

Rotating vortex lattice in a Bose-Einstein condensate trapped in combined quadratic and quartic radial potentials

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A dense vortex lattice in a rotating dilute Bose-Einstein condensate is studied with the Thomas-Fermi approximation. The upper critical angular velocity Ω_{c2} occurs when the intervortex separation b becomes comparable with the vortex-core radius ξ . For a radial harmonic trap, the loss of confinement as $\Omega \rightarrow \omega_{\perp}$ implies a singular behavior. In contrast, an additional radial quartic potential provides a simple model for which Ω_{c2} is readily determined. Unlike the case of a type-II superconductor at fixed temperature, the onset of Ω_{c2} arises not only from decreasing b but also from increasing ξ caused by the vanishing of the chemical potential as $\Omega \rightarrow \Omega_{c2}$.

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I. INTRODUCTION

The recent observations [1–4] of vortex arrays and vortex lattices in rotating dilute trapped Bose-Einstein condensates (BEC) raise the question of analogies with type-II superconductors. What limits the number of vortices? Is there an upper critical rotation speed Ω_{c2} similar to the upper critical field H_{c2} [5] familiar from type-II superconductors? In the superconducting case, the field H_{c2} occurs when the distance $b \sim \sqrt{\Phi_0/\pi B}$ between vortices becomes comparable with the vortex-core size $\xi \sim 10\text{--}100$ nm in typical conventional superconducting alloys. The measured flux quantum $\Phi_0 = h/2e$ then implies $H_{c2} \sim \Phi_0/\pi\xi^2 \sim 0.1\text{--}10$ T (in this limit, the distinction between H and B becomes negligible). For rotating superfluid ^4He , where $\xi \sim$ a few angstrom and $b \sim \sqrt{\hbar/M_4\Omega}$ [6], essentially the same condition yields the unattainably large value $\Omega_{c2} \sim \hbar/M_4\xi^2 \sim 10^{15}$ rad/s.

A similar criterion also applies to rotating Bose condensates. In this case, the vortex-core radius $\xi = \hbar/\sqrt{2M\mu} \sim 0.1$ μm is macroscopic, where μ is the chemical potential, suggesting that experimental study of Ω_{c2} might well be possible. As seen below, the situation in rotating dilute condensates is even more favorable, because the chemical potential decreases at large rotation speeds, increasing the vortex-core size and thus reducing Ω_{c2} .

For a trap with a radial harmonic potential $V_{\perp}(r_{\perp}) = \frac{1}{2}M\omega_{\perp}^2 r_{\perp}^2$, the behavior becomes singular when $\Omega \rightarrow \omega_{\perp}$ because the outward centrifugal force counteracts the inward force from the harmonic trap [7–9]. Specifically, the effective radial trap frequency $(\omega_{\perp}^2 - \Omega^2)^{1/2}$ produces an observable centrifugal distortion of the condensate [10–12] for currently attainable values of $\Omega/\omega_{\perp} \lesssim 1$. As seen below, this behavior means that ω_{\perp} effectively acts like Ω_{c2} for pure harmonic radial confinement, and direct experimental study of the limiting behavior for $\Omega \rightarrow \omega_{\perp}$ would be difficult. Thus, it is convenient to consider an additional stiffer radial potential, which eliminates the singularity when $\Omega = \omega_{\perp}$. In this case, Ω_{c2} exceeds ω_{\perp} and the limit $\Omega \rightarrow \Omega_{c2}$ is relatively smooth. The analysis is especially simple for a quartic radial

potential $V_4(r_{\perp}) = \frac{1}{4}kr_{\perp}^4$ and this example is analyzed in detail.

