Conclusive teleportation of a *d*-dimensional unknown state

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We formulate a conclusive teleportation protocol for a system in *d*-dimensional Hilbert space utilizing the positive operator-valued measurement. The conclusive teleportation protocol ensures some perfect teleportation events when the channel is only partially entangled, at the expense of lowering the overall average fidelity. We discuss how much information remains in the inconclusive parts of the teleportation.

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Recently, Mor and Horodecki [1] proposed a conclusive teleportation protocol for a quantum state in two-dimensional Hilbert space for a partially entangled quantum channel. They employed a positive operator-valued measurement (POVM) [2] as the joint measurement at the sending station. When the measurement is successful by a random chance, the initially unknown quantum state is teleported perfectly, which is called a *conclusive* event. In contrast, for an *inconclusive* event, the teleportation loses its quantum characteristics. To indicate the success of the measurement, an extra single bit has to be sent to the receiver together with the two-bit information via a classical channel. Bandyopadhyay [3] and Li *et al.* [4] have also proposed protocols to implement conclusive teleportation in two-dimensional Hilbert space. Bandyopadhyay uses a combination of orthogonal Greenberger-Horne-Zeilinger measurements and POVM's while Li *et al.* take general transformations at the receiving station leaving the measurement orthogonal.

Quantum information theory has been extensively developed for a two-level spin-1/2 system, namely, quantum bit or *qubit*. It is only recently that *d*-dimensional quantum systems have attracted a considerable research effort. In particular, discrimination between linearly independent quantum states in *d* dimensions and its application to entanglement concentration have been studied by Chefles and Barnett [5], quantum cloning by Zanardi [6], and quantum teleportation by Zubairy [7] and Stenholm and Bardroff [8]. Rungta *et al.* called the *d*-dimensional quantum system the *qudit* and investigated its entanglement and separability [9]. The qudit was extensively studied as a finite-dimensional version of a continuous variable state by Gottesman *et al.* [10].

In this paper, we formulate the conclusive teleportation in *d*-dimensional Hilbert space for partially entangled quantum channel, utilizing rank-one positive operator-valued measurement for the joint measurement at the sending station and unitary transformation at the receiving station. We show that some teleportation information can be extracted from the inconclusive outcome by evaluating the teleportation fidelity. The maximum information the receiver recovers for the inconclusive outcome is limited by classical theory [11]. It is found that the overall fidelity for conclusive teleportation is less than that for the standard teleportation, which implies that the conclusive teleportation enables perfect teleportation events at the expense of lowering the overall average fidelity.

A quantum state $|\phi\rangle_1$ in the *d*-dimensional Hilbert space $\mathcal{H}$,

$$|\phi\rangle_1 = \sum_{i=1}^{d} c_i |i\rangle_1, \tag{1}$$

is teleported to a remote place via a partially entangled channel in the $(d\times d)$-dimensional Hilbert space, $\mathcal{H}\otimes\mathcal{H}$. We assume that the quantum channel is prepared with a *pure entangled pair* of particles 2 and 3 in the state

$$|\psi\rangle_{23} = \sum_{i=1}^{d} a_i |ii\rangle_{23}, \tag{2}$$

where $\{|i\rangle\}$ is an orthonormal basis set in *d*-dimensional Hilbert space $\mathcal{H}$. All real coefficients a_i are nonzero. For a maximally entangled quantum channel, $a_i = d^{-1/2}$. The total quantum state for the unknown particle 1 and the entangled pair 2 and 3 is given by the direct product of the unknown state $|\phi\rangle_1$ and the quantum channel state $|\psi\rangle_{23}$,

$$|\Psi\rangle_{123} = |\phi\rangle_1 \otimes |\psi\rangle_{23}. \tag{3}$$

The sender performs a joint measurement on particles 1 and 2 so that we expand the composite system based on states for particles 1 and 2.

