Vortex state in superfluid trapped Fermi gases at zero temperature

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We discuss various aspects of the single vortex state of a dilute superfluid atomic Fermi gas at $T=0$. The energy of the vortex in a trapped gas is calculated and we provide an expression for the thermodynamic critical rotation frequency of the trap for its formation. Furthermore, we propose a method to detect the presence of a vortex by calculating the effect of its associated velocity field on the collective mode spectrum of the gas.

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I. INTRODUCTION

Inspired by the impressive progress in recent years in the field of Bose-Einstein condensation (BEC) in dilute atomic gases, increasing attention is being devoted to examining the behavior of a gas of fermionic atoms at the same ultralow temperatures. Experimentally, the trapping and cooling of fermionic alkali-metal atoms has been demonstrated, reaching temperatures as low as $\sim T_F/4$ for ^{40}K [1] and ^{6}Li [2–4] with T_F denoting the Fermi temperature. It is well known from condensed matter and nuclear physics that a gas composed of two different internal states of the same fermionic particle, which interact via an attractive interaction, is unstable to formation of so-called Cooper pairs, thus becoming a superfluid. Since the possibility of such a superfluid transition for trapped Fermi gases was proposed [5], a lot of theoretical work has focused on various properties of this system [6]. At the same time a major experimental goal has become to observe the formation of the superfluid state.

One of the intriguing properties of a superfluid is the possibility of forming quantized vortices. For a Bose-Einstein condensate, the study of vortices has produced several interesting results [7]. Recently, some aspects of the vortex state of a trapped superfluid Fermi gas close to the critical temperature T_c of the superfluid phase transition were considered [8]. In this paper we are interested in the properties of the single vortex state of clouds of trapped Cooper-paired fermions at $T=0$, and in particular in understanding under which conditions a vortex forms, what is its energy, and how it can be detected. We consider large systems where the coherence length ξ of the superfluid is much smaller than the extent of the cloud. In this limit, we are able to derive an analytical estimate of the energy of a vortex in a trap, thereby predicting the critical rotation frequency for its formation. Also, we propose a way of observing the vortex by calculating its effect on the collective mode spectrum of the gas. The paper is organized as follows. First, in Sec. II, we examine for which values of the characteristic parameters the vortex is well localized within the gas. In Sec. III we present a simple model for calculating the energy of a vortex in a uniform superfluid Fermi system. Using the result of Sec. III, in Sec. IV we calculate the energy of a singly quantized vortex in a trapped gas and use the result to obtain the value of the thermodynamic critical rotation frequency for its formation. The problem of observing the vortex state is considered in Sec. V, where we calculate how the presence of a vortex influences the collective mode spectrum of the gas. Finally, we summarize our results in Sec. VI. Given the uncertainties intrinsic in any simple model of the vortex, such as the one presented in Sec. III, in the Appendix we briefly discuss another possible way of describing a vortex in a uniform gas, based on an approximate zero-temperature Ginzburg-Landau approach [9]. We then calculate what this alternative method gives for the energy of the vortex, and compare the two results to show that they do not differ in any significant way.

II. BASIC CONSIDERATIONS

In the dilute ultracold limit the effective interaction between identical fermionic atoms vanishes due to the Pauli principle, and that between different ones can be well described by one parameter only, the *s*-wave scattering length *a*. For a negative scattering length, the interaction is attractive and if the number of particles in the two internal states is the same the $T=0$ ground state of the gas is a superfluid. The critical temperature T_c for the transition to such a superfluid state in a dilute gas was first determined for a uniform system by Gorkov and Melik-Barkhudarov [10], and using a more modern approach by Heiselberg *et al.* [11]. The predicted value is

$$k_B T_c = \frac{\gamma}{\pi}\left(\frac{2}{e}\right)^{7/3} \epsilon_F e^{-1/\lambda}, \tag{1}$$

where λ stands for $2k_F|a|/\pi$, ϵ_F is the Fermi energy common to the two species of fermions, k_F is the associated Fermi wave number, and $\gamma \simeq 1.781$ is related to Euler's constant C by $\gamma = e^C$. The pairing gap Δ at $T=0$ is, as usual in BCS theory, related to the critical temperature by the relation $\Delta_0 = \pi\gamma^{-1} k_B T_c$ [12,13].

