G = H_1 \mathcal{C}_{1N} \cdots \mathcal{C}_{13} \mathcal{C}_{12}, \tag{13}$$

where the subscripts of $\mathcal{C}_{ij}$ denotes the control ion i and the target j. Let the state (10) be the input of this gate G. The output is the product state $|j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_N\rangle (j_k \in \{0,1\})$. Local measurements on this output product state after the gate G has been applied correspond to the desired generalized Bell-state measurements.

One problem that can arise in creating multimode entanglement is that, during the interaction time between ions, the state of the vibrational mode may change and no longer exist as coherent state. However, thereby exists a remedy that uses the interaction Hamiltonian for the ith ion with the c.m. mode [21]

$$\mathcal{H}_i = -i(\alpha_i a_1^\dagger - \alpha_i^* a_1)(1-\sigma_{iz}) \tag{14}$$

during $\mathcal{C}$ gate operations, and avoids deleterious changes to the vibrational mode state [25]. The evolution operator $\exp(-it\mathcal{H}_i)$ incorporates the unitary operators

$$\exp[\pm i k_x X_1 (1-\sigma_{iz})/2]$$

and

$$\exp[\pm i k_p P_1 (1-\sigma_{jz})/2] \quad (i \neq j),$$

where $X_1 = (a_1 + a_1^\dagger)\sqrt{2}, P_1 = (a_1 - a_1^\dagger)/\sqrt{2}i$, and k_x, k_y are real numbers. From these two unitary operators, $\exp[\pm i k_p P_1 (1-\sigma_{jx})/2]$ can be realized by a single-qubit rotation. As $[1-\sigma_{iz}, 1-\sigma_{jx}] = 0$, we can use the technique in Ref. [25] to realize

$$\begin{aligned} &\exp[-ik_x k_p (1-\sigma_{iz})(1-\sigma_{j,x})/4] \\ &\quad = \exp[ik_x X_1 (1-\sigma_{iz})/2] \exp[ik_p P_1 (1-\sigma_{j,x})/2] \\ &\qquad \times \exp[-ik_x X_1 (1-\sigma_{iz})/2] \\ &\qquad \times \exp[-ik_p P_1 (1-\sigma_{jx})/2] \end{aligned} \tag{15}$$

by an appropriate choice of Raman-laser pulse phases. Letting $k_x k_p = \pi$, we obtain

$$\mathcal{C}_{ij} = \exp\left[-i\frac{\pi}{4}(1-\sigma_{iz})(1-\sigma_{jx})\right]. \tag{16}$$

The vibrational degree of trapped ions acts as a databus, and it does not change after the gate operation, i.e., the vibrational modes remain as coherent states.

III. QUANTIFYING THE ENTANGLEMENT

The method for preparing the MECS has been discussed; now we quantify the entanglement for this state. Multipartite entanglement continues to be an important topic, and here we choose to study entanglement by certain well-accepted measures, which suffice for the MECS.

Whereas we have shown how to generate the special case of even and odd MECSs, in this section we consider the generalized balanced MECS for studying entanglement. By "balanced," we refer to the constraint that each element of the superposition of multipartite coherent states in the MECS is equally weighted with all other elements of the superposition. An unbalanced ECS is more difficult to construct [9], and the extension of the following work to the unbalanced MECS is straightforward; because of the greater challenge in creating unbalanced MECSs and the ready generalization of the analysis below to unbalanced MECSs, we do not include this analysis of unbalanced MECS here.

Thus far, we have considered the even and odd balanced MECS as generated by a unitary evolution in a larger Hilbert space followed by measurements on the other degrees of freedom. A unitary evolution can be used to generate the bipartite ECS, on the other hand [8,12], which does not yield the even and odd variety of ECS. The N-partite MECS with an arbitrary relative phase θ is given by

$$\begin{aligned} |\alpha, \theta, N\rangle_{\mathrm{ECS}} = \mathcal{N}(&|\alpha\rangle_1 \otimes |\alpha\rangle_2 \otimes \cdots \otimes |\alpha\rangle_N + e^{i\theta} |-\alpha\rangle_1 \\ &\otimes |-\alpha\rangle_2 \otimes \cdots \otimes |-\alpha\rangle_N) \end{aligned} \tag{17}$$

