PHYSICAL REVIEW A, VOLUME 65, 010301(R)

Entanglement sharing among quantum particles with more than two orthogonal states

Kenneth A. Dennison and William K. Wootters
Department of Physics, Williams College, Williamstown, Massachusetts 01267

(Received 14 July 2001; revised manuscript received 28 August 2001; published 5 December 2001)

Consider a system consisting of n d-dimensional quantum particles (qudits), and suppose that we want to optimize the entanglement between each pair. One can ask the following basic question regarding the sharing of entanglement: what is the largest possible value $E_{max}(n,d)$ of the *minimum* entanglement between any two particles in the system? (Here we take the entanglement of formation as our measure of entanglement.) For $n=3$ and $d=2$, that is, for a system of three qubits, the answer is known: $E_{max}(3,2)=0.550$. In this paper we consider first a system of d qudits and show that $E_{max}(d,d)\geqslant 1$. We then consider a system of three particles, with three different values of d. Our results for the three-particle case suggest that as the dimension d increases, the particles can share a greater fraction of their entanglement capacity.

DOI: 10.1103/PhysRevA.65.010301 PACS number(s): 03.67.−a, 03.65.Ta, 03.65.Ud

Quantum entanglement, as exhibited, for example, in the singlet state $(1/\sqrt{2})(|01\rangle-|10\rangle)$ of a pair of qubits, has been the object of much study in recent years because of its connection with quantum communication and quantum computation [1]. Though entanglement is a kind of correlation, it is known to be fundamentally different from ordinary classical correlation. One of the characteristic differences is this: whereas arbitrarily many classical systems can be perfectly correlated with each other—the temperature fluctuations in ten different cities could, in principle, be exactly parallel—any entanglement that may exist between two quantum particles seems to limit the degree to which either of the particles can be entangled with anything else [2,3]. For example, if two qubits A and B are in the singlet state, then neither of them can have any entanglement with a third qubit C, simply because such entanglement would require the pair AB to be in a *mixed* state, whereas the singlet state is pure. Coffman *et al.* [3] have generalized this example (still considering only qubits) by allowing A and B to be only partially entangled, in which case one finds an inequality expressing a trade-off between the AB entanglement and the AC or BC entanglement.

As this sort of limitation may be a fundamental property of entanglement, one would like to express it more generally. In particular, one would like to capture quantitatively the limitation on the sharing of entanglement among arbitrarily many particles of arbitrary dimension. The following problem offers one approach to such a quantitative expression. Consider a system of n d-dimensional quantum particles (qudits), and suppose that one wants each particle to be highly entangled with each of the other particles. We expect that there will have to be compromises, since increasing the entanglement of any given pair will probably work against the entanglements of other pairs. It makes sense, then, to ask how large one can make the *minimum* pairwise entanglement, the minimum being taken over all pairs [4]. In this paper we address this problem, taking as our measure of entanglement the entanglement of formation [5,6], which for a pair of qudits ranges from zero to $\log_2 d$. For a collection of n qudits, let us call the maximum possible value of the minimum pairwise entanglement $E_{max}(n,d)$. This function, if it can be found, will give us a specific quantitative bound on the degree to which entanglement can be shared among a number of particles. We focus in this paper on two special cases: $n=d$ and $n=3$. As we will see, our results for $n=3$, combined with earlier work on the problem, suggest that in a well-defined sense the limitation on entanglement sharing becomes less restrictive with increasing values of the dimension d.

Before reviewing what is currently known about $E_{max}(n,d)$, let us recall the definition of entanglement of formation. For a pure state $|\Phi\rangle$ of a bipartite quantum system, the entanglement $E(\Phi)$ is defined [7] as

$$E(\Phi)=-\sum_i r_i \log_2 r_i, \tag{1}$$

where the r_i's are the eigenvalues of the density matrix of either subsystem. (For a pure bipartite state the density matrices of both subsystems necessarily have the same eigenvalues.) A mixed state ρ can always be written in many different ways as a probabilistic mixture of distinct but not necessarily orthogonal pure states

$$\rho=\sum_j p_j|\Phi_j\rangle\langle\Phi_j|. \tag{2}$$

The entanglement of formation of ρ is defined [5,6] as the average entanglement of the pure states of the decomposition, minimized over all possible decompositions:

$$E_f(\rho)=\inf\sum_j p_j E(\Phi_j). \tag{3}$$

As we have mentioned above, the entanglement of formation between a pair of qudits ranges from zero to $\log_2 d$. Let us refer to the maximum value $\log_2 d$ as the *entanglement capacity* of a pair of qudits.