When applying this result to a gas trapped by a harmonic oscillator potential, as in the cases of experimental interest today, some requirements have to be met. The first one, just as for the uniform case, is that the density is everywhere so low that the gas is dilute, i.e., $k_F(\mathbf{r})|a| \ll 1$. We have introduced a local Fermi wavenumber $k_F(\mathbf{r})$. This corresponds to using the Thomas-Fermi approximation, which is valid if $\epsilon_F \gg \hbar\omega_T$, where ω_T is the frequency of the oscillator (which for the time being we assume to be isotropic). This condition is always satisfied if the particle number is sufficiently large, since for a harmonic potential $\epsilon_F=(6N_\sigma)^{1/3}\hbar\omega_T$, with N_σ being the number of particles of one species.

Another condition for applicability of Eq. (1) is that $k_BT_c \gg \hbar\omega_T$ [14]. When this latter condition is not satisfied, the shell structure of the harmonic oscillator is crucial when determining the superfluid properties of the gas, and Eq. (1) in general breaks down.

In a superfluid Fermi gas at zero temperature the size of the vortex core of a singly quantized vortex is given approximately by the BCS coherence length $\xi_{BCS}=\hbar v_F/\pi\Delta_0$, where $v_F=\hbar k_F/m_a$ is the Fermi velocity and m_a the mass of a single atom. It is clear that in order for a vortex to appear at all the BCS coherence length (size of the vortex core) at the center of the cloud has to be smaller than the size of the cloud itself, which in the Thomas-Fermi approximation is given by $R_{TF}=(2\epsilon_F/m_a\omega_T^2)^{1/2}$. If this were not so the superfluid properties of the system would be more like those of a nucleus (for which $\xi \gtrsim R$) than those of a bulk superfluid.

Substituting the appropriate expressions one can immediately see that $\xi_{BCS}/R_{TF}=\pi^{-1}\hbar\omega_T/\Delta_0$, so that requiring $\xi_{BCS} \ll R_{TF}$ corresponds to demanding that $\Delta_0 \gg \pi^{-1}\hbar\omega_T$. This condition is automatically satisfied if $k_BT_c \gg \hbar\omega_T$, but is not at all obviously realized in possible practical circumstances. Indeed, if we assume the validity of Eq. (1) and of the related value of Δ_0, and we use the expression $\epsilon_F=(6N_\sigma)^{1/3}\hbar\omega_T$ for the Fermi energy, we obtain

$$N_\sigma \gg \frac{(e/2)^7}{6\pi^3}e^{3/\lambda}. \tag{2}$$

We may then immediately see that unless λ, and therefore $k_F|a|$, is sufficiently close to 1, the exponential is very large and the condition in Eq. (2) is not satisfied, implying that the coherence length is much larger than the radius of the cloud and the rotation pattern very different from a vortex state. If, however, $k_F|a|$ is too close to 1 ($\sim$0.3–0.4 or more) the formula becomes unreliable because the gas is no longer dilute and effects due to induced interactions, which strongly modify the value of Δ_0 obtained in the dilute limit, must be taken into account. Determining the corrections to the dilute gas results due to these strong coupling effects is a complicated issue and some preliminary studies have been presented only recently [15]. It is not the purpose of the present paper to consider these effects and we thus limit our study to regions of densities in which Eq. (1) is reasonably reliable, keeping $k_F|a| \lesssim 0.4$. There is then a range of applicability of Eq. (1) for a trapped gas which depends on the number of particles N_σ and the scattering length. In order to find this region we impose the equality in Eq. (2), and we plot in Fig. 1 the critical number of atoms $N_{\sigma,c}$ for which $\xi_{BCS}/R_{TF}=1$ as a function of $k_F|a|$. Well above the curve we are in the regime where the local density approximation can be applied and a vortex may form, and below it the superfluid has a character more related to that of a nucleus. Since the value of $k_F|a|$ can be simply increased by keeping the number of particles fixed and tightening the external trapping potential, we see that if N_σ is sufficiently large ($\gtrsim 10^5$) these systems have the interesting possibility of going from one regime to the other.

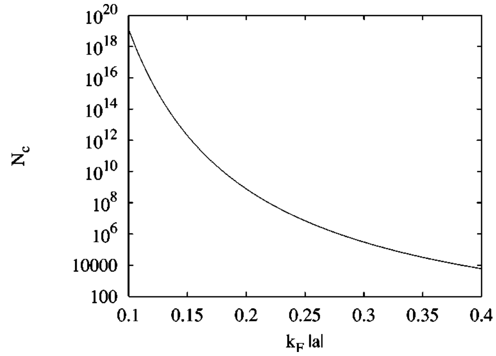

FIG. 1. Critical number of atoms per spin species for which $\xi_{BCS}/R_{TF}=1$ in an isotropic trap. Well above the line the local density approximation, and thus Eq. (1), applies; below the line the system is intrinsically finite sized.