with

$$\mathcal{N} \equiv [2 + 2p^N \cos\theta]^{-1/2}, \tag{18}$$

the normalization constant, and

$$p \equiv e^{-2|\alpha|^2} = \langle -\alpha | \alpha \rangle. \tag{19}$$

The MECS satisfies the equation

$$a_1 a_2 \ldots a_N |\alpha, \theta, N\rangle_{\mathrm{ECS}} = \alpha^N |\alpha, \theta, N\rangle_{\mathrm{ECS}}, \tag{20}$$

where $a_i (i = 1,2,\ldots,N)$ is the annihilation operator of mode i. Two interesting limits arise for $|\alpha| \to \infty$ and $|\alpha| \to 0$. In the asymptotic limit $|\alpha| \to \infty$, the two states $|\alpha\rangle$ and $|-\alpha\rangle$ approach orthogonality and the state $|\alpha, \theta, N\rangle_{\mathrm{ECS}}$ approaches the multipartite maximally entangled state, the Greenberger-Horner-Zeilinger (GHZ) state,

$$\begin{aligned} |\psi^{\mathrm{GHZ}}\rangle_N = \frac{1}{\sqrt{2}}(&|\mathbf{0}\rangle_1 \otimes |\mathbf{0}\rangle_2 \otimes \cdots \otimes |\mathbf{0}\rangle_N + e^{i\theta} |\mathbf{1}\rangle_1 \otimes |\mathbf{1}\rangle_2 \\ &\otimes \cdots \otimes |\mathbf{1}\rangle_N). \end{aligned} \tag{21}$$

An orthogonal basis can be constructed such that $|\mathbf{0}\rangle_i \equiv |\alpha \to \infty\rangle_i$ and $|\mathbf{1}\rangle_i \equiv |\alpha \to -\infty\rangle_i$, where we have symbolically identified the large-$|\alpha|$ limit. In the asymptotic limit $|\alpha| \to 0$, the state $|\alpha, \pi, N\rangle_{\mathrm{ECS}}$ reduces to the so-called W state [27],

$$\begin{aligned} |W\rangle_N = N^{-1/2}(&|1\rangle \otimes_1 |0\rangle_2 \otimes \cdots \otimes |0\rangle_N + |0\rangle_1 \otimes |1\rangle_2 \otimes \ldots \\ &\otimes |0\rangle + \cdots + |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |1\rangle_N). \end{aligned} \tag{22}$$

Here $|n\rangle_i (n = 0,1)$ denote the Fock states of mode i.

We employ concurrence [6] as a measure of bipartite entanglement for the state $|\alpha, \theta, N\rangle_{\mathrm{ECS}}$. For ρ_{12}, the density matrix for a pair of qubits 1 and 2, the concurrence is [6]

$$\mathfrak{C}_{12}=\max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4,0\} \tag{23}$$

for $\lambda_1\geqslant\lambda_2\geqslant\lambda_3\geqslant\lambda_4$, the square roots of the eigenvalues of the operator

$$\varrho_{12}\equiv\rho_{12}(\sigma_y\otimes\sigma_y)\rho_{12}^{*}(\sigma_y\otimes\sigma_y), \tag{24}$$

where $\sigma_y=\left(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix}\right)$ is a Pauli matrix.

Nonzero concurrence occurs if and only if qubits 1 and 2 are entangled. Moreover, $\mathfrak{C}_{12}=0$ only for an unentangled state and $\mathfrak{C}_{12}=1$ only for a maximally entangled state. This concurrence measure can be extended to multipartite systems, and we apply it to pure-state and mixed-state entanglement for MECS below.

A. Pure-state entanglement

In this section we consider bipartite splitting of the multipartite system, i.e., splitting the entire system into two subsystems, one subsystem containing any k $(1\leqslant k\leqslant N-1)$ particles and the other containing the remaining $N-k$ particles. Let $\mathfrak{C}_{(k,N-k)}$ denote the concurrence between the two subsystems. Applying the general result for concurrence of bipartite nonorthogonal states [26] to the MECS(17) yields

$$\mathfrak{C}_{(k,N-k)}(\theta)=\frac{\sqrt{(1-p^{2k})(1-p^{2(N-k)})}}{1+p^N\cos\theta} \tag{25}$$

from which the condition for the maximally entangled ECS [i.e., $\mathfrak{C}_{(k,N-k)}(\theta)=1$] is given by

