Tunneling and traversal of ultracold atoms through vacuum-induced potentials

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We examine the passage of ultracold two-level atoms through the potential produced by the vacuum of the cavity field. We find that the phase time may be considered as the appropriate measure of the time required for the atom to traverse the cavity. The phase tunneling time for ultracold atoms exhibits both super- and sub-classical time and we show how this behavior may be understood in terms of the momentum dependence of the phase of transmission amplitude. The passage of the atom through the cavity is unique, as it involves a coherent addition of the transition amplitudes corresponding to both barrier and well.

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I. INTRODUCTION

An important question of great interest in several disciplines of physics has been—what is the tunneling time or traversal time of a quantum-mechanical particle through a potential. Various definitions have been proposed and the subject has been reviewed extensively [1–3]. In this paper, we examine the passage time of a cold atom through a high-quality cavity. The question is a complicated one as we have here a coupling with three different types of the degrees of freedom—(a) atom's center of mass motion, (b) atom's electronic states, and (c) photons. An analysis in the dressed-state basis reveals that the interaction of a moving atom with a single-mode vacuum field in a high-quality cavity is equivalent to a combination of a potential barrier and a well. The connection to potential problems is provided by the existing results in the context of micromazers [4,5]. These potentials belong to the category of vacuum-induced potentials and should be distinguished from the optical potentials produced by a far-off resonant field interacting with an atom [6]. Having realized that the cavity field may act like a potential for an ultracold atom, one could calculate the time the atom takes to traverse the cavity using methods similar to those used, for example, in the context of tunneling electrons through potential barriers and the propagation of light through a dispersive medium. The motional effects in the context of cavity QED are beginning to be seen. Münstermann *et al.* [7] have already reported the evidence of the effect of a quantized motion of atoms in the asymmetries of transmission of a weak light field through a cavity.

II. MODEL SYSTEM AND SUMMARY OF ATOM-FIELD INTERACTION

We consider an ultracold, two-level atom in its excited state to be incident on a single-mode cavity of length L. The frequency of the cavity field has been tuned to the frequency ω of the atomic transition between the excited state $|e\rangle$ and the ground state $|g\rangle$. In a reference frame rotating with frequency ω, the Hamiltonian of the atom-field interaction including the quantized motion of center-of-mass (c.m.) of the atom, is given by

$$H_I=\frac{p_z^2}{2m}+\hbar g u(z)(\sigma a^\dagger+a\sigma^\dagger), \tag{1}$$

where g is the atom-field coupling constant and σ $(\sigma^\dagger)$ are the lowering (raising) operators for the atomic transition. The operators a $(a^\dagger)$ annihilate (create) a photon of frequency ω. For simplicity, the mode function $u(z)$ of the cavity is assumed to be a mesa function $\theta(z)\theta(L-z)$. The operator $(\sigma a^\dagger+a\sigma^\dagger)$ is easily diagonalizable. It has eigenstates $|\phi^0\rangle$, $|\phi^\pm_{n+1}\rangle$ with eigenvalues 0, $\pm\sqrt{n+1}$, respectively. The dressed eigenstates may be expanded in terms of eigenstates of the free Hamiltonian as $|\phi^0\rangle=|g,0\rangle$ and $|\phi^\pm_{n+1}\rangle=1/\sqrt{2}(|e,n\rangle\pm|g,n+1\rangle)$.

Since we need the transmission amplitude of the excited atom for further discussion, we summarize the main results of Meyer *et al.* [4]. Consider the initial atom-field state to be $|e,n\rangle$, i.e., the atom is in the excited state and the cavity field contains fixed number (n) of photons. If we expand the combined state of the atom-cavity system as

$$|\Psi(z,t)\rangle=\chi_+(z,t)|\phi^+_{n+1}\rangle+\chi_-(z,t)|\phi^-_{n+1}\rangle, \tag{2}$$

then the time-dependent Schrödinger equation becomes

$$i\hbar\frac{\partial\chi_\alpha}{\partial t}=h_\alpha\chi_\alpha, \quad \alpha=\pm. \tag{3}$$

Here, $h_\pm=p_z^2/2m\pm\hbar g u(z)\sqrt{n+1}$ are operators acting in the space of the center-of-mass variables. Clearly, the cavity with fixed number of photons creates a barrier and a well potential for the external motion of the atom corresponding to the dressed states $|\phi^\pm_{n+1}\rangle$, respectively, as discussed in Ref. [4].

