Conformal symmetry and the nonlinear Schrödinger equation

Pijush K. Ghosh
Department of Physics, Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan
(Received 27 February 2001; published 10 December 2001)

We show that the width of the wave packet of a class of generalized nonlinear Schrödinger equations (NLSE) trapped in an arbitrary time-dependent harmonic well in any dimensions is universally determined by the same Hill's equation. This class of generalized NLSE is characterized by a dynamical O(2,1) symmetry in absence of the trap. As an application, we study the dynamical instabilities of the rotating as well as nonrotating Bose-Einstein condensates in one and two dimensions. We also show exact extended parametric resonance in a nonrelativistic Chern-Simons theory producing a gauged NLSE.

DOI: 10.1103/PhysRevA.65.012103 PACS number(s): 03.65.Ge, 05.45.Yv, 03.75.Fi, 11.15.−q

The nonlinear Schrödinger equation (NLSE) appears in many branches of present-day physics and mathematics [1]. The optical soliton [2], a solution of the NLSE, has even been observed experimentally [3]. The one-dimensional NLSE is exactly solvable. There exists several other generalized NLSE in one dimension that are also exactly solvable. However, very little exact and analytical results are known for higher-dimensional generalizations of these models, although they are very much relevant in many branches of modern science. The purpose of this paper is to present an exact, analytical description of the dynamics of the width of the wave packet for a class of generalized NLSE in arbitrary dimensions trapped in a time-dependent harmonic well. This class of generalized NLSE is characterized by a dynamical O(2,1) symmetry in absence of the trap.

Consider the following Lagrangian in arbitrary $d+1$ dimensions,

$$\mathcal{L}=i\psi^*\partial_\tau\psi-\frac{1}{2m}|\nabla\psi|^2-gV(\psi,\psi^*,\mathbf{r}). \tag{1}$$

The coupling constant g has the inverse-mass dimension in the natural units with $c=\hbar=1$. The real potential V does not depend on any dimensional coupling constant. This allows to have a scale-invariant theory. We demand the invariance of the action $\mathcal{A}=\int d\tau d^d\mathbf{r}\mathcal{L}$ under the following time-dependent transformations [4,5],

$$\mathbf{r}\to\mathbf{r}_h=\dot{\tau}(t)^{-1/2}\mathbf{r},\quad \tau\to t=t(\tau),\quad \dot{\tau}(t)=\frac{d\tau(t)}{dt},$$

$$\psi(\tau,\mathbf{r})\to\psi_h(t,\mathbf{r}_h)=\dot{\tau}^{\,d/4}\exp\left(-i\frac{\ddot{\tau}}{4\dot{\tau}}r_h^2\right)\psi(\tau,\mathbf{r}), \tag{2}$$

with the scale factor τ given by

$$\tau(t)=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \alpha\delta-\beta\gamma=1. \tag{3}$$

Particular choices of $\tau(t)=t+\beta,\alpha^2t$, and $t/(1+\gamma t)$, correspond to time translation, dilation, and special conformal transformation. The generators of these transformations (as given below) close under an O(2,1) algebra. Although V is restricted to have specific forms due to the requirement of the symmetry, one can still make infinitely many choices of it. It might be noted here, apart from its dependence on the condensate ψ, the potential can also be explicitly dependent on the space coordinates. We keep V arbitrary, but, consistent with the O(2,1) symmetry in Eq. (1), unless mentioned otherwise.

Let us now introduce two moments I_1 and I_2 in terms of the density ρ and the current $\mathbf{j}$ as

$$\rho(\tau,\mathbf{r})=\psi^*\psi,\quad \mathbf{j}(\tau,\mathbf{r})=-\frac{i}{2m}(\psi^*\nabla\psi-\psi\nabla\psi^*),$$

$$I_1(\tau)=\frac{m}{2}\int d^d\mathbf{r}\,r^2\rho,\quad I_2(\tau)=\frac{m}{2}\int d^d\mathbf{r}\,\mathbf{r}\cdot\mathbf{j}. \tag{4}$$