Section II introduces the basic Gross-Pitaevskii free energy and condensate wave function $\Psi = |\Psi|e^{iS}$ for a rotating vortex lattice in a trapped BEC. In equilibrium, the vortex lattice must experience self-consistent solid-body rotation, which means that neither the phase S nor the superfluid velocity $\mathbf{v}_s = (\hbar/M)\nabla S$ can be spatially periodic in the laboratory frame. The transformation to the frame rotating with angular velocity $\mathbf{\Omega}$ yields a solid-body velocity $\mathbf{v}_{sb} = \mathbf{\Omega} \times \mathbf{r}$ that cancels the overall rotation. Thus, the resulting relative velocity $\mathbf{v}_s - \mathbf{v}_{sb}$ is, indeed, spatially periodic for an unbounded vortex lattice. This feature facilitates a simple description of the vortex lattice in a large trapped BEC. The case of a trap with harmonic radial confinement is studied in Sec. III, and the more interesting case of an additional quartic radial confining potential is studied in Sec. IV.

II. GENERAL FORMALISM

It is convenient to start from the Gross-Pitaevskii (GP) [13,14] free energy in the rotating frame

$$F = \int dV \left[\frac{\hbar^2}{2M} |\nabla \Psi|^2 + V_{\perp}(r_{\perp}) |\Psi|^2 + \frac{1}{2} M \omega_z^2 z^2 |\Psi|^2 + \frac{1}{2} g |\Psi|^4 - \Psi^* \mathbf{\Omega} \cdot \mathbf{r} \times \mathbf{p} \Psi \right], \quad (1)$$

where $V_{\perp}(r_{\perp})$ is the transverse radial confining potential and $g = 4\pi a \hbar^2/M$ relates the interparticle coupling constant to the s -wave scattering length a (here assumed positive). The representation $\Psi = |\Psi|e^{iS}$ emphasizes the hydrodynamic aspects of the behavior, with condensate density $n_0 = |\Psi|^2$ and superfluid velocity $\mathbf{v}_s = \hbar \nabla S/M$. The quantity $-\Psi^* \mathbf{\Omega} \cdot \mathbf{r} \times \mathbf{p} \Psi$ can be written as $\frac{1}{2} i \hbar \Omega \partial |\Psi|^2 / \partial \phi - M \mathbf{v}_{sb} \cdot \mathbf{v}_s |\Psi|^2$ and the first term makes no contribution to the spatial integral. Straightforward manipulations of Eq. (1) yield

$$F = \int dV \left[\frac{1}{2} M (\mathbf{v}_s - \mathbf{v}_{sb})^2 |\Psi|^2 + \frac{\hbar^2}{2M} (\nabla |\Psi|)^2 + V_{\perp}(r_{\perp}) |\Psi|^2 + V_{\text{cent}}(r_{\perp}) |\Psi|^2 + \frac{1}{2} M \omega_z^2 z^2 |\Psi|^2 + \frac{1}{2} g |\Psi|^4 \right], \quad (2)$$

where $V_{\text{cent}}(r_{\perp}) = -\frac{1}{2} M \Omega^2 r_{\perp}^2$ is an effective repulsive centrifugal potential.

The simplest variational model is to take the phase S as that for a classical unbounded vortex lattice (with vortices aligned along \hat{z}), which has been analyzed in detail by Tkachenko [15] (as a model for a vortex lattice in rotating superfluid ^4He). In this case, the total superfluid velocity includes the divergent locally axisymmetric circulating flow near each vortex core. The resulting singular kinetic energy in Eq. (2) is cut off by a self-consistent core structure of characteristic dimension $\xi = \hbar / \sqrt{2M\mu}$, where μ is the chemical potential [13,14,16].

This classical hydrodynamic phase S cannot be a spatially periodic function in the x - y plane because the vortex array induces a self-consistent rotation. Instead, this phase obeys definite quasiperiodic conditions [15,17] that depend on the details of the lattice structure. For the same reasons, $\mathbf{v}_s \propto \nabla S$ is also not spatially periodic. In contrast, the quantity $\mathbf{v}_s - \mathbf{v}_{sb}$ (namely, the superfluid velocity as seen in the rotating frame) is spatially periodic, because the subtracted term \mathbf{v}_{sb} cancels the effect of the quasiperiodic terms in ∇S , considerably simplifying the subsequent analysis.