In the remainder of this work we assume that we are in the upper region of Fig. 1 and therefore that $\xi_{BCS} \ll R_{TF}$. In this region a vortex forms in the cloud if it is stirred at an angular velocity greater than a critical one ω_{c1}, which we shall calculate using a thermodynamic approach.

III. VORTEX IN A UNIFORM GAS

Let us for the time being suppose that the vortex we want to describe is in a uniform gas. In particular, we may take the system to be in a cylinder of radius $R_c \gg \xi_{BCS}$.

Associated with the vortex there is a superfluid velocity flow which decreases with the distance ρ from the vortex axis: $\mathbf{v}_v(\rho)=\mathbf{e}_\phi\kappa\hbar/2m_a\rho$, where κ is the number of quanta of circulation of the vortex. In a simple model this velocity field extends from $\rho\sim\kappa\xi_{BCS}$ to $\rho=R_c$. At distances shorter than $\sim\kappa\xi_{BCS}$, the kinetic energy associated with the rotation becomes high enough to break the Cooper pairs, and thus the fluid inside a cylinder of radius $\sim\kappa\xi_{BCS}$ about the vortex axis can be thought of as being in a normal (nonsuperfluid) state. The energy per unit length associated with the vortex is then given by the sum of two contributions. One is the kinetic energy due to the flow,

$$\mathcal{E}_{kin}=\int_{\kappa\xi_{BCS}}^{R_c} 2\pi\rho\, d\rho\, m_a n_\sigma\left[\frac{\kappa\hbar}{2m_a\rho}\right]^2 = \frac{\pi\kappa^2\hbar^2 n_\sigma}{2m_a}\ln\frac{R_c}{\kappa\xi_{BCS}}, \tag{3}$$

and the other is the loss in condensation energy about the vortex axis,

$$\mathcal{E}_{cond}\sim\pi\kappa^2\xi_{BCS}^2\epsilon_{cond} = \frac{\pi\kappa^2\hbar^2 n_\sigma}{2m_a}\frac{3}{\pi^2}, \tag{4}$$

where $\epsilon_{cond}=3\Delta_0^2 n_\sigma/4\epsilon_F$ is the condensation energy per unit volume due to the pairing [12], and the usual expression for ξ_{BCS} has been employed. Notice that we have introduced here the one-species particle density n_σ, and we have supposed that this is a constant throughout the system, since contrary to the boson case it is not the particle density but only the pairing field that changes close to the vortex axis [12].

The total energy per unit length of a vortex is therefore given in this simple model by

$$\mathcal{E}_v = \mathcal{E}_{kin} + \mathcal{E}_{cond} \simeq \frac{\pi\kappa^2\hbar^2 n_\sigma}{2m_a}\ln\left(1.36\frac{R_c}{\kappa\xi_{BCS}}\right). \quad (5)$$

The important feature of this result is that for large systems (i.e., for which $R_c \gg \kappa\xi_{BCS}$) the most relevant contribution is the logarithmic one arising from the kinetic integration. The value of the constant inside the logarithm will depend on the choice of the model used to describe the vortex. A more reliable value would be obtained from a numerical solution of the Bogoliubov–de Gennes equations, although it is unlikely that it will differ significantly from the one found here, since one expects it in any case to be of order 1. As an example of what a different approach may yield, in the Appendix we show the result for the total energy of the vortex obtained using a zero-temperature Ginzburg-Landau model. As we shall see one obtains, as foreseen, the same expression as in Eq. (5), with coefficient 1.65 instead of 1.36 inside the logarithm.

