Generation of maximally entangled photonic states with a quantum-optical Fredkin gate

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When a quantum-optical Fredkin gate is embedded into a Mach-Zehnder interferometer, state reduction techniques permit the generation of maximally entangled states of the radiation field when Fock states are input to the device. These states exactly reach the Heisenberg limit of phase sensitivity. We investigate the consequences of injecting more general states, and particularly coherent states, into the apparatus. Applications to interferometry and photolithography are discussed.

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Beyond long-standing interest in their fundamental properties, nonclassical states of light are of essential importance in the emerging field of quantum information [1]. Another potentially important application is the reduction of quantum noise in the measurement of relative phase shifts between two paths of an interferometer. A conventional Mach-Zehnder interferometer (MZI) with a coherent input laser field provides a phase-difference uncertainty $\Delta\varphi$ proportional to $1/\sqrt{\bar{N}}$, where $\bar{N}$ is the average number of photons supplied by the laser during the time interval of the measurement [1]. Although a higher sensitivity can, in principle, be obtained by increasing the laser power, there are practical issues, such as heating of the optical elements of the interferometer, which ultimately limit this approach. An attractive alternative is to employ a species of nonclassical light that produces greater sensitivity for a given average photon number. For example, if a squeezed vacuum is simultaneously injected into the normally unused port of the interferometer, the phase uncertainty then becomes proportional to $e^{-r}/\sqrt{\bar{N}}$ where $r>0$ is the squeeze parameter [2]. The optimal, and ultimate, sensitivity for phase measurements is given by the Heisenberg limit, for which $\Delta\phi=1/\bar{N}$ [3].

Although there are known methods to approach the Heisenberg limit asymptotically, Bollinger *et al.* [4] recognized that it is obtained exactly with the quantum input-state

$$\frac{1}{\sqrt{2}}(|N\rangle_a|0\rangle_b+e^{i\xi}|0\rangle_a|N\rangle_b), \tag{1}$$

where $\bar{N}=N$, and the labels a and b represent the two internal beams of the interferometer. Equation (1) is known as a maximally entangled state (MES) for the definite number of photons N [5]. In addition to their potential for high-precision interferometry, the MES are also predicted to surpass the diffraction limit for imaging applications such as photolithography [6]. Clearly, besides material issues, the first challenge to the optical use of MES is the identification of methods for generating such states.

In this paper we propose a method for the generation of optical MES and related pure-state superpositions of MES. The essential feature of this technique is the replacement of the ordinary first beam splitter of an MZI by a conditional beam splitter (CBS) whose central component is a quantum-optical Fredkin gate as will be described in detail below. The necessity for a CBS arises from considerations of optical mixing at an ordinary beam splitter, which shows that such a device cannot produce MES for photon numbers greater than two [7]. The essential ingredients for a CBS are the incorporation of a nonlinear cross-Kerr interaction between one arm of the MZI and state reduction using an auxiliary or control beam. These requirements are simpler than previous schemes that demand competition between different types of nonlinear interactions [8]. An interaction whose mathematical form is very similar to the one described in this paper has previously been discussed in connection with a proposal to generate MES in a system of N trapped two-level ions [9], the MES in that case being of importance for ultrahigh-resolution spectroscopy.

With the Schwinger realization of angular momentum operators in terms of Bose operators, interferometers and beam splitters can be represented as abstract rotations [1,7]. These angular momentum operators are constructed in terms of the input field operators as

$$\hat{J}_1=\tfrac{1}{2}(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger),\quad \hat{J}_2=\tfrac{1}{2i}(\hat{a}^\dagger\hat{b}-\hat{a}\hat{b}^\dagger),$$

$$\hat{J}_3=\tfrac{1}{2}(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}). \tag{2}$$

The traditional angular momentum basis kets $|j,m\rangle$ correspond exactly to the photon number ket product state $|N\rangle_a|M\rangle_b$ provided $j=\frac{1}{2}(N+M)$ and $m=\frac{1}{2}(N-M)$. An important general property of the angular momentum system is

$$\exp(\pm i\pi\hat{J}_2)|j,m\rangle=(-1)^{j\pm m}|j,-m\rangle, \tag{3}$$

which correspondingly swaps the product state $|N\rangle_a|M\rangle_b$ with the product state $|M\rangle_a|N\rangle_b$. The key to the construction of an MES is the ability to perform this swap operation of Eq. (3) in a controlled manner; this is what a conditional beam splitter accomplishes.