According to Feynman's picture of a rotating superfluid [18], the vortices have a uniform areal density $n_v = 2M\Omega/h$, ensuring that the mean vorticity of the vortex lattice mimics that of solid-body rotation $\nabla \times \mathbf{v}_{sb} = 2\Omega$. Thus, the area per vortex $n_v^{-1} \equiv \pi b^2$ can be taken to define an intervortex separation $b = \sqrt{\hbar/M\Omega}$; this characteristic length sets the scale of the vortex lattice. For the present purposes, the detailed lattice structure is unimportant [for example, the free energies of the triangular and square configurations have the same logarithmic contributions $\propto \ln(b/\xi)$ and differ only in the additive constants [19]]; it will be convenient to use a Wigner-Seitz approximation in which each polygonal unit cell is replaced by an equivalent circular cell of radius b . Evidently, the ratio ξ^2/b^2 characterizes the fractional volume occupied by the "normal" vortex cores.

For any reasonable transverse confining potential $V_{\perp}(r_{\perp})$, the single particle ground state will have some characteristic transverse dimension d_{\perp} , along with the characteristic axial dimension $d_z = \sqrt{\hbar/M\omega_z}$ that is set by the axial harmonic confining potential in Eqs. (1) and (2). Let N_0 be the number of atoms in the condensate at low temperature. In the Thomas-Fermi (TF) limit [20], the condensate density is taken as locally constant, which holds when the dimensionless interaction parameter $N_0 a / d_{\perp}$ is large. In this case, the repulsive Hartree (mean-field) interactions expand the condensate relative to its noninteracting transverse and axial dimensions d_{\perp} and d_z . In particular, the radial and axial condensate radii R_{\perp} and R_z are simply the classical turning points for a particle of energy μ . Thus, $R_z = \sqrt{2\mu/M\omega_z^2}$ de-

pends only on μ , but R_{\perp} also depends on Ω because of the repulsive centrifugal potential $V_{\text{cent}}(r_{\perp})$ in Eq. (2).

For a rotating condensate with chemical potential μ , the GP equation derived from Eq. (2) implies the corresponding TF density

$$|\Psi(r_{\perp}, z)|^2 = \frac{\mu}{g} \left[1 - \frac{V_{\perp}(r_{\perp})}{\mu} + \frac{\Omega^2 r_{\perp}^2}{2\mu} - \frac{M\omega_z^2 z^2}{2\mu} - \frac{M(\mathbf{v}_s - \mathbf{v}_{sb})^2}{2\mu} \right], \quad (3)$$

obtained by omitting the gradient of the condensate density. If the vortex cores do not overlap significantly (so that $\xi^2/b^2 \ll 1$), the condensate density can be approximated by a product $|\Psi(r_{\perp}, z)|^2 \approx |\Psi_{\text{TF}}(r_{\perp}, z)|^2 u^2(r_{\perp})$ of the TF density

$$|\Psi_{\text{TF}}(r_{\perp}, z)|^2 = \frac{\mu}{g} \left[1 - \frac{V_{\perp}(r_{\perp})}{\mu} + \frac{\Omega^2 r_{\perp}^2}{2\mu} - \frac{M\omega_z^2 z^2}{2\mu} \right] \quad (4)$$

in the absence of the vortex lattice and a factor

$$u^2(r_{\perp}) = 1 - \frac{M(\mathbf{v}_s - \mathbf{v}_{sb})^2}{2\mu} \quad (5)$$

that is spatially periodic in the transverse plane and must be cut off near the vortex core to ensure that u^2 remains positive. In any given unit cell, it has the local form

$$u^2(r_{\perp}) = 1 - \frac{\xi^2}{r_{\perp}^2} + \frac{2\xi^2}{b^2} - \frac{\xi^2 r_{\perp}^2}{b^4}, \quad (6)$$

where a circular Wigner-Seitz cell has been used with $\xi \leq r_{\perp} \leq b$. In effect, the resulting density is that of a vortex-free condensate with narrow holes along the axes of the vortex lattice [4,12,21].