$$\cos\theta=-\frac{1-\sqrt{(1-p^{2k})(1-p^{2(N-k)})}}{p^N}. \tag{26}$$

Hence,

$$1-\sqrt{(1-p^{2k})(1-p^{2(N-k)})}\geqslant p^N, \tag{27}$$

with equality only for $2k=N$. It follows that the condition for $\mathfrak{C}_{(k,N-k)}(\theta)=1$ is simply

$$\cos\theta=-1,2k=N. \tag{28}$$

Thus, for an even number N, $k=N/2$, and $\theta=\pi$, the state $|\alpha,\theta,N\rangle_{\mathrm{ECS}}$ is maximally entangled in the sense that $\mathfrak{C}_{(N/2,N/2)}(\pi)=1$. For other nonorthogonal $(p\neq 0)$ cases, the state is not maximally entangled.

B. Mixed-state entanglement

Now we study the bipartite reduced-density matrix ρ_{kl}, which is obtained by tracing out all other systems except systems k and l. There are $N(N-1)/2$ different density matrices ρ_{kl}. However, for our state $|\alpha,\theta,N\rangle_{\mathrm{ECS}}$, all particles are equally entangled with each other and all the reduced-density matrices ρ_{kl} are identical. Therefore, it is sufficient to consider ρ_{12} and to generalize from this case. For convenience, we first make a local transformation $(-1)^{a_2^{\dagger}a_2}$ on the state $|\alpha,\theta,N\rangle_{\mathrm{ECS}}$. This local transformation does not change the amount of entanglement in the state. Then, by tracing out systems $3,4,\ldots,N$ in the transformed state, we obtain the reduced-density matrix describing systems 1 and 2 as

$$\begin{aligned}\rho_{12}=&\mathrm{Tr}_{3,4,\ldots,N}(|\alpha,\theta,N\rangle_{\mathrm{ECS}}\langle\alpha,\theta,N|)=\mathcal{N}^2(|\alpha\rangle|-\alpha\rangle\\&\times\langle\alpha|\langle-\alpha|+|-\alpha\rangle|\alpha\rangle\langle-\alpha|\langle\alpha|)+e^{i\theta}q|-\alpha\rangle|\alpha\rangle\\&\times\langle\alpha|\langle-\alpha|+e^{-i\theta}q|\alpha\rangle|-\alpha\rangle\langle-\alpha|\langle\alpha|\end{aligned} \tag{29}$$

with $q\equiv p^{N-2}$.

In order to diagonalize the density matrix, we choose an orthogonal basis $\{|\mathbf{0}\rangle,|\mathbf{1}\rangle\}$, distinguished from the electronic-state basis of the same notation employed earlier for the entanglement-swapping operation by using boldface symbols. This orthogonal basis is defined as

$$|\mathbf{0}\rangle\equiv|\alpha\rangle,|\mathbf{1}\rangle\equiv(|-\alpha\rangle-p|\mathbf{0}\rangle)/\mathcal{M}, \tag{30}$$

where $\mathcal{M}=\sqrt{1-p^2}$. It then follows that

$$|-\alpha\rangle=\mathcal{M}|\mathbf{1}\rangle+p|\mathbf{0}\rangle. \tag{31}$$

Substituting Eqs. (30) and (31) into Eq. (29), we obtain the density matrix

$$\rho_{12}=\mathcal{N}^2\begin{pmatrix}2p^2(1+q\cos\theta) & p\mathcal{M}(1+qe^{i\theta}) & p\mathcal{M}(1+qe^{-i\theta}) & 0\\ p\mathcal{M}(1+qe^{-i\theta}) & \mathcal{M}^2 & \mathcal{M}^2qe^{-i\theta} & 0\\ p\mathcal{M}(1+qe^{i\theta}) & \mathcal{M}^2qe^{i\theta} & \mathcal{M}^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{32}$$

in the standard basis $\{|\mathbf{00}\rangle,|\mathbf{01}\rangle,|\mathbf{10}\rangle,|\mathbf{11}\rangle\}$. From Eqs. (23) and (32), the square roots of eigenvalues of ϱ_{12} in Eq. (24) are

$$\lambda_1=\mathcal{N}^2\mathcal{M}^2(1+q),\quad \lambda_2=\mathcal{N}^2\mathcal{M}^2(1-q),$$

$$\lambda_3=\lambda_4=0. \tag{33}$$

Although ρ_{12} is complicated, the expressions for the square roots of the eigenvalues are rather simple. The concurrence is thus not complicated and follows directly from Eqs. (23) and (33), namely,

$$\mathfrak{C}_{12}=\frac{\mathcal{M}^2q}{1+p^N\cos\theta}=\frac{p^{N-2}-p^N}{1+p^N\cos\theta}. \tag{34}$$