For a pair of qubits, there is an explicit formula for the entanglement of formation of an arbitrary mixed state [8]. It is given in terms of another measure of entanglement called the concurrence [8,9], which at this point has a standard definition only for qubits.[1] In terms of the concurrence C, which ranges from zero to one, the entanglement of formation of a pair of qubits is $E_f(\rho)=\mathcal{E}[C(\rho)]$, where the function $\mathcal{E}$ is defined by

$$\mathcal{E}(C)=h\left[\frac{1}{2}(1+\sqrt{1-C^2})\right], \tag{4}$$

h being the binary entropy function $h(x)=-x\log_2 x-(1-x)\log_2(1-x)$. Note that $\mathcal{E}(C)$ is a monotonically increasing function, with $\mathcal{E}(0)=0$ and $\mathcal{E}(1)=1$. We will not be focusing particularly on qubits in this paper, but Eq. (4) will be useful both for summarizing previous work on the problem and, in a different context, for presenting our own results.

We now list the results that have been obtained so far regarding $E_{max}(n,d)$.

(1) $E_{max}(2,d)=\log_2 d$. This equation simply says that if there are only two particles, they can saturate their entanglement capacity; they do not have to share the entanglement with other particles.

(2) $E_{max}(3,2)=\mathcal{E}(2/3)=0.550$. Dür *et al.* [4] obtained this result by proving that the optimal pairwise entanglement for a system of three qubits is achieved in the state $(1/\sqrt{3})(|100\rangle+|010\rangle+|001\rangle)$.

(3) $E_{max}(n,2)\geqslant\mathcal{E}(2/n)$. Koashi *et al.* [10] showed that for a system of n qubits, if the state is such that the density matrix of each pair of particles is the same, then the maximum pairwise concurrence is $2/n$. It is conceivable (though it seems unlikely) that by removing the symmetry constraint one might be able to achieve a greater pairwise entanglement; therefore, we write this result as an inequality rather than an equality.

In this paper we add two new items to the above list: (i) For $n=d$, that is, for a system of d qudits, we find for each value of d a specific state in which each pair of particles has exactly 1 "ebit" of entanglement between them. (For $d=2$, our state reduces to the singlet state of a pair of qubits.) This will show that $E_{max}(d,d)$ is at least 1 for all values of d. (ii) For $n=3$, that is, for a system of three particles, we add to the known result for qubits ($d=2$) and to our own result for qutrits [$d=3$ in item (i)] a third example with $d=7$. Our results for the three-particle case suggest that as d increases, the particles can share not just more entanglement, but a greater *fraction* of their entanglement capacity.

A system of *d* qudits

Before writing down our special state of d qudits with arbitrary d, we illustrate our construction in the special case of three qutrits. Let the particles be called A, B, and C, and let the indices i, j, and k label the elements of orthogonal bases for the three particles, each index taking the values 0, 1, and 2. Our special state for this system is

$$|\xi\rangle=\frac{1}{\sqrt{6}}\sum_{i,j,k}\epsilon_{ijk}|ijk\rangle, \tag{5}$$

where ϵ_{ijk} is antisymmetric under interchange of any two indices and $\epsilon_{012}=1$. This is the singlet state of three qutrits with respect to the group SU(3); i.e., it is the unique three-qutrit state (up to an overall phase factor) that is invariant under arbitrary transformations of the form $U\otimes U\otimes U$ where $U\in\mathrm{SU}(3)$. The density matrix $|\xi\rangle\langle\xi|$ is symmetric under interchange of any two particles, so that each pair of particles is equally entangled. To find the pairwise entanglement, we write down the reduced density matrix of any pair; for definiteness we choose the first two particles, A and B:

$$\rho^{AB}_{ij,i'j'}=\frac{1}{6}\sum_k\epsilon_{ijk}\epsilon_{i'j'k}=\frac{1}{6}(\delta_{ii'}\delta_{jj'}-\delta_{ij'}\delta_{ji'}), \tag{6}$$