We are dealing with a conservative system and the total number of particles $N(\tau)=\int d^d\mathbf{r}\rho$ is a constant of motion. The moment I_2 is related to the speed of the growth of the condensate. The moment I_1 describes the square of the width of the wave packet [6]. This quantity plays the central role in the analysis of the collapse of the condensates of the NLSE with or without time-independent harmonic trap [7–10]. It is also used in the context of the Bose-Einstein condensation (BEC) [11] to study the low-energy excitations and in optics [12] to determine the beam-parameter evolution. The dynamics of I_1, when the system (1) is immersed in an external time-dependent harmonic trap, is the central subject of the investigation of this paper. We show that the dynamics of I_1 is universally determined by the same solvable Hill's equation, independent of the space dimensionality, integrability, and nature (short range, long range, local, nonlocal, linear, nonlinear) of the interaction. The universality in the description of the dynamics of the width for this class of theory has been observed partially through a time-dependent variational analysis in [13]. We present here exact, analytical, and complete treatment.

The system (1) has a dynamical O(2,1) symmetry. The generators, the Hamiltonian H, the dilatation generator D, and the conformal generator K are

$$H=\int d^d\mathbf{r}\left[\frac{1}{2m}|\nabla\psi|^2+gV(\psi,\psi^*,\mathbf{r})\right],$$

$$D=\tau H-I_2,$$

$$K=-\tau^2H+2\tau D+I_1. \tag{5}$$

These generators are constant in time and lead to the following equations:

$$\frac{dH}{d\tau}=0, \quad \frac{dI_1}{d\tau}=2I_2, \quad \frac{dI_2}{d\tau}=H. \tag{6}$$

For time-independent solutions, both I_1 and I_2 do not depend on τ. As a consequence, the static solutions of a system with O(2,1) symmetry carry zero energy [14]. We also note that $I_2=D=0$ and $K=I_1$ for static solutions of Eq. (1). Defining the width of the wave packet, $X=\sqrt{I_1}$, it is easy to find a decoupled equation for X from Eq. (6),

$$\frac{d^2X}{d\tau^2}=\frac{Q}{X^3}, \quad Q=I_1H-I_2^2>0, \quad \frac{dQ}{d\tau}=0. \tag{7}$$

The constant of motion Q is the Casimir operator of the O(2,1) symmetry. Equation (7) can be interpreted as the equation of motion of a particle moving in an inverse-square potential. Interestingly enough, this system also has a dynamical O(2,1) symmetry. This reduced system of a particle in an inverse-square potential is a well-studied problem and the solution is given by [4]

$$X^2=(a+b\tau)^2+\frac{Q}{a^2}\tau^2, \tag{8}$$

where a and b are the integration constants.

Consider the time-dependent transformations in Eq. (2) with arbitrary scale factor $\tau(t)$. This is no more symmetry transformations of Eq. (1) for general τ. The action $\mathcal{A}$ is transformed into a new one, $\mathcal{A}_h=\int dt d^d\mathbf{r}_h\mathcal{L}_h$, containing a time-dependent harmonic trap. The new Lagrangian $\mathcal{L}_h$ now reads

$$\mathcal{L}_h=i\psi_h^*\partial_t\psi_h-\frac{1}{2m}|\boldsymbol{\nabla}_h\psi_h|^2-gV(\psi_h,\psi_h^*,\mathbf{r}_h)$$
$$-\frac{1}{2}m\omega(t)r_h^2|\psi_h|^2. \tag{9}$$

The time-dependent frequency $\omega(t)$ of the harmonic trap is determined by

$$\ddot{b}+\omega(t)b=0, \quad b(t)=\dot{\tau}^{-1/2}. \tag{10}$$

Once the solution of the equation of motion of Eq. (1) is known, the same can be obtained for Eq. (9) by using the transformation (2) and the Eq. (10) or the vice versa. Equation (10) describes the motion of a particle in a time-dependent harmonic trap. For $\tau(t)=(1/\omega_0)\tan(\omega_0 t)$, it gives $\omega=\omega_0$. For the special choice of Eq. (3), the frequency ω obviously vanishes. The general solution of Eq. (10) for the physically relevant periodic $\omega(t)$ is well known and will be discussed below.