The CBS device we have in mind incorporates, as mentioned previously, a Fredkin gate [10] whose quantum-optical realization was discussed some time ago by Milburn [11] and more recently by researchers in quantum computation [12,13]. In Fig. 1, a Fredkin gate lies in the interior of the dashed lines. The unitary operator $\hat{U}_F$ describing its ac-

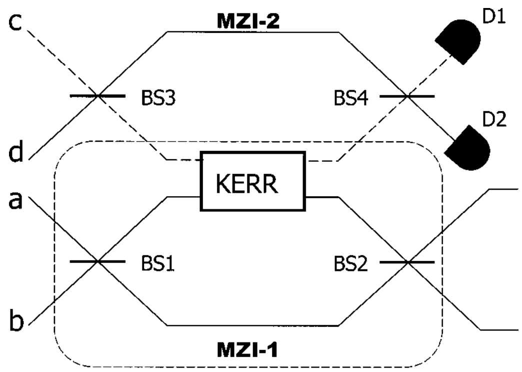

FIG. 1. A Fredkin gate is a logic device that implements a controlled swap. The optical Fredkin gate (enclosed by the dashed box) is a Mach-Zehnder interferometer (MZI-1) comprised of two beam splitters (BS1 and BS2) and a cross-Kerr medium along one of its arms. When the control beam of a Fredkin gate (mode c, dashed line) is embedded in a second Mach-Zehnder interferometer (MZI-2), one obtains a device that can produce maximally entangled states of the radiation field upon state reduction at output modes at detectors D1 or D2. This device can be viewed as a pair of nonlinearly coupled interferometers.

tion is readily composed using the beam splitter su(2) angular momentum algebra. The first beam splitter (BS1) is represented by the rotation operator

$$\hat{U}_1 = \exp\left(i\frac{\pi}{2}\hat{J}_1\right), \tag{4}$$

which in turn represents a particular construction or choice of the internal phases [1,7]. For convenience we choose BS2 to act conjugate to BS1, i.e., $\hat{U}_2 = \hat{U}_1^\dagger = \exp[-i(\pi/2)\hat{J}_1]$. Finally, the cross-Kerr interaction between modes b and c is described by the operator

$$\hat{U}_k = \exp(i\chi\hat{b}^\dagger\hat{b}\hat{c}^\dagger\hat{c}). \tag{5}$$

We assume that the self-modulation terms $\hat{a}^{\dagger 2}\hat{a}^2$ and $\hat{b}^{\dagger 2}\hat{b}^2$ can be eliminated by an appropriate choice of resonances [14]. The parameter χ is proportional to the third-order nonlinear susceptibility $\chi^{(3)}$ and the length of the medium. The unitary operator of the Fredkin gate is then

$$\hat{U}_F = \hat{U}_1^\dagger\hat{U}_k\hat{U}_1 = \exp(i\chi\hat{c}^\dagger\hat{c}\hat{J}_0)\exp(i\chi\hat{c}^\dagger\hat{c}\hat{J}_2), \tag{6}$$

where standard shifting properties of the angular momentum operators have been used and $\hat{J}_0 = (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})/2$ measures the total number of input photons to the system. The selection $\chi = \pi$ for the strength of the nonlinear interaction is assumed henceforth, as it provides the swap operator we seek. When the control beam c carries one photon, the Fredkin gate effects a state-swap operation; otherwise if the control beam is empty of photons, the input state is unchanged.