δ being the Kronecker delta. Alternatively, we can write ρ^{AB} without indices as

$$\rho^{AB}=\frac{1}{6}(I-F), \tag{7}$$

where I is the identity operator and F is the operator that interchanges particles A and B: $F=\Sigma_{ij}|ij\rangle\langle ji|$. The two-qutrit state ρ^{AB} is an example of a Werner state, that is, a state that is invariant under all transformations of the form $U\otimes U$ where U is unitary. Werner states can be defined for any $d\times d$ system, and one can show [11] that in any dimension the Werner states are precisely those states that can be written as $\rho=aI+bF$, a and b being real numbers and F being defined as above. Vollbrecht and Werner [11] have shown that the entanglement of formation of any Werner state is given by $E_f(\rho)=\mathcal{E}[c(\rho)]$, where $c(\rho)=-\mathrm{Tr}\,\rho F$ and $\mathcal{E}$ is the function defined in Eq. (4). [When $c(\rho)$ is nonnegative, it plays the role of a concurrence for Werner states.] In our case, $c(\rho^{AB})=1$, so that the entanglement is $E_f(\rho^{AB})=\mathcal{E}(1)=1$. Thus each pair of qutrits has exactly one ebit of entanglement. This value is, by the way, the maximum possible entanglement of any Werner state.

It is a simple matter to generalize the above construction to a system of d qudits. In that case, we have d indices $i_1,i_2,\ldots,i_d$, each taking values from 0 to $d-1$. Our special state for this system is the SU(d) singlet state[2]

$$|\xi\rangle=\frac{1}{\sqrt{d!}}\sum_{i_1\ldots i_d}\epsilon_{i_1\ldots i_d}|i_1\ldots i_d\rangle, \tag{8}$$

[1]The concurrence of a pure state of two qubits is simply $2\sqrt{\det\rho_A}$, where ρ_A is the reduced density matrix of one of the qubits. The concurrence of a mixed state ρ of two qubits is given by $\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}$, where $\lambda_1\geqslant\lambda_2\geqslant\lambda_3\geqslant\lambda_4$ are the square roots of the eigenvalues of $\rho(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)$, ρ^* being the complex conjugate of ρ in the standard basis and σ_y being the usual Pauli matrix [8,9]. We recall these formulas here for the sake of completeness but will not need them in the present paper.

[2]This state has been used recently by Hillery and Bužek in a scheme designed to probe a quantum gate that realizes an unknown unitary transformation [12].

where $\epsilon_{i_1 \ldots i_d}$ is completely antisymmetric and $\epsilon_{0,1,\ldots,d-1} = 1$. One can show directly that the reduced density matrix of each pair of particles is again a Werner state

$$\rho^{AB} = \frac{1}{d(d-1)}(I-F), \tag{9}$$

and that the entanglement of formation of this state is one ebit. We thus conclude that $E_{max}(d,d) \geqslant 1$. We write an inequality here simply because our state $|\xi\rangle$ may not optimize the pairwise entanglement; one might be able to do better. However, having put some effort into looking for better states with $d=3$, we regard it as likely that our state is optimal in that case.

A three-particle system

It is interesting to compare our result for three qutrits with the previously studied example of three qubits [4]. For a triple of qubits E_{max} is 0.550, and we have just seen that for a triple of qutrits, E_{max} is at least 1. However, a straightforward comparison of these numbers is not particularly illuminating, because qubits and qutrits have different entanglement capacities. We can perhaps make a fairer comparison by considering the ratio of E_{max} to the relevant entanglement capacity. For qubits, this ratio is $0.550/\log_2 2 = 0.550$, whereas for qutrits it is $1/\log_2 3 = 0.631$. Thus by this measure, qutrits are better able to share entanglement than qubits: they can share a greater fraction of their entanglement capacity. It is interesting to ask whether this trend will continue for larger values of d. That is, will $E_{max}(3,d)/\log_2 d$ continue to increase with increasing d?

To address this question, we consider one further case with three particles, namely, the case $d=7$. We choose the value 7 because it allows us to construct a reasonably simple and symmetric state that exhibits large pairwise entanglement. In fact, we consider a one-parameter *family* of states, having the following form:

$$|\zeta\rangle = \frac{1}{\sqrt{7}} \sum_{j=0}^{6} \left(a|j,j,j\rangle + b \sum_{k \in Q} |j+k, j+2k, j+4k\rangle\right). \tag{10}$$

Here Q is the set $\{1,2,4\}$, and all the arithmetic shown in the ket labels is mod 7, the basis states of each particle being labeled by the integers $0,\ldots,6$. We take a and b to be real and positive, with $a^2 + 3b^2 = 1$ to ensure normalization. Thus the state $|\zeta\rangle$ is completely specified once the value of a is given.