We assume the initial state of the cavity to be vacuum $(n=0)$ state. The initial wave packet of a moving free atom may be written in the form $\psi(z,t)=\exp(-ip_z^2t/2m\hbar)\int dk\,A(k)e^{ikz}=\int dk\,A(k)e^{-i(\hbar k^2/2m)t}e^{ikz}$. We assume that $A(k)$'s are such that $\psi(z,t)$ at $z=0$ peaks in time at the instant $t=0$. Thus, in the presence of the cavity, the wave packet at $z=0$ (entry of the cavity) has its peak (in time) at $t=0$. We therefore write the initial wave function of the atom-field system as $|\Psi(z,0)\rangle=\psi(z,0)|e,0\rangle$. The wave function of the atom-field system after the interaction may be obtained by expanding $|e,0\rangle$ in the dressed-state basis $|\phi_1^{\pm}\rangle$ [4]. We use the reflection and transmission amplitudes $\rho^{\pm}$, $\tau^{\pm}$ of Ref. [5] for the potential barrier (superscript $+$) and well (superscript $-$), respectively. Carrying out the time evolution for the dressed states $|\phi_1^{\pm}\rangle$, we get the following transmitted wave function in the region $z \geqslant L$:

$$|\Psi_T(z,t)\rangle = \int dk\, A(k) e^{-i(\hbar k^2/2m)t}[T_{e,0}(k)|e,0\rangle + T_{g,1}(k)|g,1\rangle] e^{ikz}, \tag{4}$$

where

$$T_{e,0} = \frac{1}{2}(\tau^+ + \tau^-), \quad T_{g,1} = \frac{1}{2}(\tau^+ - \tau^-), \tag{5}$$

are the transmission amplitudes for the excited and ground state of the atom, respectively. Note that the transmission amplitudes for the excited or ground state of the atom depend on the coherent addition of amplitudes of the barrier and well. The reflected wave function of the atom-field system in the region $z \leqslant 0$ is obtained by replacing the transmission amplitudes $\tau^{\pm}$ by reflection amplitudes $\rho^{\pm}$ of the barrier and well.

III. PHASE TIME FOR ULTRACOLD ATOMS PASSING THROUGH A HIGH-QUALITY CAVITY—ANALOG OF SUB- AND SUPERLUMINAL PROPAGATION

In the previous section, we have seen that dynamics of an ultracold atom passing through the cavity is reduced to the problem of reflection and transmission of an atom incident on the cavity-induced potentials. In this section, we study in detail the transmission of the atom in the initial excited state through the cavity initially in vacuum state. The transmission amplitude $T_{e,0} \equiv |T_{e,0}| e^{i\phi(k)}$, given by Eq. (5), depends on the vacuum coupling energy $\hbar g \equiv \hbar^2 k_o^2/2m$. We consider a Gaussian wave-packet $A(k) = \exp(-(k-\overline{k})^2/\sigma^2)$ of width σ and mean-momentum $\overline{k}$ for the incident atom. With this substitution for $A(k)$, the transmitted wave function including the normalization factor, is given for $z \geqslant L$ by

$$|\Psi_T(z,t)\rangle = \frac{1}{(2\pi)^{3/4}} \sqrt{\frac{2}{\sigma}} \int_{-\infty}^{\infty} dk \exp(-(k-\overline{k})^2/\sigma^2) \times e^{-i(\hbar k^2/2m)t} |T_{e,0}| e^{i\phi(k)} e^{ikz} |e,0\rangle. \tag{6}$$