In order to create superpositions of the form of Eqs. (1) or (4), mode c must be further embedded in an auxiliary interferometer (MZI-2) as illustrated in Fig. 1. We assume that the beam splitters of MZI-2 are described by $\hat{U}_1$ and $\hat{U}_1^\dagger$ as previously considered. Assuming a single photon input state to MZI-2, the state after its first the beam splitter is

$$|\psi\rangle_{\text{MZI-2}} = \frac{1}{\sqrt{2}}(|1\rangle_c|0\rangle_d + i|0\rangle_c|1\rangle_d). \tag{7}$$

We use the convention that the first state represents the counterclockwise path of MZI-2 and the second represents the clockwise path. Clearly it is the former that plays the role of the c-mode passing through the Kerr medium. If we also inject into the a- and b-modes the state $|N\rangle_a|0\rangle_b$ then we shall have

$$\hat{U}_F|N\rangle_a|0\rangle_b|\psi\rangle_{\text{MZI-2}} = \frac{1}{\sqrt{2}}[e^{i\xi_{N0}}|0\rangle_a|N\rangle_b|1\rangle_c|0\rangle_d + i|N\rangle_a|0\rangle_b|0\rangle_c|1\rangle_d], \tag{8}$$

where $\xi_{N0} = -N\pi/2$. The second beam splitter of MZI-2 effects the transformations

$$|1\rangle_c|0\rangle_d \rightarrow \frac{1}{\sqrt{2}}(|1\rangle_c|0\rangle_d - i|0\rangle_c|1\rangle_d),$$

$$|0\rangle_c|1\rangle_d \rightarrow \frac{1}{\sqrt{2}}(|0\rangle_c|1\rangle_d - i|1\rangle_c|0\rangle_d). \tag{9}$$

So the final state of the system is

$$\tfrac{1}{2}\{[|N\rangle_a|0\rangle_b + e^{i\xi_{N0}}|0\rangle_a|N\rangle_b]|1\rangle_c|0\rangle_d + i[|N\rangle_a|0\rangle_b + e^{i\xi_{N0}}|0\rangle_a|N\rangle_b]|0\rangle_c|1\rangle_d\}. \tag{10}$$

Whenever detector D1 (D2) fires registering the state $|1\rangle_c|0\rangle_d(|0\rangle_c|1\rangle_d)$, the output state for modes a-b reduces to

$$|\psi_{1(2)}\rangle_{ab} = \frac{1}{\sqrt{2}}[|N\rangle_a|0\rangle_b \pm e^{i\xi_{N0}}|0\rangle_a|N\rangle_b]. \tag{11}$$

We note that the resultant states differ only by a relative phase of π. If the input state was instead $|N\rangle_a|M\rangle_b$, the same procedure readily provides the more general output state

$$|\psi_{1(2)}\rangle_{ab} = \frac{1}{\sqrt{2}}[|N\rangle_a|M\rangle_b \pm e^{i\xi_{NM}}|M\rangle_a|N\rangle_b], \tag{12}$$

where $\xi_{NM} = \pi(M-N)/2$. This completes the proof that a quantum-optical Fredkin gate, combined with an auxiliary MZI to enable state reduction, produces MES states in the form of Eq. (1) and more generally Eq. (12) with $\exp(i\xi_{NM}) = i^{M-N}$.

As the primary motivation for generating states of the form of Eq. (11) is their potential application to interferometry, we now summarize the advantages of these states for that purpose. Consider a replacement of the first beam splitter within a standard MZI by the CBS described above. The internal state of this interferometer, just after the CBS, is described by Eq. (11) and for the sake of definiteness we select $|\psi_1\rangle_{ab}$ using state reduction at detector D1. The operator

$$\hat{U}_S=\exp\left(i\frac{\pi}{2}\hat{J}_1\right)\exp(i\varphi\hat{J}_3) \tag{13}$$

describes the rest of the interferometer, which effects a relative phase difference φ between the two paths, and recombines them at the second (standard) beam splitter. The detection technique of Bollinger *et al.* [4] measures the parity of one of the output beams, which for the b mode is represented in operatorial form as

$$\hat{O}=(-1)^{\hat{b}^+\hat{b}}=\exp[i\pi(\hat{J}_0-\hat{J}_3)]. \tag{14}$$

This technique amounts to direct detection at one output port and raising -1 to that power. This clearly requires photon detectors with resolutions at the level of a single photon but it must be said that the same is true for proposals to measure the number difference operator $\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}$. Using standard shifting properties of angular momentum, it follows that

$$\langle\hat{O}\rangle={}_{ab}\langle\psi_1|\hat{U}_S^\dagger\hat{O}\hat{U}_S|\psi_1\rangle_{ab}=(-1)^N\cos(N\varphi), \tag{15}$$

from which we readily calculate the phase uncertainty

$$\Delta\varphi=\Delta\hat{O}/\left|\frac{\partial\langle\hat{O}\rangle}{\partial\varphi}\right|=\frac{1}{N}. \tag{16}$$

This result, unlike other proposed schemes for approaching the Heisenberg limit [1,2,15], is independent of the phase shift φ.