In the standard teleportation protocol suggested by Bennett *et al.* [12], the joint measurement is based on Bell states, which form an orthonormal basis set of the maximally entangled states $\{|\psi_\alpha^m\rangle\}$ for a spin-1/2 system. This basis set can be obtained by applying a set of local unitary operations $\{\hat{U}^\alpha\}$ on the given maximally entangled state $|\psi^m\rangle$: $|\psi_\alpha^m\rangle = \hat{U}^\alpha \otimes \mathbb{1}|\psi^m\rangle$ [12,13]. For a spin-1/2 system the set of unitary operators is given by $\{\mathbb{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$. These results are extended to the *d*-dimensional quantum teleportation [13]. The orthonormal basis set should satisfy the completeness,

$$\mathbb{1}_{d\times d} = \sum_{\alpha=1}^{d^2} |\psi_\alpha^m\rangle\langle\psi_\alpha^m| = \sum_{\alpha=1}^{d^2} \hat{U}^\alpha \otimes \mathbb{1}|\psi^m\rangle\langle\psi^m|\hat{U}^{\alpha^\dagger}\otimes\mathbb{1}, \tag{4}$$

where $\mathbb{1}_{d\times d}=\mathbb{1}\otimes\mathbb{1}$ with the identity operator $\mathbb{1}$ in $\mathcal{H}$. The completeness can be written in the matrix form based on the orthonormal bases of $\{|ij\rangle\}$,

$$\delta_{ik}\delta_{jl}=\frac{1}{d}\sum_{\alpha=1}^{d^2} U_{ij}^{\alpha}U_{kl}^{\alpha *}, \tag{5}$$

where $U_{ij}^{\alpha}\equiv\langle i|\hat{U}^{\alpha}|j\rangle$. Note that Eq. (5) depends only on the unitary operators. In addition the orthonormality of the entangled basis states $\delta_{\alpha\beta}=\langle\psi_{\alpha}^{m}|\psi_{\beta}^{m}\rangle$ leads to the orthogonal condition for the unitary operators,

$$\mathrm{Tr}\,\hat{U}^{\alpha\dagger}\hat{U}^{\beta}=d\quad\delta_{\alpha\beta}. \tag{6}$$

In the conclusive teleportation protocol, the basis set of entangled states is obtained by using the same local unitary operators $\{\hat{U}^{\alpha}\}$ on the state of the quantum channel (2),

$$|\psi_{\alpha}\rangle=\hat{U}^{\alpha}\otimes\mathbb{1}|\psi\rangle \tag{7}$$

for $\alpha=1,2,\dots,d^2$. Note that d^2 state vectors (7) form a basis set because they are linearly independent as shown later. The basis states $|\psi_{\alpha}\rangle$ are not necessarily orthogonal. Only when the channel is maximally entangled with $a_i=d^{-1/2}$ the set of the basis states $\{|\psi_{\alpha}\rangle\}$ becomes $\{|\psi_{\alpha}^{m}\rangle\}$, which represents the von Neumann orthogonal measurement.

It is possible to expand the state $|\psi_{\alpha}\rangle$ in the orthonormal bases $|ij\rangle$,

$$|\psi_{\alpha}\rangle=\sum_{i,j=1}^{d}\Gamma(\alpha;i,j)|ij\rangle, \tag{8}$$

where $\Gamma(\alpha;i,j)$ is calculated from Eqs. (2) and (7) as

$$\Gamma(\alpha;i,j)=U_{ij}^{\alpha}a_j. \tag{9}$$

The map Γ performs the basis transformation from $\{|ij\rangle\}$ to $\{|\psi_{\alpha}\rangle\}$. We define a new map Γ^{-1} as

$$\Gamma^{-1}(i,j;\alpha)=\frac{1}{d}U_{ij}^{\alpha *}a_j^{-1}, \tag{10}$$

which is well defined because $a_j\neq 0$ for all j in Eq. (2). It is then straightforward to show that Γ^{-1} is the inverse map of Γ such that

$$\sum_{\alpha=1}^{d^2}\Gamma^{-1}(i,j;\alpha)\Gamma(\alpha;k,l)=\delta_{ik}\delta_{jl} \tag{11}$$

and

$$\sum_{ij=1}^{d}\Gamma(\alpha;i,j)\Gamma^{-1}(i,j;\beta)=\delta_{\alpha\beta}, \tag{12}$$

which result from Eqs. (9) and (10) with use of the completeness equation (5) and the orthogonality of unitary operators, Eq. (6). The existence of Γ^{-1} shows that the d^2 state vectors $\{|\psi_{\alpha}\rangle\}$ form a basis set. Using $\Gamma^{-1}(i,j;\alpha)$, the inverse relation to Eq. (8) follows:

$$|ij\rangle=\sum_{\alpha=1}^{d^2}\Gamma^{-1}(i,j;\alpha)|\psi_{\alpha}\rangle. \tag{13}$$