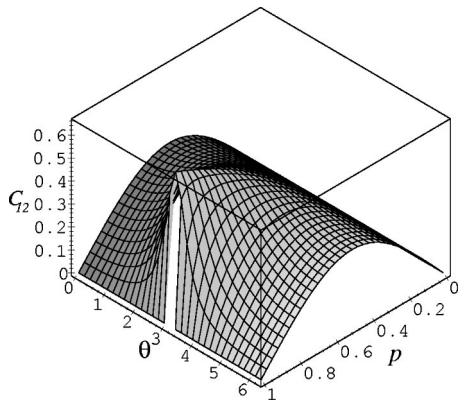

FIG. 1. The concurrence versus θ and p for $N=3$.

This expression for concurrence must be calculated carefully for $p=1$ and $\theta=\pi$. Of course, $p=1$ implies that $\alpha=0$, which is the limit in which MECS approaches the vacuum state. In this limit, we apply l'Hôpital's rule to Eq. (34) to obtain $\lim_{p\to 1}\mathfrak{C}_{12}=2/N$ for $N>2$. This result is in accordance with the known maximal degree of entanglement between any pair of qubits in an N-qubit system, attained for qubits prepared in the pure symmetric state referred to as the W state [27,28] and presented for MECS in Eq. (22). The limit $|\alpha|\to 0$ yielding a nonzero concurrence can thus be understood in the context of producing a symmetric state [26].

The first application of this formula for concurrence is to determine when systems 1 and 2 are disentangled, i.e., $\mathfrak{C}_{12}=0$. One case arises for $p=0$, which corresponds to the orthogonal case. As described earlier, this case is only valid in the asymptotic limit of infinite $|\alpha|$. Another case of complete disentanglement arises for $N\to\infty$ and $0<p<1$, yielding a concurrence of $\mathfrak{C}_{12}=0$. The third case arises for $\theta\neq\pi$ and in the limit $|\alpha|\to 0$. In summary, there is no bipartite entanglement in three cases: the asmptotic limit of infinite-amplitude coherent states, the asymptotic limit of an infinite number of entangled systems, and the case of the MECS for which the coherent state is just the vacuum state.

For $N=2$ the concurrence (34) reduces to

$$\mathfrak{C}_{12}=\frac{1-p^2}{1+p^2\cos\theta}, \tag{35}$$

which is the concurrence for the pure state $|\alpha,\theta,2\rangle_{\mathrm{ECS}}$. The bipartite concurrence for a bipartite ECS provides arbitrarily strong entanglement for appropriate parameter choices. When $\theta=\pi$, the concurrence $\mathfrak{C}_{12}=1$, and the state becomes the the antisymmetric state $|\Psi^-\rangle$, Eq. (5). Bipartite entanglement of multipartite systems offers reduced entanglement, however. For $N\neq 2$, the reduced density-matrix describes a mixed state, and the degree of entanglement is given by Eq. (34).

Figure 1 gives a plot of the concurrence verus θ and p. As seen from the figure, the maximum value of the concurrence occurs when $\theta=\pi$ for fixed values of p and N. From Eq. (34), the maximum value is obtained as

TABLE I. A summary of concurrences for some special states.

$\lvert\alpha\rvert$	p	N	θ	$\mathfrak{C}$	State
0	1	>2	π	$2/N$	$\lvert\mathrm{W}\rangle_N$
0	1	2	π	1	$\lvert\Psi^-\rangle$
∞	0	2	any	1	$\lvert\Phi\rangle$
∞	0	>2	any	0	$\lvert\psi^{\mathrm{GHZ}}\rangle_N$
0	1	$\geqslant 2$	$\neq\pi$	0	$\lvert 0\rangle^{\otimes N}$
$\neq 0,\infty$	$0<p<1$	∞	any	0	$\lvert\alpha,\theta,\infty\rangle_{\mathrm{ECS}}$