For what follows we shall not need to know the precise value of this coefficient, which may be better determined in the future, and we shall therefore leave it unspecified and state our result as

$$\mathcal{E}_v \simeq \frac{\pi\kappa^2\hbar^2 n_\sigma}{2m_a}\ln\left(D\frac{R_c}{\kappa\xi_{BCS}}\right), \quad (6)$$

with the understanding that D is some constant of order 1. From Eq. (6) it is already clear that one vortex with $\kappa=\tilde{\kappa}\neq 1$ has greater energy than $\tilde{\kappa}$ vortices with $\kappa=1$ since in any case we need to have $\kappa\xi_{BCS} \ll R_c$. This implies that vortices with $\kappa\neq 1$ are unstable for a homogeneous system [16]. In analogy with the results for Bose-Einstein condensates [17], we expect that a vortex with multiple circulation is unstable toward the formation of several vortices with unit circulation also in the presence of a trap, but specific checks, which are beyond the scope of this work, may be needed in particular when ξ_{BCS} becomes comparable to (but still less than) the size of the cloud.

With this solution, recalling that the thermodynamic critical velocity for formation of a first vortex is given by $\omega_{c1} = \mathcal{E}_v/\mathcal{L}_v$ [18], and using the fact that the total angular momentum per unit length of the system with a vortex with unit circulation is $\mathcal{L}_v \simeq \hbar\pi R_c^2 n_\sigma$, corresponding to $\hbar$ per Cooper pair, we can immediately state what the critical velocity is in a uniform system, which is of course a well known result:

$$\omega_{c1} = \frac{\hbar}{2m_a R_c^2}\ln\left(D\frac{R_c}{\xi_{BCS}}\right). \quad (7)$$

The result should be compared with the critical velocity found for a Bose-Einstein condensate. This is completely analogous if the mass of a single bosonic atom is replaced with that of a Cooper pair ($2m_a$), and the boson coherence length by the BCS one.

Notice that since $\xi_{BCS} \propto \Delta_0^{-1}$ from the measurement of ω_{c1} one could in principle deduce the value of Δ_0 if D is known. This possibility is usually lost, however, in a nonuniform system since several values of Δ_0 are integrated over.

IV. VORTEX FORMATION IN TRAPPED GASES

In this section we calculate the energy of a vortex in a trapped gas, specializing to the trapping configurations used in a typical experiment. Since our goal is to calculate the thermodynamic critical frequency ω_{c1} for a trapped gas, we shall consider only the singly quantized vortex case. Extension to the more general case of a vortex with circulation κ is straightforward for those κ for which $\kappa\xi_{BCS}$ is still much smaller than the extent of the cloud.

The atoms are generally confined in a cylindrically symmetric harmonic potential of the form

$$V_{ext}(\mathbf{r}) = \frac{1}{2}m\omega_z^2[z^2 + \lambda_T^2(x^2+y^2)] \quad (8)$$

and the density profile of the gas is, within the Thomas-Fermi approximation, given by

$$n_\sigma(\rho,z) = n_{\sigma,0}\left(1 - \frac{\lambda_T^2\rho^2 + z^2}{R_z^2}\right)^{3/2}. \quad (9)$$

Here $n_{\sigma,0} = n_\sigma(0,0)$ is the density at the center of the cloud and we have taken the profiles of the two species to be identical. The anisotropy of the trap is controlled by the coefficient λ_T. The energy of a cloud with a vortex along the z axis can be calculated with the procedure devised by Lundh *et al.* [19]. One can divide the cloud into vertical slices of height dz and use the result (6) for a cylinder of radius ρ_1 such that $\xi_{BCS} \ll \rho_1 \ll R_\perp = R_z/\lambda_T$, within which one can assume that the gas is approximately uniform. The energy per unit length associated with the vortex in a slice at z is then given by

$$\mathcal{E}_v(z) = \frac{\pi\hbar^2 n_\sigma(0,z)}{2m_a}\ln\left(D\frac{\rho_1}{\xi_{BCS}(z)}\right) + \int_{\rho_1}^{R_\perp(z)} 2\pi\rho\, d\rho\, m_a n_\sigma(\rho,z)\left[\frac{\hbar}{2m_a\rho}\right]^2, \quad (10)$$

where $R_\perp(z) = (1 - z^2/R_z^2)^{1/2} R_z/\lambda_T$ is the value of ρ up to which the cloud extends for a given z, and $n_\sigma(0,z)$ is the density on the z axis at height z. The second term in Eq. (10) gives the kinetic energy of the superfluid outside the cylinder of radius ρ_1.