The above generation scheme for MES is gated by the ability to generate input Fock states of the radiation field, which is itself a challenging exercise especially for high number of photons. Is it possible to obtain $1/N$ sensitivity levels by using more conventional input states to the Fredkin gate, e.g., coherent states of the radiation field? In the following we will show that this is indeed possible at least over some ranges of the interferometric phase φ. To this end, consider the most general pure input state for the a mode with the b mode in the vacuum state

$$|\Psi_{\text{in}}\rangle=\left(\sum_{N=0}^{\infty}C_N|N\rangle_a\right)|0\rangle_b, \tag{17}$$

and with the single photon state injected into MZI-2 as before. Assuming D1 fires, the output a-b modes are in the state

$$|\psi_1\rangle_{ab}=\frac{1}{\sqrt{2}}(1+|C_0|^2)^{-1/2}\sum_{N=0}^{\infty}C_N(|N\rangle_a|0\rangle_b+e^{i\xi_{N0}}|0\rangle_a|N\rangle_b). \tag{18}$$

For an input coherent state $|\alpha\rangle_a$ we have $C_N=\exp(-|\alpha|^2/2)\alpha^N/\sqrt{N!}$ and the output state is

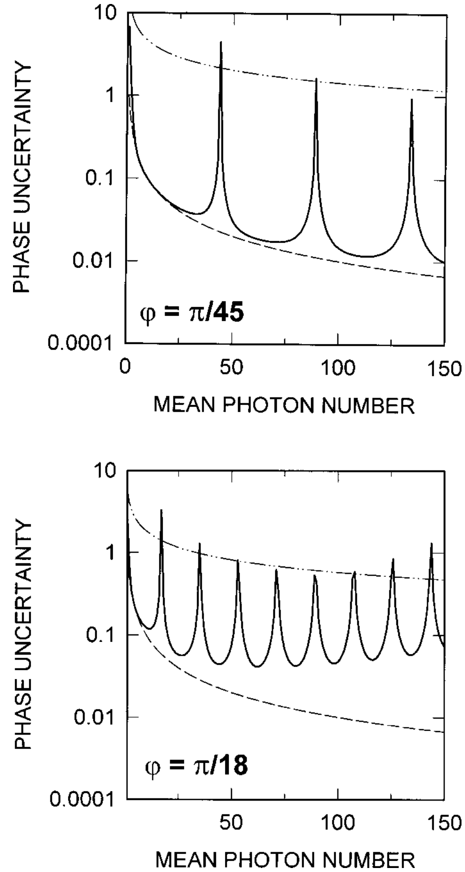

FIG. 2. (a) The phase uncertainty of a maximally entangled coherent state discussed in Eq. (19) is plotted (solid line) as a function of mean photon number $|\alpha|^2$ for fixed $\varphi=\pi/45\,(4°)$. The Heisenberg limit represented by the dashed line and the classical limit is given by the dashed-dotted line. (b) Same as (a) but for $\varphi=\pi/18\,(10°)$.

$$|\psi_1\rangle_{ab}=\frac{1}{\sqrt{2}}(1+e^{-|\alpha|^2})^{-1/2}(|\alpha\rangle_a|0\rangle_b+|0\rangle_a|-i\alpha\rangle_b), \tag{19}$$

which obviously entangles the coherent state with the vacuum. For this state,

$$\langle\hat{O}\rangle=\frac{\exp[-\bar{N}(1-\cos\varphi)]}{1+\exp(-\bar{N})}\cos(\bar{N}\sin\varphi), \tag{20}$$

where $\bar{N}=|\alpha|^2$ is the average photon number of the input coherent state. Thus it appears that the sensitivity will now be dependent on the value of the phase φ. But if we take $\varphi=2\pi s+\delta$, $s=0,1,2,\ldots,$ where δ is small, we have

$$\langle\hat{O}\rangle\approx\frac{\exp(-\bar{N}\delta^2/2)}{1+\exp(-\bar{N})}\cos(\bar{N}\delta). \tag{21}$$