The chemical potential is determined by the general condition that $N_0 = \int dV |\Psi(r_{\perp}, z)|^2$. Here the factorized form yields

$$\begin{aligned} N_0 &\approx \int dV |\Psi_{\text{TF}}(r_{\perp}, z)|^2 u^2(r_{\perp}) \\ &= \frac{\mu M R_z}{3\pi a \hbar^2} \int d^2 r_{\perp} \left[1 + \frac{M\Omega^2 r_{\perp}^2}{2\mu} - \frac{V_{\perp}(r_{\perp})}{\mu} \right]^{3/2} u^2(r_{\perp}). \end{aligned} \quad (7)$$

For a dense vortex lattice, the intervortex separation b is small compared to the TF transverse radius R_{\perp} , so that the first factor varies slowly on the length scale b . Thus, u^2 can be averaged over any single unit cell, and the Wigner-Seitz approximation [17] in Eq. (6) yields

$$\langle u^2 \rangle = \frac{1}{\pi b^2} \int_{\text{cell}} d^2 r_{\perp} u^2(r_{\perp}) \approx 1 - 2 \frac{\xi^2}{b^2} \left[\ln \left(\frac{b}{\xi} \right) - \frac{3}{4} \right]; \quad (8)$$

as anticipated, $\langle u^2 \rangle \approx 1$ for $\xi \ll b$. In the limit of a large condensate with many vortices, Eq. (7) can, therefore, be approximated by

$$N_0 = \frac{\mu M R_z \langle u^2 \rangle}{3 \pi a \hbar^2} \int d^2 r_\perp \left[1 + \frac{M \Omega^2 r_\perp^2}{2\mu} - \frac{V_\perp(r_\perp)}{\mu} \right]^{3/2}. \quad (9)$$

III. RADIAL HARMONIC CONFINING POTENTIAL

The simplest situation is the harmonic radial potential $V_\perp(r_\perp) = \frac{1}{2} M \omega_\perp^2 r_\perp^2$, in which case Eq. (9) implies the TF condensate radius

$$R_\perp = \sqrt{\frac{2\mu}{M(\omega_\perp^2 - \Omega^2)}}. \quad (10)$$

Familiar manipulations yield

$$\frac{\mu}{\hbar \omega_\perp} = \frac{1}{2} \left(\frac{15 N_0 a \lambda}{d_\perp \langle u^2 \rangle} \right)^{2/5} \left(1 - \frac{\Omega^2}{\omega_\perp^2} \right)^{2/5}, \quad (11)$$

where $\lambda = \omega_z / \omega_\perp$ is the axial anisotropy parameter.

Except for a very narrow region $\Omega / \omega_\perp \lesssim 1$ close to the singular limit, the vortex cores occupy negligible volume, so that $\langle u^2 \rangle \approx 1$. In this case, the condensate particle number N_0 remains essentially constant and Eq. (11) shows how the chemical potential decreases with increasing rotation speed $\bar{\Omega} \equiv \Omega / \omega_\perp$. In this same large interval, Eq. (10) implies that

$$\frac{R_\perp^2(\Omega)}{d_\perp^2} = \left(\frac{15 N_0 a \lambda}{d_\perp} \right)^{2/5} (1 - \bar{\Omega}^2)^{-3/5}. \quad (12)$$

Equivalently,

$$\frac{R_\perp(\Omega)}{R_\perp(0)} = (1 - \bar{\Omega}^2)^{-3/10}, \quad (13)$$

as seen in recent experiments [10].

The fraction of depleted condensate occupied by the vortex cores is given quite generally by $\xi^2 / b^2 = \hbar \Omega / 2\mu$ and use of Eq. (11) gives the specific result

$$\frac{\xi^2}{b^2} = \frac{\hbar \Omega}{2\mu} = \frac{\bar{\Omega}}{2} \frac{\epsilon^{2/5}}{(1 - \bar{\Omega}^2)^{2/5}}, \quad (14)$$

where

$$\epsilon \equiv \frac{d_\perp \langle u^2 \rangle}{15 N_0 a \lambda} \approx \frac{d_\perp^5}{R_\perp^5(0)} \ll 1 \quad (15)$$