With $n_\sigma(\rho,z)$ given by Eq. (9) we then get

$$\mathcal{E}_v(z)=\frac{\pi\hbar^2 n_{\sigma,0}}{2m_a}\left[\frac{n_\sigma(0,z)}{n_{\sigma,0}}\ln\left(D\frac{\rho_1}{\xi_{BCS}(z)}\right)+\int_{\rho_1}^{R_\perp(z)}\left(1-\frac{\lambda_T^2\rho^2+z^2}{R_z^2}\right)^{3/2}\frac{d\rho}{\rho}\right]. \tag{11}$$

This result differs from the boson case in Ref. [19] in the power 3/2 instead of 1 in the density distribution. Using the fact that

$$\int (1-x^2)^{3/2}\frac{dx}{x}=\sqrt{1-x^2}+\ln\left(\frac{x}{1+\sqrt{1-x^2}}\right)+\frac{1}{3}(1-x^2)^{3/2}, \tag{12}$$

and that unless z is very close to R_z one can assume $\rho_1 \ll R_\perp(z)$, we finally obtain

$$\mathcal{E}_v(z)=\frac{\pi\hbar^2 n_{\sigma,0}}{2m_a}\left(1-\frac{z^2}{R_z^2}\right)^{3/2}\ln\left(\frac{2}{e^{4/3}}D\frac{R_\perp(z)}{\xi_{BCS}(z)}\right). \tag{13}$$

In order to proceed with the z integration we need to know the explicit dependence of ξ_{BCS} on z. In the dilute gas approximation where Eq. (1) is valid this is given by

$$\xi_{BCS}(z)=\frac{2}{\pi}\left(\frac{e}{2}\right)^{7/3}\left(1-\frac{z^2}{R_z^2}\right)^{-1/2}\times\exp\left[\frac{1}{\lambda_0}\left(1-\frac{z^2}{R_z^2}\right)^{-1/2}\right]k_{F,0}^{-1}, \tag{14}$$

where $k_{F,0}=(2m_a\epsilon_F/\hbar^2)^{1/2}$ and $\lambda_0=2k_{F,0}|a|/\pi$ are the local Fermi wave number and λ, respectively, evaluated at the center of the cloud. Inserting this value into Eq. (13), using the expression for $R_\perp(z)$, and integrating over z, we get after some cumbersome but straightforward calculations

$$E_v=\frac{\pi\hbar^2 n_{\sigma,0}}{2m_a}\frac{4}{3}R_z\left[\frac{9\pi}{32}\ln\left(D\frac{2^{4/3}\pi\epsilon_F}{e^{5/2}\hbar\omega_\perp}\right)-\frac{1}{\lambda_0}\right]. \tag{15}$$

Note that Eq. (15) predicts the energy cost of the vortex to be negative for small $k_{F,0}|a|$ and $\epsilon_F/\hbar\omega_\perp$ not too large. This is clearly an unphysical result reflecting the fact that, in the limit of relatively few particles trapped and small $k_{F,0}|a|$, the condition $\xi_{BCS}\ll R_\perp$ is violated, making Eq. (15) invalid. In the regime $\xi_{BCS}\ll R_\perp$, Eq. (15) yields positive vortex energies as expected. If we ignore the nonrotating particles at the core of the vortex, the total angular momentum of a unit circulation vortex state is $L_v=N_\sigma\hbar$ and the critical rotation frequency $\omega_{c1}=E_v/L_v$ for the formation of a vortex in a trap given by Eq. (8) is

$$\omega_{c1}=\omega_\perp\frac{16}{3\pi}\frac{l_\perp^2}{R_\perp^2}\left[\frac{9\pi}{32}\ln\left(\frac{2^{1/3}\pi D}{e^{5/2}}\frac{R_\perp^2}{l_\perp^2}\right)-\frac{1}{\lambda_0}\right], \tag{16}$$

with $l_\perp=\sqrt{\hbar/m\omega_\perp}$ being the harmonic oscillator length in the radial direction. For realistic parameters of the gas, this critical frequency is rather small: choosing for D the value obtained in the Appendix, taking $k_{F,0}|a|=0.4$, $\omega_\perp=\omega_z=\omega_T$ (i.e., $\lambda_T=1$), and $\epsilon_F=200\hbar\omega_T$ corresponding to an isotropic trap with $N_\sigma\sim 1.3\times 10^6$, we obtain $\omega_{c1}\simeq 0.0035\omega_\perp$. The reason for the critical frequency being so small is that the angular momentum per atom is $\hbar/2$, yielding $L_v=N_\sigma\hbar$, whereas the energy given by Eq. (15) only scales as $N_\sigma^{2/3}$. The Fermi pressure expands the cloud and reduces the density at the center of the trap. Since the energy of the vortex mainly comes from regions close to the vortex axis where the superfluid velocity is high, the energy of the vortex is correspondingly reduced.