For small width σ, the integrand in Eq. (6) has nonvanishing value only in a small range of wave-numbers k centered about the mean $\overline{k}$. Then, the envelope of the transmitted wave-packet $|\langle e,0|\Psi_T(z,t)\rangle|^2$ will be maximum when the total phase $\Theta(k)$ of the integrand exhibits extremum at the wave-number $k = \overline{k}$. Since we have assumed that the peak of incident wave packet enters the cavity at time $t=0$, this stationary phase condition gives the time at which the wave packet at the exit of the cavity $z=L$ is peaked as follows:

$$\left.\frac{\partial \Theta(k)}{\partial k}\right|_{k=\overline{k}} = \left.\frac{\partial}{\partial k}[kL + \phi(k) - (\hbar k^2/2m)t]\right|_{k=\overline{k}} = 0, \tag{7}$$

which yields the phase tunneling time t_{ph}

$$t_{ph} = \left[\frac{m}{\hbar k}\left(\frac{\partial \phi}{\partial k} + L\right)\right]_{k=\overline{k}}. \tag{8}$$

The integral in Eq. (6) may be evaluated approximately by making the Taylor expansion of the phase of transmission amplitude about the mean-wave number $k = \overline{k}$. Keeping terms up to second order in the expansion and assuming $\sigma \ll \overline{k}$ to approximate $|T_{e,0}(k)| \approx |T_{e,0}(\overline{k})|$, the transmitted wave function is given at $z=L$ by

$$|\Psi_T(z,t)\rangle|_{z=L} \approx \frac{1}{(2\pi)^{3/4}} \sqrt{\frac{2}{\sigma}} \exp[i(\overline{k}L + \phi(\overline{k}) - \overline{E}t/\hbar)] |T_{e,0}(\overline{k})| \times \sqrt{\frac{2\pi}{\left(\frac{2}{\sigma^2} + i\alpha\right)}} \exp\left(\frac{-\overline{E}(t-t_{ph})^2}{m\left(\frac{2}{\sigma^2} + i\alpha\right)}\right) |e,0\rangle, \tag{9}$$

where $\overline{E} = \hbar^2 \overline{k^2}/2m$ is the average energy of the incident atom and the parameter $\alpha = \hbar t/m - \partial^2\phi/\partial k^2|_{k=\overline{k}}$ accounts for the spreading of the wave packet as it propagates. The maximum amplitude of the transmitted wave packet, i.e., of $|\langle e,0|\Psi_T(L,t)\rangle|^2$ occurs at time $t = t_{ph}$ given by the stationary phase assumption. It is very important to note that the phase time has no significance when either the Taylor expansion of the phase does not converge or additional terms more than the second-order term are important in the expansion. In this general case, the transmitted wave packet will be deformed from the Gaussian shape and the concept of following the peak of the wave packet is meaningless. When there is no cavity $|T_{e,0}(k)| = 1$, $\phi(k) = 0$, then the phase time in Eq. (8) becomes $t_{ph} = mL/\hbar\overline{k} \equiv t_{cl}$, which is the classical time needed for the peak of a free-atomic wave packet to traverse a distance of length L. The phase tunneling time that a particle takes to traverse a *potential barrier*, has been studied extensively by Hartman [1]. The tunneling time for a barrier is less than the time a free particle takes to traverse the same distance in free space. Here, we report such a superclassical traversal of the ultracold atom through the vacuum-induced potentials. Note that the temperature of the atom will be in the range 10^{-7}–10^{-8} K if the coupling constant g ($\equiv \hbar k_o^2/2m$) is in the range of 100–10 kHz and if the mean momentum $\overline{k}/k_o = 0.1$. It should be borne in mind that both barrier and well contribute to the traversal time of ultracold atoms. Using Eq. (8), we plot in Fig. 1 the phase time as a function of the mean wave-number $\overline{k}$ for the length of the cavity $k_o L = 10\pi$. The important result here is that the phase time exhibits the resonant behavior of transmission probability and that the phase time is less than the classical time t_{cl}. In a different context, viz., in the tunneling time of

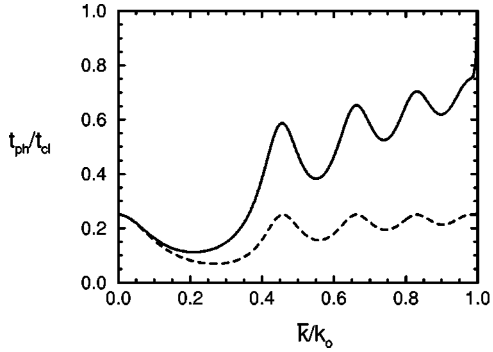

FIG. 1. Dependence of the dimensionless phase time (solid curve) for transmission in the excited state on the mean wave-number $\overline{k}/k_o$ of the incident atom for the parameter $k_oL=10\pi$. The phase time follows the resonant behavior of the transmission probability $|T_{e,0}|^2$ (dashed curve).