is small in the TF limit. Equation (14) shows that the ratio ξ^2 / b^2 increases linearly with $\bar{\Omega}$ but remains small (of order $\epsilon^{2/5}$) until $1 - \bar{\Omega} \lesssim \epsilon / 8\sqrt{2}$, when the ratio grows rapidly toward 1. If Ω_{c2} is defined like H_{c2} for a type-II superconductor, Ω_{c2} corresponds to the limit $\xi \sim b$. As a result, a radial harmonic trap potential leads to singular behavior when $\Omega \rightarrow \omega_\perp$ [9]. Specifically, the disappearance of the con-

densate associated with the sudden expansion of the ‘‘normal’’ vortex cores presumably implies a phase transition, which occurs essentially simultaneously with the loss of confinement as $\bar{\Omega} \rightarrow 1$ [7–9,12].

This straightforward analysis indicates that any study of a dense vortex lattice near Ω_{c2} for a radial-harmonic trap will encounter significant difficulties arising from the softening of the effective trap potential. As a result, it is interesting to consider a stiffer radial potential, and the next section treats one possible example.

IV. RADIAL QUARTIC CONFINING POTENTIAL

As a specific example, let $V_\perp(r_\perp)$ include a quartic-radial-trap potential $V_4(r_\perp) = \frac{1}{4} k r_\perp^4$ in addition to the quadratic potential $\frac{1}{2} M \omega_\perp^2 r_\perp^2$; Lundh [22] has recently examined this and other power-law potentials in a theoretical study of the formation of a multiply quantized vortex in a rotating condensate. For a two-dimensional ideal gas confined with pure $V_4(r_\perp)$, the balance between the ground-state kinetic energy $\sim \hbar^2 / M d_4^2$ and ground-state potential energy $\sim k d_4^4$ implies a ground-state mean radius $d_4 \sim (\hbar^2 / M k)^{1/6}$ and ground-state energy $E_4 = \hbar \omega_4 \sim (\hbar^4 k / M^2)^{1/3}$. For definiteness, it is convenient to ignore numerical constants of order unity and take

$$E_4 = \hbar \omega_4 = \frac{\hbar^2}{M d_4^2} = k d_4^4 \quad (16)$$

as the relevant dimensional quantities.

For a rotating condensate in this combined radial trap, Eq. (9) becomes

$$N_0 = \frac{\mu M R_z \langle u^2 \rangle}{3 a \hbar^2} \int_0^{R_\perp^2} du \left[1 - \frac{M(\omega_\perp^2 - \Omega^2)}{2\mu} u - \frac{k}{4\mu} u^2 \right]^{3/2}, \quad (17)$$

where $u = r_\perp^2$, and R_\perp^2 is the turning point, where the integrand vanishes. The presence of the quartic potential means that the external rotation speed Ω can now exceed ω_\perp . In this regime ($\bar{\Omega} > 1$), the particle density actually attains a local minimum on the axis of symmetry, but this effect is probably difficult to detect.

The integral in Eq. (17) can be expressed in terms of a dimensionless parameter

$$\eta \equiv \frac{M(\omega_\perp^2 - \Omega^2)}{2\sqrt{k\mu}}. \quad (18)$$

A detailed analysis yields the final form

$$\int_0^{R_\perp^2} du \left[1 - \frac{M(\omega_\perp^2 - \Omega^2)}{2\mu} u - \frac{k}{4\mu} u^2 \right]^{3/2} = 2 \sqrt{\frac{\mu}{k}} f(\eta), \quad (19)$$

where

$$f(\eta) = \frac{3\pi}{16}(1+\eta^2)^2 \left[1 - \frac{2}{\pi} \arcsin\left(\frac{\eta}{\sqrt{1+\eta^2}}\right) \right] - \frac{3\eta^3}{8} - \frac{5\eta}{8}. \quad (20)$$

It has the limiting forms

$$f(\eta) \approx \begin{cases} (5\eta)^{-1} & \text{for } \eta \gg 1, \\ \frac{3}{16}\pi - \eta & \text{for } |\eta| \ll 1, \\ \frac{3}{8}\pi\eta^4 & \text{for } \eta \ll -1. \end{cases} \quad (21)$$