The completeness relation with respect to the entangled states $\{|\psi_{\alpha}\rangle\}$ is nontrivial due to their being nonorthogonal for a partially entangled quantum channel. Instead, we modify the completeness for the set of orthonormal bases $\{|ij\rangle\}$ using Eq. (13) as

$$\mathbb{1}_{d\times d}=\sum_{ij=1}^{d}|ij\rangle\langle ij|=\sum_{\alpha=1}^{d^2}\sum_{ij=1}^{d}\Gamma^{-1}(i,j;\alpha)|\psi_{\alpha}\rangle\langle ij|. \tag{14}$$

The total state $|\Psi\rangle_{123}$ in Eq. (3) can now be written with help of the modified completeness equation (14) as

$$\begin{aligned}|\Psi\rangle_{123}&=\left[\sum_{\alpha ij}\Gamma^{-1}(i,j;\alpha)|\psi_{\alpha}\rangle_{12}\langle ij|\right]|\phi\rangle_1\otimes|\psi\rangle_{23}\\&=\frac{1}{d}\sum_{\alpha=1}^{d^2}|\psi_{\alpha}\rangle_{12}\otimes\hat{U}^{\alpha\dagger}|\phi\rangle_3.\end{aligned} \tag{15}$$

The second equality is given by Eq. (10) and the orthonormality of the bases $\{|ij\rangle\}$. The von Neumann orthogonal measurement with the maximally entangled bases of $\{|\psi_{\alpha}^{m}\rangle\}$ cannot exactly discern the nonorthogonal state vectors $|\psi_{\alpha}\rangle$ and the teleportation is no longer perfect for the partially entangled quantum channel.

A POVM for the joint measurement is defined as a partition of unity by the positive operators, which are in general nonorthogonal. A set of POVM operators $\{\hat{M}_{\alpha}\}$ with $n>d^2$ outcomes satisfy the measurement conditions of positivity and completeness,

$$\sum_{\alpha=1}^{n}\hat{M}_{\alpha}=\mathbb{1}_{d\times d}. \tag{16}$$

The conclusive teleportation has a crucial step to discern nonorthogonal basis states $|\psi_{\beta}\rangle$ for $\beta=1,2,\dots,d^2$ in Eq. (15). For this purpose, joint POVM operators are designed such that

$$\langle\psi_{\beta}|\hat{M}_{\alpha}|\psi_{\beta}\rangle\propto\delta_{\alpha\beta}\quad\text{for}\ \ \alpha\leqslant d^2. \tag{17}$$

This implies that when the measurement outcome is due to any of $\hat{M}_{\alpha}$ for $\alpha\leqslant d^2$, we can conclusively discern the nonorthogonal states in Eq. (15). On the other hand, the measurement bears inconclusive results when the outcome is of $\hat{M}_{\alpha}$ for $\alpha>d^2$.

Any set of POVM operators can be decomposed into rank-one general projectors [2]. We present the conclusive measurement operators by general projectors in the form of [14]

$$\hat{M}_{\alpha}=\lambda|\tilde{\psi}_{\alpha}\rangle\langle\tilde{\psi}_{\alpha}|\quad\text{for}\ \ \alpha\leqslant d^2, \tag{18}$$

where the real parameter $\lambda \geqslant 0$. The completeness can be guaranteed by adding an inconclusive measurement operator $\hat{M}_{d^2+1}$,

$$\hat{M}_{d^2+1} = \mathbb{1}_{d\times d} - \sum_{\alpha=1}^{d^2} \hat{M}_\alpha . \tag{19}$$

Note that $\hat{M}_{d^2+1}$ is, in general, a mixture of rank-one projectors and it can also be decomposed into rank-one general projectors. The positive real parameters λ in Eq. (18) are constrained such that $\hat{M}_{d^2+1}$ satisfies the positivity under the completeness equation (16),

$$\langle \psi | \hat{M}_{d^2+1} | \psi \rangle \geqslant 0 \quad \text{for } \forall |\psi\rangle. \tag{20}$$