We have chosen Q to consist of the *quadratic residues* mod 7, that is, the elements of $\{1,2,3,4,5,6\}$ that can be written as x^2 mod 7 for some integer x. The properties of quadratic residues [13] tend to minimize the overlap, in each particle's state space, between terms in Eq. (10) with different values of j. (For this it is important that 7 is a prime of the form $4N-1$ with integral N.)

A particular symmetry of $|\zeta\rangle$ shows immediately that the pairs AB, BC, and CA are all equally entangled. If we define a new summation index k' by $k = 2k'$ mod 7, then Eq. (10) becomes

$$|\zeta\rangle = \frac{1}{\sqrt{7}} \sum_{j=0}^{6} \left(a|j,j,j\rangle + b \sum_{k' \in Q} |j+2k', j+4k', j+k'\rangle\right), \tag{11}$$

where we have used the invariance of Q under multiplication by 2 mod 7. But Eq. (11) differs from Eq. (10) in that the ket labels have been cyclically permuted. Thus $|\zeta\rangle$ is invariant under a cyclic permutation of the particles, and it follows that each pair is equally entangled.

To write down the density matrix of one of the pairs, say BC, it is helpful to reexpress Eq. (10) in yet another form, changing the index j in the jk sum to $j' = j+k$ and then relabeling j' as j:

$$|\zeta\rangle = \frac{1}{\sqrt{7}} \sum_{j=0}^{6} \left(a|j,j,j\rangle + b \sum_{k \in Q} |j, j+k, j+3k\rangle\right). \tag{12}$$

The density matrix of BC is the trace of $|\zeta\rangle\langle\zeta|$ over particle A, which we can write as

$$\rho_{BC} = \frac{1}{7} \sum_{j=0}^{6} |s_j\rangle\langle s_j|, \tag{13}$$

where $|s_j\rangle$, defined by

$$|s_j\rangle = a|j,j\rangle + b \sum_{k \in Q} |j+k, j+3k\rangle, \tag{14}$$

is the state of BC associated with the state $|j\rangle$ of A.

In order to find the entanglement of formation $E_f(\rho_{BC})$, we need to consider pure-state decompositions of ρ_{BC} and find their average entanglements. Now, any pure state $|\beta\rangle$ in such a decomposition must be a linear combination of the seven orthogonal states $|s_j\rangle$ that make up ρ_{BC}; that is, it must lie in the seven-dimensional subspace $\mathcal{H}$ spanned by $\{|s_j\rangle\}$:

$$|\beta\rangle = \sum_j \beta_j |s_j\rangle, \tag{15}$$

where $\Sigma_j |\beta_j|^2 = 1$. The problem of finding $E_f(\rho_{BC})$ is simplified by two facts: (i) $E(\rho_{BC})$ cannot be smaller than the smallest entanglement of any $|\beta\rangle \in \mathcal{H}$; that is, $E_f(\rho_{BC}) \geqslant \min_\beta E(\beta)$. (ii) Given any state $|\beta\rangle \in \mathcal{H}$, one can generate an entire decomposition of ρ_{BC} in which *every* element has the same entanglement as $|\beta\rangle$. (We prove this assertion in the following paragraph.) Together, these two facts imply that the entanglement of formation of ρ_{BC} is *equal* to $\min_\beta E_f(\beta)$. Thus it is sufficient to find a single minimally entangled pure state in the subspace $\mathcal{H}$ occupied by ρ_{BC}.

To generate a decomposition of ρ_{BC} from a given state $|\beta\rangle \in \mathcal{H}$, we apply a set of local unitary transformations to