electrons passing through a finite superlattice, a similar resonant behavior is found [3]. Another remarkable behavior of phase time is that it may even be *negative*. Negative phase time implies that the peak of the transmitted wave packet emerges even before the peak of the incident wave packet enters the interaction region. This may be understood from the interference between the incident wave and the wave that is reflected at the end of the cavity. From Eq. (8), we see that when the derivative of the phase of transmission amplitude is negative and its absolute value is greater than the length L of the cavity, the phase time becomes negative. Put another way, when the phase function $\phi(k)+kL$ has negative slope, the phase time takes negative values. In Fig. 2, we show the phase time for the parameter $k_oL=\pi/2$. It is seen from the graph that for ultracold atoms ($\overline{k}/k_o \ll 1$), the phase time is negative. For fast atoms ($\overline{k}/k_o \gg 1$), the phase time approaches the classical time as the transmission probability

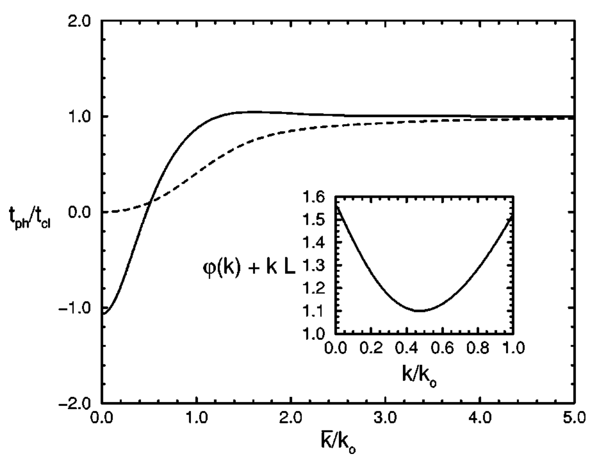

FIG. 2. Dimensionless phase time (solid curve) for transmission in the excited state as a function of the mean wave-number $\overline{k}/k_o$ of the incident atom for the parameter $k_oL=\pi/2$. The dashed curve represents the probability of transmission of the atom in the initial excited state ($|T_{e,0}|^2$) through the cavity. The inset shows the phase function $\phi+kL$ as a function of the wave-number k/k_o of the excited atom for the same parameter.

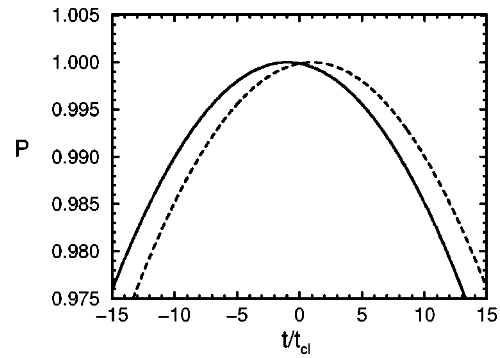

FIG. 3. Normalized probability density $P\equiv|\langle e,0|\Psi_T(z,t)\rangle|^2/\sigma$ at $z=L$ as a function of dimensionless time t/t_{cl}. The solid (dashed) curve represents P after transmission through the cavity (free space). The parameters used for the calculation are $k_oL=\pi/2$, $\sigma/k_o=0.01$, and $\overline{k}/k_o=0.1$. Both the solid and dashed curves are normalized to unity.

becomes closer to unity. The phase time being negative is very similar to the concept of negative group velocity in the case of electromagnetic pulse propagation. Here, the variation of the refractive index of the medium with respect to the frequency has a steep negative slope leading to superluminal propagation [8]. To understand the negative phase time, we have also plotted the phase function $\phi(k)+kL$ in the inset of Fig. 2. The graph shows the expected negative slope for ultracold atoms.