Equations (17) and (19) can be combined with $R_z = \sqrt{2\mu/M}\omega_z^2$ to express the chemical potential in terms of the parameter η

$$\left(\frac{\mu}{\hbar\omega_\perp}\right)^2 = \frac{N_0 a}{d_\perp \langle u^2 \rangle} \frac{\omega_z}{\omega_\perp} \frac{3}{2\sqrt{2}f(\eta)} \left(\frac{d_\perp}{d_4}\right)^3, \quad (22)$$

where d_\perp/d_4 is a dimensionless measure of the relative strength of the quartic potential (note that $d_\perp/d_4 \rightarrow 0$ in the limit that the quartic coupling constant k becomes small). In addition, Eq. (18) can be rewritten to express the rotation speed as a function of μ and η

$$\bar{\Omega}^2 \equiv \frac{\Omega^2}{\omega_\perp^2} = 1 - 2\eta \sqrt{\frac{\mu}{\hbar\omega_\perp}} \left(\frac{d_\perp}{d_4}\right)^3. \quad (23)$$

Note that η is large and positive for a nonrotating condensate ($\Omega=0$), η vanishes for $\bar{\Omega}=1$, and η becomes negative for $\bar{\Omega}>1$ (this limit can occur only for nonzero positive d_\perp/d_4). These two equations provide a parametric representation of the dependence of the dimensionless chemical potential $\bar{\mu} = \mu/\hbar\omega_\perp$ on the rotation speed, generalizing the result in Eq. (11) for a pure harmonic potential to include the effect of an additional quartic potential.

It is not difficult to see from Eqs. (21) and (22) that μ vanishes like η^{-2} as $\eta \rightarrow -\infty$, so that the right-hand side of Eq. (23) remains finite in the same limit. Thus, the chemical potential vanishes when the rotation frequency attains its maximum value. This maximum rotation speed can be identified as the upper critical value Ω_{c2} because the vortex-core size $\xi = \hbar/\sqrt{2M\mu}$ diverges as $\mu \rightarrow 0$. A detailed analysis (approximating $\langle u^2 \rangle \approx 1$) shows that

$$\Omega_{c2}^2 \approx \omega_\perp^2 + \omega_4^2 \left(\frac{32\sqrt{2} N_0 a}{\pi} \frac{\omega_z}{d_4} \frac{\omega_4}{\omega_\perp} \right), \quad (24)$$

where the positive quantity $\Omega_{c2}^2 - \omega_\perp^2$ has been expressed solely in terms of the parameters associated with the quartic term in the confining potential. Figure 1 shows the dimensionless chemical potential $\bar{\mu}$ as a function of the dimensionless rotation speed $\bar{\Omega}$ for two illustrative cases (note that $\bar{\mu}$ remains positive for $\bar{\Omega}=1$).

In the presence of the additional quartic confining potential, the TF radius R_\perp is given by

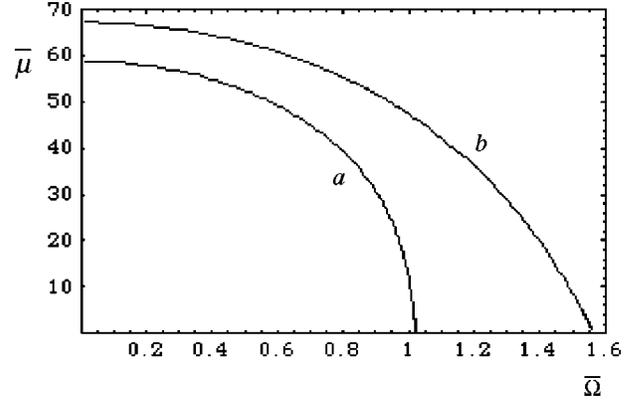


FIG. 1. Dimensionless chemical potential $\bar{\mu} \equiv \mu/\hbar\omega_\perp$ for the combined quadratic and quartic radial confining potential as a function of the dimensionless angular velocity $\bar{\Omega} = \Omega/\omega_\perp$ with $(N_0 a/d_\perp)(\omega_z/\omega_\perp) = 10^4$. The two curves correspond to the values (a) $d_4/d_\perp = 5$ (b) $d_4/d_\perp = 2$. Note that $\bar{\mu}$ remains finite at $\bar{\Omega} = 1$ for both values of d_4/d_\perp .