Substituting Eqs. (18) and (19) into Eq. (20), the positivity condition for the inconclusive operator leads to

$$\sum_{\alpha=1}^{d^2} \lambda |\langle \tilde{\psi}_\alpha | \psi \rangle|^2 \leqslant 1 \quad \text{for} \quad \forall \ |\psi\rangle. \tag{21}$$

For the purpose of the discrimination (17), the generally nonorthogonal and unnormalized states $\{|\tilde{\psi}_\alpha\rangle\}$ in Eq. (18) are conditioned to satisfy the relation

$$\langle \tilde{\psi}_\alpha | \psi_\beta \rangle = \delta_{\alpha\beta} . \tag{22}$$

To find its explicit form, we expand $|\tilde{\psi}_\alpha\rangle$ in the orthogonal basis $|ij\rangle$,

$$|\tilde{\psi}_\alpha\rangle = \sum_{i,j=1}^{d} \tilde{\Gamma}(\alpha;i,j) |ij\rangle. \tag{23}$$

The map $\tilde{\Gamma}$ is obtained from the relation between Γ and Γ^{-1} in Eqs. (9) and (10) as

$$\begin{aligned} \tilde{\Gamma}(\alpha;i,j) &= \Gamma^{-1*}(i,j;\alpha) \\ &= \frac{1}{d} U_{ij}^{\alpha} a_j^{-1} . \end{aligned} \tag{24}$$

Then, the unnormalized states $|\tilde{\psi}_\alpha\rangle$ satisfy the condition (22).

The probability for each conclusive event $\alpha \leqslant d^2$ is given by

$$p_\alpha = {}_{123}\langle \Psi | \hat{M}_\alpha | \Psi \rangle_{123} = \frac{\lambda}{d^2}. \tag{25}$$

For the outcome α with the probability p_α, the particle 3 is in the state $\hat{U}^{\alpha\dagger}|\phi\rangle_3$. Performing the unitary operation $\hat{U}^\alpha$ by the receiver completes the conclusive teleportation. The overall probability P_{con} for conclusive events is given by the summation of p_α over conclusive events,

$$P_{con} = \sum_{\alpha=1}^{d^2} p_\alpha = \frac{\lambda}{d}. \tag{26}$$

The inconclusive event occurs with the probability

$$p_{d^2+1} = {}_{123}\langle \Psi | \hat{M}_{d^2+1} | \Psi \rangle_{123} = 1 - P_{con} . \tag{27}$$

The maximum of P_{con} is determined by the range of the positive real parameter λ, which is constrained by the positivity of inconclusive operator. Instead of calculating Eq. (21), the diagonal elements of the inconclusive operator are used. With use of the orthogonality of unitary operators, Eq. (5), the inconclusive measurement operator is found as

$$\hat{M}_{d^2+1} = \sum_{i,j=1}^{d} \left(1 - \frac{\lambda}{a_j^2 d} \right) |ij\rangle\langle ij|. \tag{28}$$

This is positive only when $\lambda \leqslant a_j^2 d$ for all $j = 1,2,\ldots,d$. Let a_s^2 be the smallest among a_j^2, then the condition $0 < \lambda \leqslant a_s^2 d$ ensures the positivity. We have thus found that the maximal conclusive-event probability,

$$P_{con}^m = a_s^2 d, \tag{29}$$

which is in agreement with the results obtained by Chefles and Barnett [5].

The proposed POVM enables to discern nonorthogonal states $|\psi_\alpha\rangle$ with finite probability $p_\alpha = \lambda/d^2$ for each conclusive event. When it is employed for the joint measurement, we have a nonzero probability to teleport faithfully. When the measurement outcome is inconclusive, we simply repeat the protocol till a conclusive result is obtained. Our protocol for the conclusive teleportation has two distinct components from standard teleportation: POVM for joint measurement and additional classical communication whether the event is conclusive or not. The additional classical communication requires a single classical bit.