$|\beta\rangle$; such transformations are guaranteed not to change the entanglement. We start by defining two basic single-particle transformations S and T:

$$S|j\rangle=\omega|j\rangle; \quad T|j\rangle=|j+1\rangle, \tag{16}$$

where $\omega=e^{2\pi i/7}$ and, as always, the addition in the ket label is mod 7. In terms of these basic operations, we define a pair of two-particle transformations U and V:

$$U=S^5\otimes S^3; \quad V=T\otimes T. \tag{17}$$

One can show that $U|s_j\rangle=\omega^j|s_j\rangle$ and $V|s_j\rangle=|s_{j+1}\rangle$, from which it follows that

$$\frac{1}{49}\sum_{m=0}^{6}\sum_{p=0}^{6} V^pU^m|\beta\rangle\langle\beta|U^{-m}V^{-p} \tag{18}$$

$$=\frac{1}{49}\sum_{j,j'}\sum_{m,p}\omega^{(j-j')m}\beta_j\beta_{j'}^*|s_{j+p}\rangle\langle s_{j'+p}| \tag{19}$$

$$=\frac{1}{7}\sum_j |\beta_j|^2\sum_p |s_{j+p}\rangle\langle s_{j+p}|=\rho_{BC}. \tag{20}$$

We have thus produced the desired decomposition of ρ_{BC}.

It remains, then, to find the smallest possible value of $E_f(\beta)$. For the special case where $a=b=1/2$, each state $|s_j\rangle$ has exactly two bits of entanglement, but it happens that certain linear combinations of the states $|s_j\rangle$ have slightly smaller entanglement. Using numerical minimization, we find that for this case, $\min E_f(\beta)=1.9933$.

Of course we are free to choose the value of a as we please, and it turns out that we maximize the entanglement of formation by choosing $a=0.461$, in which case $b=0.512$. For this value of a, we find numerically that the minimum $E_f(\beta)$ is 1.9944, obtained both for the simple case $|\beta\rangle=|s_j\rangle$ and for certain nontrivial linear combinations. [One such combination has coefficients β_j equal to $(0.120,0.197,0.689,0.259,-0.468,-0.275,-0.332)$, and the others we have found are all related to this one by permutations and phase changes.] We conclude, then, that for the state $|\zeta\rangle$ with $a=0.461$, the entanglement of formation between each pair of particles is 1.9944, and, therefore, $E_{max}(3,7)\geq 1.9944$.

TABLE I. Lower bounds on E_{max}.

d	$E_{max}^{(bound)}(3,d)$	$E_{max}^{(bound)}(3,d)/\log_2 d$
2	0.550	0.550
3	1.000	0.631
7	1.994	0.710

This result gives us another data point as we consider the dependence of the ratio $E_{max}(3,d)/\log_2 d$ on the dimension d. Table I summarizes what we know so far about the case $n=3$. (For $d=3$ and $d=7$, the values given are lower bounds.)

In the limit as d goes to infinity, we wonder what value, if any, the ratio $E_{max}(3,d)/\log_2 d$ approaches. It is conceivable that the limit is 1, but it is equally conceivable that it is some smaller constant. Either answer would be interesting. If the limit of $E_{max}(n,d)/\log_2 d$ is 1 for all values of n, then one could reasonably say that entanglement can be shared freely in an infinite dimensional state space.

We note that although in this paper we have focused on the entanglement of formation, there exist other sound measures of entanglement, and it is surely a good idea, in trying to quantify the restrictions on the sharing of entanglement, to keep in mind alternatives such as the relative entropy of entanglement [14] and the generalized concurrence of Rungta *et al.* [15]. At the present stage of investigation, it is not clear which measure or measures will yield the most elegant quantitative expressions of the limitations on entanglement sharing.

[1] See, for example, M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000), and entanglement citations therein.

[2] D. Bruß, Phys. Rev. A **60**, 4344 (1999).

[3] V. Coffman, J. Kundu, and W.K. Wootters, Phys. Rev. A **61**, 052306 (2000); e-print quant-ph/9907047.

[4] W. Dür, G. Vidal, and J.I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[5] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A **54**, 3824 (1996).

[6] C.H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J.A. Smolin, and W.K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).

[7] C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).

[8] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[9] S. Hill and W.K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).

[10] M. Koashi, V. Bužek, and N. Imoto, Phys. Rev. A **62**, 050302 (2000).

[11] K.G.H. Vollbrecht and R.F. Werner, e-print quant-ph/0010095.

[12] M. Hillery and V. Bužek, e-print quant-ph/0104107.

[13] F.J. MacWilliams and N.J.A. Sloane, *The Theory of Error-Correcting Codes* (North-Holland, Amsterdam, 1978), pp. 45, 46.

[14] V. Vedral, M.B. Plenio, M.A. Rippin, and P.L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).

[15] P. Rungta, V. Bužek, C.M. Caves, M. Hillery, and G.J. Milburn, e-print quant-ph/0102040.