V. OBSERVATION OF THE VORTEX

As we already mentioned, contrary to the situation for Bose-Einstein condensates, the presence of a vortex in the Fermi gas does not alter the density profile significantly [12]. One cannot therefore reveal the vortex simply by looking at the density profile. Use of the laser probing method of Ref. [20] has been suggested to detect the local decrease of the pairing near the center of the vortex [8]. Here we examine a different method based on measuring the collective mode spectrum of the gas. In the case of no vortex present, excitations of the gas carrying equal and opposite angular momentum along the z axis are degenerate in energy. The velocity field associated with a vortex aligned with this axis lifts the degeneracy since the rotational symmetry is removed; the velocity flow of the excitation is either parallel or antiparallel to that of the vortex giving rise to an energy splitting of the modes [21–23]. Since the collective mode frequencies of the gas can be measured with a fairly high precision, the possibility of detecting the presence of the vortex by its spectroscopic signatures is a promising method. Indeed, this method has proven to be very useful in the case of a vortex in a Bose-Einstein condensate [24]. The calculations will be carried out for an isotropic trap with $R_\perp=R_z=R_{TF}$ and $\omega_\perp=\omega_z=\omega_T$.

In the $\xi_{BCS}\ll R_{TF}$ limit considered in this paper, the collective modes of the superfluid gas for $T=0$ can be calculated using a hydrodynamic theory. The relevant continuity and superfluid velocity equations read [25]

$$\partial_t n(\mathbf{r},t)=-\nabla\cdot\mathbf{j}(\mathbf{r},t),$$

$$\partial_t\mathbf{v}_s(\mathbf{r},t)=-\frac{1}{m_a}\nabla[m_a|\mathbf{v}_s|^2/2+\mu_F+V_{ext}] \tag{17}$$

with $n_s(\mathbf{r},t)$, $n_n(\mathbf{r},t)$, and $n(\mathbf{r},t)=n_s(\mathbf{r},t)+n_n(\mathbf{r},t)$ being the superfluid, normal, and total density of the gas, respectively. The total current is $\mathbf{j}(\mathbf{r},t)=n_s(\mathbf{r},t)\mathbf{v}_s(\mathbf{r},t)+n_n(\mathbf{r},t)\mathbf{v}_n(\mathbf{r},t)$, where $\mathbf{v}_s(\mathbf{r},t)$ is the superfluid velocity and $\mathbf{v}_n(\mathbf{r},t)$ the velocity of the normal fluid. For $\xi_{BCS}\ll R_{TF}$, the extent of the vortex core is small compared to the size of the cloud, and the main effect of the vortex on the collective mode spectrum is the presence of the vortex velocity field $\mathbf{v}_v(\mathbf{r})=\mathbf{e}_\phi\kappa\hbar/2m_a\rho$. Writing $n(\mathbf{r},t)=n_0(\mathbf{r})+\delta n(\mathbf{r},t)$ and

$\mathbf{v}_s(\mathbf{r},t)=\mathbf{v}_v(\mathbf{r})+\mathbf{u}(\mathbf{r},t)$, where $n_0(\mathbf{r})$ is the equilibrium density profile with the vortex alone (which we take to be coincident with the Thomas-Fermi one without vortex), and linearizing in $\delta n(\mathbf{r},t)$ and $\mathbf{u}(\mathbf{r},t)$, Eq. (17) can be written as

$$\left(\omega-\frac{\kappa\hbar m}{2m_a\rho^2}\right)^2 m_a n_0(\mathbf{r})\left[\frac{\partial n_0(\mathbf{r})}{\partial P_0(\mathbf{r})}\right]_{T=0}\Phi(\mathbf{r},t) = -\boldsymbol{\nabla}\cdot[n_0(\mathbf{r})\boldsymbol{\nabla}\Phi(\mathbf{r},t)]. \tag{18}$$