The dynamical O(2,1) symmetry of $\mathcal{L}$ is not present for $\mathcal{L}_h$. We replace $(\mathbf{r},\psi,\tau)$ by $(\mathbf{r}_h,\psi_h,t)$ in the definition of I_1, I_2, and H and denote the resulting expressions in terms of the "curly form" of the associated variables. Under the transformation (2), the Eq. (6) have the following form:

$$\dot{\mathcal{I}}_1(t)=2\mathcal{I}_2(t),$$
$$\dot{\mathcal{I}}_2(t)=\mathcal{H}(t)-\omega(t)\mathcal{I}_1(t),$$
$$\dot{\mathcal{H}}(t)=-2\omega(t)\mathcal{I}_2(t). \tag{11}$$

Defining a new variable $\mathcal{X}(t)=\sqrt{\mathcal{I}_1(t)}$, it is easy to find a decoupled equation for $\mathcal{X}$,

$$\ddot{\mathcal{X}}+\omega(t)\mathcal{X}=\frac{\mathcal{Q}}{\mathcal{X}^3},$$
$$\mathcal{Q}=\mathcal{I}_1\mathcal{H}-\mathcal{I}_2^2>0, \quad \dot{\mathcal{Q}}=0. \tag{12}$$

We have the surprising result that the dynamics of the width of the wave packet of the system (9) is universally determined by the Eq. (12). This result is independent of the integrability of the model. We also have the freedom of choosing a large class of V as long as the dynamical O(2,1) symmetry in absence of the harmonic trap is maintained. The knowledge of the time evolution of $\mathcal{X}$ allows us to determine the time evolution of $\mathcal{H}$,

$$\mathcal{H}=\dot{\mathcal{X}}^2+\frac{\mathcal{Q}}{\mathcal{X}^2}. \tag{13}$$

For the time-independent trap, $\omega(t)=\omega_0$, the Hamiltonian $H_h=\mathcal{H}+\omega_0\mathcal{I}_1$ corresponding to the Lagrangian $\mathcal{L}_h$ is a constant of motion. The Hamiltonian H_h is related to the generator of the compact SO(2) rotation of SO(2,1).

Equation (12) can be interpreted as that of a particle moving in a time-dependent harmonic trap and a inverse-square potential. Due to the underlying O(2,1) symmetry in absence of the trap, this equation can be obtained directly from Eq. (7) through the use of the transformations (2). The dynamics of the width $\mathcal{X}$ can thus be constructed exactly from Eq. (8),

$$\mathcal{X}(t)=b(t)X(\tau(t)), \tag{14}$$

with the knowledge of the scale factors $\tau(t)$ and $b(t)$ from Eq. (10) for a particular choice of $\omega(t)$. We provide below a familiar form of solution of Eq. (12), since it appears in many branches of physics including the cylindrically symmetric two-dimensional NLSE [10]. The general solution of Eq. (12) is given by,

$$\mathcal{X}^2(t)=u^2(t)+\frac{\mathcal{Q}}{W^2}v^2(t), \quad W(t)=u\dot{v}-v\dot{u}, \tag{15}$$

where $u(t)$ and $v(t)$ are two independent solutions of the equation,

$$\ddot{x}+\omega(t)x=0, \tag{16}$$

satisfying $u(t_0)=\mathcal{X}(t_0)$, $\dot{u}(t_0)=\dot{\mathcal{X}}(t_0)$, $\dot{v}(t_0)=0$, and $v(t_0)\neq 0$. For periodic $\omega(t)$ with the period T, the above equation is known as the Hill's equation and is a text-book material [15]. We just mention here the general stability criteria in terms of the quantity $\delta=|u(T)+\dot{v}(T)|$ with the normalization $\mathcal{X}(0)=0,\dot{\mathcal{X}}(0)=1$, and $v(0)=1$. The solution is stable for $\delta<2$, while it is unstable for $\delta>2$. We remark that the same stability criteria is valid for Eq. (10).

The central result of this paper is contained in Eq. (12). Although our main concern in this paper is on NLSE, we remark that the same result is true for a class of linear Schrödinger equations with Calogero-type inverse-square interaction in arbitrary dimension [16]. An example of arbitrary d dimensional nonlinear potential consistent with O(2,1) symmetry is given by

$$V(\psi,\psi^*,\mathbf{r})=\int d^d\mathbf{r}'\,\psi^*(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')|\psi(\mathbf{r})|^2, \tag{17}$$

with $U(r)$ having the following scaling property. For $\mathbf{r}\to\epsilon\mathbf{r}$, $U(\mathbf{r})\to U(\epsilon\mathbf{r})=\epsilon^{-2}U(\mathbf{r})$. We now discuss a few specific examples of NLSE with different choices of V that are relevant in the contemporary literature.