In the regime where $\bar{N}$ is large but where $\bar{N}\delta^2/2$ is still small, we have $\langle\hat{O}\rangle\approx\cos(\bar{N}\delta)$ and thus $\Delta\varphi\approx1/\bar{N}$. In Fig. 2 we plot the phase uncertainty $\Delta\varphi$ (solid line) as a function of $\bar{N}$, where $\langle \hat{O} \rangle$ is given by Eq. (20), at two different values of φ. For comparison, we also include the Heisenberg limit $\Delta\varphi = 1/\bar{N}$ (dashed line), and the standard quantum limit $\Delta\varphi = \Delta \hat{J}_3 / |\partial \langle \hat{J}_3 \rangle / \partial \varphi| = 1/(\sqrt{\bar{N}} \sin \varphi)$ (dash-dotted line) obtained by injecting a coherent state into one of the input ports of a standard MZI. In Fig. 2(a) we plot the phase uncertainties for the small angle $\varphi = \pi/45$ as a function of the average photon number $\bar{N}$ of the initial coherent state. Apart from the recurring spikes (whose periodicity is expected from the definition of $\hat{O}$), the phase uncertainty very closely follows the Heisenberg limit. This result is superior to the standard result for an MZI with an input coherent state over a wide range of average photon numbers. Phase uncertainties for a larger angle ($\varphi = \pi/18$) are shown in Fig. 2(b). We notice in this case the window of improved utility is clearly shortened as the average photon number is increased. Nevertheless, there is still a wide range over which our results are superior to those of the standard approach and indeed, in Fig. 2(b) at $\bar{N} = 150$, the phase sensitivity is still well below the standard quantum limit. It appears that this window of utility exists even for angles as high as $\varphi = \pi/7$, and it may be possible to improve the sensitivity over an even wider range of phase angles through a more judicious choice of input state (e.g., a sub-Poisson state). In any case, for applications such as gravity-wave detection, a restricted range of phase angles may not be a severe limitation if the goal is to measure small deviations from a balanced interferometer.

As mentioned previously, another important potential application of the MES is interferometric quantum photolithography. Diffraction effects in the masking approach to classical lithography limit the resolution of transferred images to the Rayleigh criterion $\lambda/2$, where λ is the optical wavelength. Boto and co-workers [6,16] showed that this limit is breached when MES states having the form of Eq. (11), interfere on the surface of a substrate capable of absorbing N photons. The deposition rate for the MES is given by $\Delta_{N,\gamma} = 1 + \cos(N\varphi + \xi_{N0})$, where $\varphi = \pi x/\lambda$ [6,16], and this provides a sharper resolution $\lambda/2N$. With a coherent state input to our proposed device, we obtain the deposition function

$$\Delta_{N,\text{coh-MES}} = \frac{e^{-|\alpha|^2}}{1+e^{-|\alpha|^2}} \frac{|\alpha|^{2N}}{N!} [1 + \cos(N\varphi + \xi_{N0})] + \frac{|\alpha|^{2N}}{N!} \frac{1-e^{-|\alpha|^2}}{1+e^{-|\alpha|^2}}, \tag{22}$$

which maximizes for $|\alpha|^2 \approx N$. However, the appearance of a background term effectively restricts the method to dilute beam intensities, which may not be practical. To create patterns in two dimensions, more general entangled states, e.g., Eq. (12) will be required and these can also be generated with the apparatus of Fig. 1.

To conclude, we comment on the feasibility and uniqueness of our proposal. The condition $\chi = \pi$ implies a large third-order nonlinear susceptibility $\chi^{(3)}$. As shown in the recent experiment of Hau *et al.* [17] and as discussed by Schmidt and Imamoglu [18], the techniques of electromagnetically induced transparency offer an avenue to meet the required level of nonlinearity. In regard to uniqueness, we note that the nonlinear MZI [19,20] originally introduced in the context of quantum-nondemolition experiments [14] is also capable of providing MES without state reduction for an equivalently large, self-Kerr, interaction. The Fredkin gate approach offers an alternative, gated method for the generation of optical MES.

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