$$\frac{R_\perp^2}{d_\perp^2} = 2 \sqrt{\frac{\mu}{\hbar\omega_\perp}} \left(\frac{d_4}{d_\perp}\right)^3 \frac{1}{\sqrt{1+\eta^2} + \eta}. \quad (25)$$

A combination with Eq. (23) provides a parametric relation for $R_\perp(\Omega)$. Figure 2 shows the dimensionless ratio R_\perp^2/d_\perp^2 as a function of the dimensionless rotation speed $\bar{\Omega}$ (note that R_\perp^2/d_\perp^2 remains finite as $\bar{\Omega} \rightarrow \bar{\Omega}_{c2}$).

Finally, the ratio of the vortex-core radius to the intervortex separation follows from

$$\frac{\xi^2}{b^2} = \bar{\Omega} \frac{\hbar\omega_\perp}{2\mu} = \frac{\bar{\Omega}}{2\bar{\mu}}, \quad (26)$$

and use of Eq. (23) to eliminate η in favor of $\bar{\Omega}$ gives the generalization of Eq. (14) for the case of a combined quadratic and quartic confining potential. Figure 3 shows the $\bar{\Omega}$

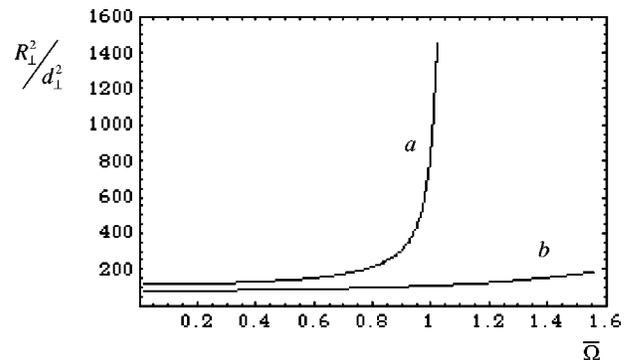


FIG. 2. Dimensionless squared radius R_\perp^2/d_\perp^2 for the combined quadratic and quartic radial confining potential as a function of the dimensionless angular velocity $\bar{\Omega} = \Omega/\omega_\perp$ with $(N_0 a/d_\perp)(\omega_z/\omega_\perp) = 10^4$. The two curves correspond to the values (a) $d_4/d_\perp = 5$ (b) $d_4/d_\perp = 2$. In contrast to the behavior seen in Eq. (10) for a rotating harmonic radial potential, the squared radius here remains finite for both values of d_4/d_\perp as $\bar{\Omega} \rightarrow \bar{\Omega}_{c2}$ given in Eq. (24).

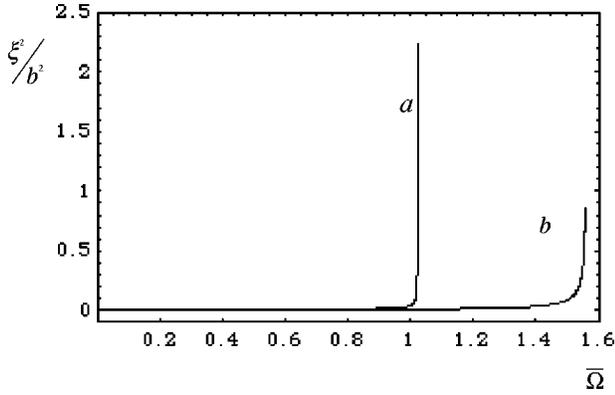


FIG. 3. Fraction of volume ξ^2/b^2 occupied by the vortex cores for the combined quadratic and quartic radial confining potential as a function of the dimensionless angular velocity $\bar{\Omega} = \Omega/\omega_\perp$ with $(N_0 a/d_\perp)(\omega_z/\omega_\perp) = 10^4$. The two curves correspond to the values (a) $d_4/d_\perp = 5$ (b) $d_4/d_\perp = 2$. Note that ξ^2/b^2 remains finite for both values of d_4/d_\perp as $\Omega \rightarrow \Omega_{c2}$ given in Eq. (24).

dependence of ξ^2/b^2 for two illustrative cases; the sharp increase near $\bar{\Omega}_{c2}$ and the finite limiting value are particularly evident.