BEC in $d=1, g>0, V=|\psi|^6$. The Gross-Pitaevskii equation (GPE) describing the repulsive Bose-Einstein condensates trapped in a time-dependent harmonic trap in one dimension can be obtained from the Lagrangian (9) [17]. Exact soliton solutions of the GPE equation have been obtained in absence of the trap [17,18]. Only approximate or numerical results are known, when the time-independent harmonic trap is included [17].

Following our analysis, the exact solutions of Eq. (9) can be obtained from those of Eq. (1) by simply using the transformation (2) and the Eq. (10) determining the time-dependent scale factor τ for a particular choice of the $\omega(t)$. We consider the case of time-independent trap with $\omega(t)=\omega_0$ and choose $g=\pi^2/(6m)$. Define the following dimensionless variables:

$$\overline{x_h}=\pi\psi_0^2 x_h, \quad \overline{t}=\frac{\pi^2\psi_0^4}{m}t, \quad \overline{\psi_h}=\frac{\psi_h}{\psi_0}, \quad \overline{\omega_0}=\frac{m\omega_0}{\pi^4\phi_0^8}, \tag{18}$$

where ψ_0^2, the asymptotic value of the density, is related to the chemical potential μ by, $\psi_0^2=\sqrt{2m\mu}/\pi$. The exact solution for $\overline{\psi_h}$ is

$$\overline{\psi_h}=\frac{1}{\cos(\overline{\omega_0}\,\overline{t})}\exp\left(-\frac{i\overline{\omega_0}}{2}\tan(\overline{\omega_0}\,\overline{t})\overline{x_h^2}-\frac{i}{\overline{\omega_0}}\tan(\overline{\omega_0}\,\overline{t})\right)\times\left[\frac{\cosh\left[\frac{2y}{\cos(\overline{\omega_0}\,\overline{t})}\right]-1}{\cosh\left[\frac{2y}{\cos(\overline{\omega_0}\,\overline{t})}\right]+2}\right]^{1/2}. \tag{19}$$

In the limit $\overline{\omega_0}\to 0$, the solution for the system without the trap is recovered [17]. Without loss of any generality, we are choosing $\beta=0$ in Eq. (12) of [17].

BEC in $d=2, V=|\psi|^4$. We get the GPE describing two-dimensional BEC in a time-dependent trap from the Lagrangian (9). No exact solution of this GPE with or without the trap is known. However, this is a well-studied system and many of the dynamical properties are already known [10,19,20]. We concentrate here on the rotating BEC and present some interesting results. Consider a further time-dependent rotation in Eq. (9) [5],

$$t\to\tilde{t}=t, \quad \mathbf{r}_h\to\tilde{\mathbf{r}}_h=\begin{pmatrix}\cos f(t) & \sin f(t)\\ -\sin f(t) & \cos f(t)\end{pmatrix}\mathbf{r}_h. \tag{20}$$

This transforms $\mathcal{A}_h$ to $\tilde{\mathcal{A}}_h=\int d\tilde{t}\,d^d\tilde{\mathbf{r}}_h\tilde{\mathcal{L}}_h$ containing an additional term proportional to the z component of the angular momentum with the coefficient given by a time-dependent frequency. In particular, the new $\tilde{\mathcal{L}}_h$ is given by

$$\tilde{\mathcal{L}}_h=i\psi_h^*\partial_{\tilde{t}}\psi_h-\frac{1}{2m}|\tilde{\boldsymbol{\nabla}}_h\psi_h|^2-gV(\psi_h,\psi_h^*,\tilde{\mathbf{r}}_h)-\frac{1}{2}m\omega(t)\tilde{r}_h^2|\psi_h|^2-\dot{f}\psi_h^*L_z\psi_h,$$

$$L_z=-i\left(\tilde{x}_h\frac{\partial}{\partial\tilde{y}_h}-\tilde{y}_h\frac{\partial}{\partial\tilde{x}_h}\right), \tag{21}$$