Here, ω is the frequency of the collective mode, $P_0(\mathbf{r})$ is the equilibrium pressure profile, and m is the magnetic quantum number of the mode. The velocity field associated with the mode has been written as $\mathbf{u}(\mathbf{r},t)=\boldsymbol{\nabla}\Phi(\mathbf{r},t)$ with $\Phi(\mathbf{r},t)=\Phi(r,\theta)\exp[i(m\phi-\omega t)]$. The term $\kappa\hbar m/2m_a\rho^2$ in Eq. (18) comes from the presence of the vortex velocity field $\mathbf{v}_v$. Without this term, Eq. (18) has been solved for a spherical symmetric trap by writing $\Phi_{nlm}(\mathbf{r})=\Phi_{nl}(r)Y_{lm}(\theta,\phi)$, yielding the spectrum $\omega_{nl0}=2\omega_T\sqrt{(n^2+2n+ln+3l/4)/3}$ with $n=0,1,2,\ldots$ [26]. From Eq. (18), we see that the frequency shift of a given mode induced by the vortex can be calculated perturbatively as

$$\omega_{nlm}^2-\omega_{nl0}^2=\frac{\kappa\hbar m\omega_{nl0}}{m_a}\frac{\langle\Phi_{nlm}|\rho^{-2}|\Phi_{nlm}\rangle}{\langle\Phi_{nlm}|\Phi_{nlm}\rangle}. \tag{19}$$

Here $\langle\Phi_{nlm}|f(\mathbf{r})|\Phi_{nlm}\rangle$ denotes the spatial average $\int_0^{R_z}w(r)dr\int d\Omega\,\Phi_{nlm}^2(\mathbf{r})f(\mathbf{r})$ with the weight function $w(r)=r^2(1-r^2/R_{TF}^2)^{1/2}$. This anomalous weight has to be introduced in place of simply r^2 because the operator in Eq. (18) without the perturbation is not Hermitian.

As pointed out in Ref. [23], the perturbative procedure works for $|m|\geqslant 2$; for $|m|<2$ it predicts an unphysical $\rho\to 0$ divergence in the density fluctuation of the mode. With no vortex present, the lowest mode for a given angular momentum l is the surface mode $\Phi_{n=0lm}(r)\propto r^l$ with frequency $\sqrt{l}\omega_T$. Recalling that $\rho=r\sin\theta$ and using the fact that $(4\pi)^{-1}\int d\Omega|Y_{lm}(\Omega)|^2/\sin^2\theta=(2l+1)/2|m|$, the matrix elements in Eq. (19) can be calculated analytically for these surface modes and we obtain for the frequency shift

$$\frac{\omega_{lm}^2-\omega_{l0}^2}{\omega_{l0}^2}=\mathrm{sgn}(m)\frac{\kappa(l+2)}{2\sqrt{l}(6N_\sigma)^{1/3}}, \tag{20}$$

with $|m|\geqslant 2$. As expected, the vortex splits the $2l+1$ degenerate modes depending on the direction of the projection of their angular momentum on the z axis. Not all the modes are split, however, since the splitting is independent of $|m|$ in analogy with the equivalent result for bosons [23]. Particularly important is the result $(\omega_{2,\pm 2}^2-\omega_{20}^2)/\omega_{20}^2=\pm\sqrt{2}\kappa/(6N_\sigma)^{1/3}$ for the quadrupolar mode $l=2$, $m=\pm 2$, since this mode is easily excited in trapped gases and has already been employed for a precise determination of the critical frequency for vortex nucleation in Bose gases.

The same result can be obtained following the sum rule approach of Ref. [21]. From that the splitting is found to be given by $\omega_{2,2}-\omega_{2,-2}=2\langle l_z\rangle/(m_a\langle\rho^2\rangle)$, where $\langle l_z\rangle$ is the expectation value of the angular momentum along the z axis per atom ($\kappa\hbar/2$ in the case of a vortex), and $\langle\rho^2\rangle$ is the expectation value of x^2+y^2 (equal to $R_{TF}/4$ for an isotropic cloud with a Thomas-Fermi density profile). From the latter result one can immediately see that the splitting of the modes of a Fermi superfluid is in general smaller than in the BEC case, the reason being that, given the same number of atoms, the radius of a fermionic cloud is usually larger due to the Pauli repulsion and thus the expectation value of $r_\perp^2$ is also correspondingly larger, and the splitting reduced. For $2N_\sigma=10^6$ particles trapped, the $m=\pm 2$ quadrupole modes are split by $\sim 1\%$. Although this is a rather small shift, it should be measurable assuming that the same high spectroscopic precision demonstrated for BEC's can be obtained for trapped Fermi gases [27].