$$\mathfrak{C}_{12}=\frac{p^{N-2}-p^N}{1-p^N}. \tag{36}$$

As a short summary we give concurrences of some special states in Table I. In Table I the state $|\Phi\rangle=2^{-1/2}[|\mathbf{0}\rangle\otimes|\mathbf{0}\rangle+\exp(i\theta)|\mathbf{1}\rangle\otimes|\mathbf{1}\rangle]$, where $|\mathbf{0}\rangle\equiv|\alpha\to\infty\rangle$ and $|\mathbf{1}\rangle\equiv|\alpha\to-\infty\rangle$. For fixed N and θ there still exist maximum values of the concurrence (see Fig. 1). From Eq. (34), the value of p at which the maximum occurs is determined by the equation

$$2p^N\cos\theta+Np^2=N-2. \tag{37}$$

As an example, we consider the tripartite case $N=3$. The above equation simplifies to

$$2p^3\cos\theta+3p^2-1=0. \tag{38}$$

For $\theta=0$ and $0<p<1$, the solution is $p=1/2$ with a corresponding maximum concurrence of 1/3. For $\theta=\pi/2$, the solution of p is $3^{-1/2}$, and the maximum value of $\mathfrak{C}_{12}$ is $2\sqrt{3}/9$.

C. Multipartite entanglement

We have thus far considered only bipartite entanglement of a multipartite system. One type of multipartite entanglement is N-way entanglement which involves all N particles. We have used the concurrence as an example of bipartite entanglement. Recently Coffman *et al.* [17] used concurrence to examine three-qubit systems, and introduced the concept of the 3-tangle, $\tau_{1,2,3}(|\psi\rangle)$ as a way to quantify the amount of three-way entanglement in three-qubit systems. Later Wong and Christensen [18] generalized the 3-tangle to the N-tangle. The N-tangle is the square of the multiqubit concurrence

$$\mathfrak{C}_{1,2,\ldots,N}\equiv|\langle\psi|\sigma_y^{\otimes N}|\psi^*\rangle| \tag{39}$$

for even qubits, with $|\psi\rangle$ being a multiqubit pure state. This concurrence works only for even numbers of qubits; $|\langle\psi|\sigma_y^{\otimes N}|\psi^*\rangle|=0$ for any odd-N-qubit pure states. Therefore, this quantity cannot act as a general measure of N-way entanglement. Next we quantify the N-way entanglment using N-tangle for $N=3$ and even N.

The 3-tangle can be calculated from concurrences $\mathfrak{C}_{1(23)}$, $\mathfrak{C}_{12}$, and $\mathfrak{C}_{13}$ because [17]

$$\tau_{1,2,3}=\mathfrak{C}_{1(23)}^2-\mathfrak{C}_{12}^2-\mathfrak{C}_{13}^2 \tag{40}$$

holds. For our state $|\alpha,\theta,3\rangle_{\mathrm{ECS}}$, the 3-tangle is simplified as

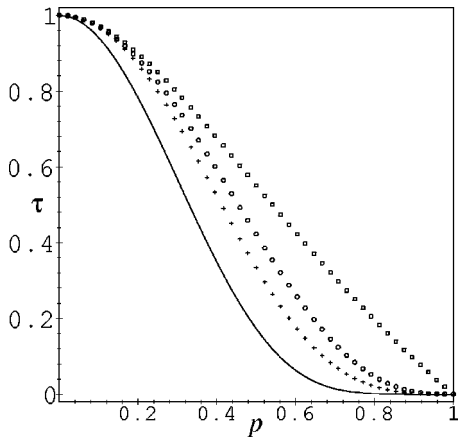

FIG. 2. The N-tangle versus p for different θ and N:$N=3,\theta=0$ (cross points), $N=3,\theta=\pi/2$ (circle points), $N=3,\theta=\pi$ (box points), and $N=6,\theta=0$ (solid line).

$$\tau_{1,2,3}=\mathfrak{C}^2_{1(23)}-2\mathfrak{C}^2_{12}. \tag{41}$$

From Eqs. (25) and (34), the 3-tangle is

$$\tau_{1,2,3}=\frac{(1-p^2)^3}{(1+p^3\cos\theta)^2}, \tag{42}$$

and as expected, $\tau_{1,2,3}=1(\tau_{1,2,3}=0)$ in the limit that $p\rightarrow 0(p\rightarrow 1)$.