where $\tilde{x}_h$ and $\tilde{y}_h$ are the components of the two-dimensional vector $\tilde{\mathbf{r}}_h$. This is the Lagrangian for rotating BEC in an external time-dependent isotropic trap in two dimensions [21]. Interestingly, once the solution of the equation of motion of Eq. (1) is known, the same can be obtained for Eq. (21), using the Eqs. (2), (10), and (20), or vice versa. Moreover, under the transformation (20), the set of equations in Eq. (11) remains the same in terms of the new variables $\tilde{t}$ and $\tilde{\mathbf{r}}_h$. Thus, the dynamics of the width $\mathcal{X}(\tilde{t})$ of Eq. (21) is again universally determined by the Eq. (12). The introduction of the last term in Eq. (21) does not change the dynamical properties of the width. There may be dynamic instabilities solely due to the rapid fluctuations in the phase of the condensate during the evolution in time. However, it is obvious from the definition of I_1 that such instabilities do not show up in the evolution of the width.

Gauged NLSE. We now show that the dynamics of the width remains unchanged even if the nontrivial gauge fields are introduced in Eq. (1) maintaining the O(2,1) symmetry. Consider a Lagrangian in $2+1$ dimensions with the gauge fields $(A_0,\mathbf{A})$ and the matter field ψ,

$$\mathcal{L}_g=i\psi^*(\partial_\tau-iA_0)\psi-\frac{1}{2m}|(\boldsymbol{\nabla}-i\mathbf{A})\psi|^2-gV(\psi,\psi^*,\mathbf{r})+\frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda, \tag{22}$$

where the last term is the Chern-Simons term. The moment I_1 for this nonrelativistic Chern-Simons (CS) theory can be interpreted as the width of the soliton or alternatively as the quadrupole moment. For $V=\frac{1}{2}|\psi|^4$, this is the Jackiw-Pi model describing gauged NLSE [22]. This is relevant in theories with anyons and in the quantum Hall effect [22–24]. The Jackiw-Pi model is exactly solvable at the self-dual point, $g=1/m|\kappa|$. Our result is valid for arbitrary V maintaining O(2,1) symmetry. For the particular case of Jackiw-Pi model, the importance of our result lies at all non-self-dual points, where the model is not integrable. The Hamiltonian is given by

$$H = \int d^2\mathbf{r}\left[\frac{1}{2m}|(\boldsymbol{\nabla} - i\mathbf{A})\psi|^2 + gV(\psi,\psi^*,\mathbf{r})\right]. \qquad (23)$$

The CS term being a topological term does not contribute to the Hamiltonian. The generators D and K have the same expressions as in Eq. (5), with the partial derivative in the expression of the current $\mathbf{j}$ in the definition of the moment I_2 replaced by the respective covariant derivative. The treatment is now identical to the case without the gauge-fields. The same transformations (2) with $d=2$ and the gauge fields transforming accordingly,

$$A_\mu^h(t,\mathbf{r}_h) = \frac{\partial x^\nu}{\partial x_h^\mu} A_\nu(\tau,\mathbf{r}),$$

$$x^\mu = (\tau,\mathbf{r}), \quad A^\mu = (A_0,\mathbf{A}), \quad \mu = 0,1,2, \qquad (24)$$

introduce a time-dependent harmonic trap [22–24]. The width of the soliton of this new Lagrangian is again universally determined by Eq. (12). Interestingly, the introduction of the gauge fields and the nontrivial CS term to the usual two-dimensional NLSE does not change the dynamics of $\mathcal{X}$. A comment is in order at this point. It is known that Eq. (12) admits parametric resonances. Thus, the solitons of the nonrelativistic CS theory should exhibit the same phenomenon. This provides an example of exact, extended parametric resonance in a gauge theory with the nontrivial CS term.