We now substantiate the above results by studying the behavior of the actual envelope of the wave function. We evaluate numerically the integral Eq. (6), which describes the propagation of a Gaussian wave packet of an excited atom through the vacuum-induced potentials. Garrett and McCumber [9] carried out a similar numerical integration for the electric-field amplitude of a Gaussian light pulse passing through an anomalous dispersive medium. In Fig. 3, we show the numerical result of the normalized probability density $|\langle e,0|\Psi_T(z,t)\rangle|^2/\sigma$ at the exit of the cavity $z=L$ as a function of the time for the parameters $k_oL=\pi/2$, $\sigma/k_o=0.01$, $\overline{k}/k_o=0.1$. The peak of the transmitted wave packet occurs at the time $t/t_{cl}\approx-0.98$, which matches with the phase time in Fig. 2 for the cold atom ($\overline{k}/k_o=0.1$). For comparison, we have also plotted the envelope of the wave packet that travels through the same distance of length L in free space. The peak of the free wave packet occurs at the expected classical time. In the case of fast atoms ($\overline{k}/k_o\gg1$), numerical integration (actual results not shown) gives the peak of the transmitted wave packet at the classical time ($t/t_{cl}\approx1$) as expected from Fig. 2. Thus, the peak of the transmitted wave packet occurs at the instant given by the expression for phase-time Eq. (8), even if that instant is earlier than the instant at which the incident wave packet enters the cavity. While this is generally true for a narrow momentum distribution characterized by $\sigma\ll\overline{k}$ of the incident atom, strong deformation of the incident wave packet sometimes makes the phase time meaningless [10].

We have so far considered only the propagation of the atomic wave packet in the initial excited state. But in a high-

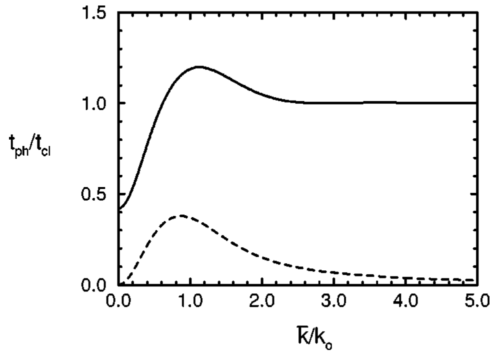

FIG. 4. Dimensionless phase time (solid curve) for transmission in the ground state as a function of the mean wave-number $\overline{k}/k_o$ of the incident atom for the parameter $k_oL=\pi/2$. The dashed curve represents the probability of transmission of the atom in the ground state ($|T_{g,1}|^2$) through the cavity.

quality cavity, the atom-field interaction leads to photon emission by the excited atom. We may also study the behavior of the transmitted wave-packet $|\langle g,1|\Psi_T(z,t)\rangle|^2$ for the ground state of the atom using Eq. (4). For the parameters of Fig. 3, the phase time for the ground-state $t_{ph}/t_{cl}\approx 0.45$ is positive but still a superclassical time. Numerical integration (results not shown) also gives the same time delay for the transmitted wave packet. In Fig. 4, we show the behavior of the phase time for the wave packet corresponding to the transmitted atom in the ground state. This behavior is to be compared with that of the phase time for the transmission in the excited state (Fig. 2). The two phase times differ considerably for cold atoms. Generally, the difference in phase times for the ground and excited states of the atom results in the splitting of the incident wave packet into two in the total transmission. But for the parameters of Fig. 3, the total transmission is dominated by the contribution from the ground state, and hence, the splitting is not seen.

IV. CONCLUSIONS

In summary, we have considered the propagation of a Gaussian wave packet of an excited two-level atom through a high-quality cavity that is initially empty. The tunneling time depends on the coherent addition of transmission amplitudes through a barrier and a well. The phase tunneling time may exhibit both super- and subclassical traversal behavior. For certain sets of parameter, the phase tunneling time for cold atoms may even be negative. All this may be understood in terms of the dispersion characteristics of the phase of the transmission amplitude and is analogous to the dispersion of the refractive index that leads to super- and subluminal propagation [8,9]. Though we have considered the vacuum state for the initial state of the field, superclassical tunneling of ultracold atoms is a common feature for a general Fock state of the cavity field.

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[10] It should be borne in mind that all this discussion is predicted on the assumption that the modulus of the transmission amplitude is a slowly varying function of *k*. Very sharp resonances have to be handled differently.