Now we examine N-way entanglement for the state $|\alpha,\theta,N\rangle_{\mathrm{ECS}}$ with N even. In the basis of Eq. (30), the state $|\alpha,\theta,N\rangle_{\mathrm{ECS}}$ can be rewritten as

$$\begin{aligned}|\alpha,\theta,N\rangle_{\mathrm{ECS}}=\mathcal{M}[&|\mathbf{0}\rangle_1\otimes|\mathbf{0}\rangle_2\otimes\cdots\otimes|\mathbf{0}\rangle_N+e^{i\theta}(\mathcal{M}|\mathbf{1}\rangle_1\\&+p|\mathbf{0}\rangle_1)\otimes(\mathcal{M}|\mathbf{1}\rangle_2+p|\mathbf{0}\rangle_2)\otimes\cdots\otimes(\mathcal{M}|\mathbf{1}\rangle_N\\&+p|\mathbf{0}\rangle_N)].\end{aligned} \tag{43}$$

The effect of writing the MECS in this basis set is to have the state expressed formally as a multiqubit state. From Eqs. (39) and (43) the N-tangle is obtained as

$$\tau_{1,2,\ldots,N}=\frac{(1-p^2)^N}{(1+p^N\cos\theta)^2} \tag{44}$$

for even N. Although this formula is obtained for even N, by comparing Eqs. (42) and(44), it is also applicable to $N=3$. The condition for maximal entanglement, $\tau_{1,2,\ldots,N}=1$, is

$$N=2,\quad \cos\theta=-1 \tag{45}$$

for $p\neq 0$. This constraint on N restricts maximal entanglement to the bipartite ECS.

In Fig. 2 we give a plot of the N-tangle versus p for various θ and N. For $p=0$ (orthogonal case), the multiqubit concurrence is equal to 1, independent of θ. We already know that the state $|\alpha,\pi,N\rangle_{\mathrm{ECS}}$ becomes the W state in the limit $p\rightarrow 1$. Now we take this limit and choose $\theta=\pi$ in Eq. (44), thereby establishing that the concurrence $\tau_{1,2,\ldots,N}=0$ in this case. Thus, we observe that multipartite entanglement, as determined by the N-tangle, is indeed zero for the W state.

IV. CONCLUSION

We have considered generation and entanglement measures for the multipartite entangled coherent state. Generating the MECS is possible by entangling vibrational degrees of freedom for trapped two-level ions with the ions' internal electronic states. By measuring the electronic states with respect to a highly entangled basis, basically an extension of the (bipartite) Bell basis, the resultant motional state is a MECS. We have quantified the entanglement of the MECS by applying the concurrence to measure bipartite entanglement(in one case, by splitting the multipartite system into two subsystems and, in the other case, by tracing over all degrees of freedom except for two subsystems). We have also employed the N-tangle to determine the overall degree of entanglement. Each of these measures tells us something important about the MECS, and the MECS versions of the GHZ and W states have been studied and elucidated.

Quantifying entanglement for MECSs provides a simple measure to evaluate the inherent resource of such states, and this is relevant to quantum-information applications where entanglement is regarded as a crucial resource. Moreover, the study of MECS highlights the subtleties of applying entanglement measures to nonorthogonal entangled states. The particular physical realization studied here is the entangled vibrational motion of ions in a trap. As the bipartite ECS has proven to be a useful alternative construct for making qubits [16], in contrast to the usual Fock-state qubits, this analysis could be valuable for encoding qubits as MECS in an ion trap.

Finally, the analysis here for MECS is readily exteneded to more general systems, including entangled squeezed states [11], entangled SU(2) and SU(1,1) coherent states [15], and so on, as follows. Essentially, Eqs. (25),(34), and (44) can be applied to the general entangled state

$$\begin{aligned}|\Psi,\Phi\rangle=\mathcal{N}'(&|\Psi\rangle_1\otimes|\Psi\rangle_2\otimes\cdots\otimes|\Psi\rangle_N+e^{i\theta'}|\Phi\rangle_1\otimes|\Phi\rangle_2\\&\otimes\cdots\otimes|\Phi\rangle_N),\end{aligned} \tag{46}$$

where is $\mathcal{N}'$ is the normalization constant, $|\Psi\rangle$ and $|\Phi\rangle$ are arbitrary linearly independent states, and $\langle\Psi|\Phi\rangle=p'$ is a real overlap. Then the corresponding concurrence for the state $|\Psi,\Phi\rangle$ is obtained by directly replacing p and θ with p' and θ' in Eqs. (25),(34), and (44). Therefore, our results for quantifying entanglement provide a useful formalism with a validity well beyond that for MECS.

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