The information transfer by conclusive teleportation is compared to standard teleportation in terms of average fidelity. The fidelity $\mathcal{F}$ is defined by the overlap between the original state $|\phi\rangle$ and the evolved state $\hat{\rho}$; $\mathcal{F} = \langle \phi | \hat{\rho} | \phi \rangle$. When the quantum channel is pure, a pure state is recovered at the receiving station after performing one teleportation procedure. The teleported pure state of the density operator $\hat{\rho}_\alpha$ is dependent on the measurement outcome, here, indexed α, at the sending station. After executing the teleportation protocol infinitely, the ensemble of teleported quantum system is represented by a density operator $\hat{\rho} = \Sigma_\alpha p_\alpha \hat{\rho}_\alpha$, where the measurement bears the outcome indexed α with the probability p_α. The fidelity can thus be given by $\mathcal{F} = \Sigma_\alpha p_\alpha \langle \phi | \hat{\rho}_\alpha | \phi \rangle$. In quantum teleportation, the original state $|\phi\rangle$ is unknown so that it is necessary to average the fidelity over all possible unknown states. The average fidelity is

$$\overline{\mathcal{F}} \equiv \frac{1}{V} \int d\vec{\Omega} \sum_\alpha p_\alpha(\vec{\Omega}) f_\alpha(\vec{\Omega}), \tag{30}$$

where $f_\alpha(\vec{\Omega}) = \langle \phi(\vec{\Omega}) | \hat{\rho}_\alpha | \phi(\vec{\Omega}) \rangle$ and an unknown pure states $|\phi(\vec{\Omega})\rangle$ is parametrized by a real vector $\vec{\Omega}$ in the parameter space of volume V [15,16].

In conclusive teleportation, we know that faithful teleportation is assured at the conclusive event. Even though the inconclusive result is not of use in quantum sense, the receiver may still try to recover some information on the original unknown state. We note that $\hat{M}_{d^2+1}$ in Eq. (28) is diagonal and a convex combination of projectors. It is thus possible to decompose the operator $\hat{M}_{d^2+1}$ further into d^2 general projectors such that the new set of POVM operators is represented by

$$\hat{M}'_\alpha = \lambda' |\tilde{\psi}'_\alpha\rangle\langle\tilde{\psi}'_\alpha|, \tag{31}$$

where

$$|\tilde{\psi}'_\alpha\rangle = \begin{cases} |\tilde{\psi}_\alpha\rangle & \text{for} \quad \alpha \leqslant d^2 \\ |ij\rangle & \text{for} \quad \alpha = d^2+(j-1)d+i, \end{cases}$$

$$\lambda' = \begin{cases} d a_s^2 & \text{for} \quad \alpha \leqslant d^2 \\ 1-a_s^2/a_j^2 & \text{for} \quad \alpha = d^2+(j-1)d+i. \end{cases} \tag{32}$$

The average fidelity is calculated by allowing the set of POVM (31) at sending station and unitary transformation at receiving station. The average fidelity for the conclusive teleportation is upper bounded by

$$\overline{\mathcal{F}} = P^m_{con}\overline{f}_{con} + (1-P^m_{con})\overline{f}_{inc}, \tag{33}$$

where $\overline{f}_{con}$ ($\overline{f}_{inc}$) denotes the average fidelity for the conclusive (inconclusive) event; $\overline{f}_{con}=1$ while $\overline{f}_{inc}$ is equal to $2/(d+1)$, which is the maximal fidelity for a classical teleportation [17]. The average fidelity for standard teleportation is upper bounded by [15]

$$\overline{\mathcal{F}}_s = \frac{1}{d+1}\left[1+\left(\sum_{i=1}^{d} a_i\right)^2\right]. \tag{34}$$

For the maximally entangled channel, both upper bounds coincide exactly. However, for the partially entangled channel,

$$\overline{\mathcal{F}} = \frac{2+d(d-1)a_s^2}{d+1} \leqslant \overline{\mathcal{F}}_s = \frac{2+\sum_{i\neq j} a_i a_j}{d+1} \tag{35}$$

since $a_s \leqslant a_j$ for all j. This shows that the average fidelity for conclusive teleportation, based on Eq. (31), is less than for standard teleportation.

We have formulated the conclusive teleportation protocol of an unknown state in the d-dimensional Hilbert space utilizing the joint POVM. The systematic scheme is presented by the discrimination of the nonorthogonal states by POVM operators. By conclusive teleportation one can teleport perfectly the unknown quantum state with finite probability. Instead of the conclusive teleportation one can choose a two-step procedure: After an entanglement concentration [5], the standard teleportation is performed using the maximally entangled quantum channel. In this case the concentration probability is the same as the success probability of conclusive teleportation. By the study of conclusive teleportation we have found that some information of the unknown state can be extracted from inconclusive events.

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