VI. CONCLUSION

In this paper we considered various aspects of the vortex state of a dilute superfluid Fermi gas at $T=0$. For a trapped system, we found that a large number of particles and a not too small scattering length yield $\xi_{BCS}\ll R_{TF}$ and the vortex is well confined within the gas. We then used a simple model to calculate the energy of a vortex in a uniform medium. Subsequently, using the fact that the structure of the vortex near the rotation axis is essentially unaffected by the trapping potential we derived an expression for the vortex energy in a trap, and we employed this energy expression to calculate the thermodynamic critical rotation frequency for the formation of a vortex. Finally, we suggested a way of observing the presence of the vortex by calculating perturbatively its influence on the collective mode spectrum of the gas. In the Appendix we report an alternative, less naive, description of the vortex in a uniform medium and find a slightly different value for its energy compared to the one obtained in Sec. III.

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APPENDIX: A GINZBURG-LANDAU DESCRIPTION OF THE VORTEX CORE

In this Appendix, we present a Ginzburg-Landau description of the vortex core in a uniform gas, and the consequent result for the total energy of a vortex in a cylindrical bucket of radius R_c. As is well known, Ginzburg-Landau theory is only valid for temperatures such that $|T-T_c|/T_c\ll 1$ but the following calculation can be used for a qualitative estimate at $T=0$ [9].

The extension of the Ginzburg-Landau theory to zero temperature for a uniform system can be done by imposing that the free energy [13]

$$F_{GL}=\int d^3r\, f_{GL}(\mathbf{r}) = \int\left[\frac{\hbar^2|\nabla\psi|^2}{4m_a}+A|\psi|^2+\frac{B}{2}|\psi|^4\right]d^3r \tag{A1}$$

be equal to the condensation energy density, which in a uniform system is given by $\epsilon_{cond}=-3\Delta_0^2 n_\sigma/4\epsilon_F$ [12]. ψ is here the well known Ginzburg-Landau order parameter. Upon minimization of Eq. (A1) with respect to ψ^* we obtain the Ginzburg-Landau equation

$$-\frac{\hbar^2}{4m_a}\nabla^2\psi+A\psi+B|\psi|^2\psi=0. \tag{A2}$$

For a uniform system the solution is $|\psi_0|^2=-A/B$, and from the fact that the Ginzburg-Landau free energy then coincides with the condensation one $f_{GL}=-3\Delta_0^2 n_\sigma/4\epsilon_F$ we obtain $A=-3\Delta_0^2/2\epsilon_F$ and $B=-A/n_\sigma$.

We now calculate the structure and energy of the vortex. A vortex along the z axis is described by writing the order parameter in cylindrical coordinates as $\psi(\mathbf{r})=f(\rho)e^{i\kappa\phi}$. Replacing this expression into Eq. (A2) we find

$$-\frac{1}{x}\frac{d}{dx}\left(x\frac{d\chi}{dx}\right)+\frac{\kappa^2}{x^2}\chi+\chi^3-\chi=0, \tag{A3}$$

where we introduced the dimensionless quantities $\chi=f/|\psi_0|$ and $x=\rho/\xi_{GL}$. We used the fact that f does not vary along the z direction if the system is uniform, and we defined the Ginzburg-Landau coherence length $\xi_{GL}^2=\hbar^2/4m_a A$, which implies $\xi_{GL}=\hbar v_F/2\sqrt{3}\Delta_0=0.907\xi_{BCS}$. This equation has exactly the same form as the Gross-Pitaevskii equation for a vortex in a uniform boson cloud [16,28]. It can be solved numerically and the results for the lowest κ ($\kappa=1,2,3$) was first obtained by Ginzburg and Pitaevskii [16]. A very good approximate solution for $\kappa=1$ can be obtained by a variational calculation yielding $\chi=x/(2+x^2)^{1/2}$ [29]. Using this solution in Eq. (A1), one finds that the energy cost associated with the vortex is given by

$$\mathcal{E}_v(z)=\pi\frac{\hbar^2}{2m_a}n_\sigma\ln\left(\frac{e^{3/4}}{\sqrt{2}}\frac{R_c}{\xi_{GL}}\right). \tag{A4}$$

This result is now identical with that for a Bose-Einstein condensate, with the mass of a single bosonic atom replaced by that of a Cooper pair $(2m_a)$, and the boson coherence length replaced by the Ginzburg-Landau one. Using $\xi_{GL}=0.907\xi_{BCS}$, we obtain $\mathcal{E}_v=\pi\hbar^2 n_\sigma\ln(DR_c/\xi_{BCS})/2m_a$ with $D=1.65$.

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