V. DISCUSSION AND CONCLUSIONS

The behavior of a dense vortex lattice in a dilute trapped rotating Bose-Einstein condensate has acquired a new interest because of recent observations of such lattices [2–4]. The experiments use radial harmonic traps, and the loss of confinement as the external rotation speed Ω approaches the trap frequency ω_\perp implies that the behavior becomes singular [9]. As a result, it would be difficult to study the approach to what is effectively the upper critical angular velocity $\Omega_{c2} \sim \hbar/M\xi^2$ when the vortex cores overlap. The present analysis considers the addition of a stiffer quartic potential $V_\perp(r_\perp) = \frac{1}{4}kr_\perp^4$, which ensures confinement for any Ω . Thus, the approach to Ω_{c2} is more gradual and could well be observed. In particular, Ω_{c2}/ω_\perp exceeds 1 and should be accessible to experiments. In principle, a similar analysis is possible for other stiff radial confining potentials, for example, other power laws or an optical dipole waveguide made from a hollow blue-detuned laser beam [23].

The Thomas-Fermi approximation assumes that the total kinetic energy of the condensate is much smaller than the energies associated with the external trapping potential and the interparticle Hartree potential. This picture certainly ap-

plies for a relatively small number of vortices when $\xi \ll b \ll R_\perp$, but it fails near Ω_{c2} when $\xi \lesssim b$ because of the density gradient near the many vortex cores. This effect will not, qualitatively, alter the value of Ω_{c2} determined for the quartic potential, but it will affect the detailed description for $\Omega \lesssim \Omega_{c2}$. This interesting question remains open.

Another question concerns the number of atoms N_0 in the condensate, which here has been assumed to remain fixed. This picture also must fail sufficiently near Ω_{c2} because the vortex cores fill the entire volume. For type-II superconductors [5] at a given temperature, the vortex-core size ξ remains fixed. In that system, an increased applied magnetic field reduces the intervortex separation b , leading to the disappearance of the superconducting component because the “normal” cores eventually overlap. In a dilute Bose-Einstein gas, however, the core size $\xi = \hbar/\sqrt{2M\mu}$ itself increases and diverges as $\Omega \rightarrow \Omega_{c2}$ because of the decreasing chemical potential. Thus, the approach to the upper critical angular velocity is more sudden in a dilute trapped gas. Unfortunately, inclusion of the quantum depletion is fairly complicated for a spatially nonuniform medium [24–28] although the general formalism is well known [29,30].

At zero temperature in the grand canonical ensemble for a rotating system in equilibrium at chemical potential μ and an angular velocity Ω , for example, the ground-state expectation value of the operator $\hat{K} = \hat{H} - \mu\hat{N} - \Omega\hat{L}_z$ is the thermodynamic potential $\bar{F}(\mu, \Omega) = \langle \hat{K} \rangle$. In the presence of a Bose-Einstein condensate with N_0 condensate atoms, this function also depends on N_0 , so that $\langle \hat{K} \rangle = \bar{F}(\mu, \Omega, N_0)$. The usual thermodynamic relation $N(\mu, \Omega, N_0) = -(\partial\bar{F}/\partial\mu)_{\Omega, N_0}$ determines the mean number of particles, which here depends not only on μ and Ω , but also on N_0 . This latter parameter can be fixed by adjusting N_0 to minimize \bar{F} , so that $(\partial\bar{F}/\partial N_0)_{\mu, \Omega} = 0$. For a uniform Bose gas in a stationary box of volume V , it is straightforward to verify that this procedure yields the familiar zero-temperature depletion of the condensate $(N - N_0)/N \approx \frac{5}{3}(na^3/\pi)^{1/2}$ as well as the first correction to the chemical potential [29–31]. It should be feasible to extend this analysis to a dilute rotating trapped Bose condensate, and this problem definitely merits further study.

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