In conclusion, we have shown that the width of the wave packet of a class of generalized NLSE is universally determined by the same Hill's equation. This class of NLSE is characterized by a dynamical O(2,1) symmetry in absence of the trap. The result is so robust that it is independent of (i) the space dimensionality, (ii) the integrability of the model, and (iii) short range, long range, local, nonlocal, linear or nonlinear nature of the many-body interaction. This result persists with its full generality even when the gauge fields are introduced maintaining the dynamical O(2,1) symmetry. The later example allows us to study an exact parametric resonance in a theory with the nontrivial gauge fields. Special cases of this class of generalized NLSE are relevant in BEC and in nonrelativistic Chern-Simons theory. It would be nice to see the importance of this class of NLSE in many more physical systems.

This work is supported by JSPS. I would like to thank T. Deguchi and T. K. Ghosh for their continuous interest in this work and valuable comments on the manuscript.

[1] V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP **34**, 62 (1972); R. Camassa, J.M. Hyman, and B.P. Luce, Physica D **123**, 1 (1998).
[2] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973); **23**, 171 (1973).
[3] L.F. Mollenauer, R.H. Stolen, and J.P. Gordon, Phys. Rev. Lett. **45**, 1095 (1980).
[4] V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cimento Soc. Ital. Fis. **34A**, 569 (1976); R. Jackiw, Ann. Phys. (N.Y.) **129**, 183 (1980); **201**, 83 (1990).
[5] S. Takagi, Prog. Theor. Phys. **84**, 1019 (1990); **85**, 463 (1991); **85**, 723 (1991).
[6] We are using the phrase ''width of the wave packet'' following Ref. [10]. It may be more appropriate for some specific cases to identify I_1 with a different physical quantity instead of the width. The central result, the exact and the universal description of the dynamics of the moment I_1, is independent of any such change in the nomenclature.
[7] V.E. Zakharov, Sov. Phys. JETP **35**, 908 (1972); V.E. Zakharov and V.S. Synakh, *ibid.* **41**, 465 (1975); M.I. Weinstein, Commun. Math. Phys. **87**, 567 (1983).
[8] T. Tsurumi and M. Wadati, J. Phys. Soc. Jpn. **66**, 3031 (1997); **66**, 3035 (1997); M. Wadati and T. Tsurumi, Phys. Lett. A **247**, 287 (1998); T. Tsurumi, H. Morise, and M. Wadati, Int. J. Mod. Phys. B **14**, 655 (2000).
[9] A.V. Rybin, G.G. Varzugin, M. Lindberg, J. Timonen, and R.K. Bullough, Phys. Rev. E **62**, 6224 (2000).
[10] J.J. Garcia-Ripoll, V.M. Perez-Garcia, and P. Torres, Phys. Rev. Lett. **83**, 1715 (1999); e-print patt-sol/9904006.
[11] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996); V.M. Perez-Garcia *et al.*, *ibid.* **77**, 5320 (1996); F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[12] S.N. Vlasov, V.A. Petrischev, and V.I. Talanov, Radiophys. Quantum Electron. **14**, 1062 (1971); V.M. Perez-Garcia *et al.*, J. Opt. B: Quantum Semiclassical Opt. **2**, 353358 (2000).
[13] T.K. Ghosh, Phys. Lett. A **285**, 222 (2001).
[14] D. Freedman and A. Newell (unpublished).
[15] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer-Verlag, New York, 1989).
[16] Pijush K. Ghosh (unpublished).
[17] E.B. Kolomeisky, T.J. Newman, J.P. Straley, and X. Qi, Phys. Rev. Lett. **85**, 1146 (2000).
[18] R.K. Bhaduri, S. Ghosh, M.V.N. Murthy, and D. Sen, J. Phys. A **34**, 6553 (2001).
[19] Yu. Kagan, E.L. Surkov, and G.V. Shlyapnikov, Phys. Rev. A **54**, R1753 (1996).
[20] L.P. Pitaevskii and A. Rosch, Phys. Rev. A **55**, R853 (1997).
[21] A.L. Fetter and A.A. Svidzinsky, e-print cond-mat/0102003.
[22] R. Jackiw and S.-Y. Pi, Phys. Rev. Lett. **64**, 2969 (1990); **66**, 2682 (1991); Phys. Rev. D **42**, 3500 (1990); **44**, 2524 (1991).
[23] Z.F. Ezawa, M. Hotta, and A. Iwazaki, Phys. Rev. Lett. **67**, 411 (1991).
[24] Pijush K. Ghosh, Phys. Rev. D **53**, 2248 (1996).