Teleportation with the entangled states of a beam splitter

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We present a teleportation protocol based upon the entanglement produced from Fock states incident onto a beam splitter of arbitrary transmissivity. The teleportation fidelity is analyzed, its trends being explained from consideration of a beam splitter's input-output characteristics.

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Entanglement is a resource with which to perform quantum information processing tasks, such as quantum computing [1–4], quantum error correction [5,6], dense coding [7,8], and quantum teleportation [9–12]. In particular, teleportation has generated a lot of interest since it was first proposed [9] and demonstrated [10,11]. There are many protocols for teleportation using both discrete and continuous variables [9,12–14], nevertheless, all are based upon the original proposal. For further related work, the reader is directed to references [15–22].

In this paper, we generalize and expand upon results of previous work [13], showing how harmonic-oscillator states entangled on a beam splitter may be used as an entanglement resource for teleportation. We describe the teleportation protocol and derive the fidelity of output showing its behavior as a function of the difference in photon number incident to the beam splitter and the transmission properties of the beam splitter. The average fidelity trends are as expected from a simple consideration of the beam splitter.

The process of teleportation may be explained in general terms as follows: There are two parties who wish to communicate quantum information between one another; a sender, Alice, and a receiver, Bob. Alice and Bob initially share one part each of a bipartite entangled system. Alice also has a particle of an unknown quantum state, this being the information she wishes to send to Bob. She sends this information by making *joint* measurements on her part of the entangled pair and the unknown particle, and then sending the results of these measurements to Bob via the classical channel. Bob may then recreate the unknown quantum state perfectly (in principle) after performing local unitary transformations on his part of the entangled pair. The important point is that, in principle, perfect transmission of quantum information is possible between spatially separated points, but only with the help of quantum entanglement.

There are many processes involved in performing teleportation; the measurements made by Alice, the transmission of the classical information, and the transformations made by Bob. If one assumes that these processes are all performed perfectly, then the only influence on the efficacy of teleportation will be the quality of the entanglement.

Consider the experiment shown schematically in Fig. 1. Two Fock states, number $\mathcal{N}$ in mode A, and $\mathcal{M}$ in mode B, are incident on a beam splitter with transmissivity described by the parameter β. Mode A goes to Alice and mode B goes to Bob. Alice makes joint number sum and phase difference measurements [13] on the target and mode A. She sends the results of these measurements to Bob via the classical channel, who then applies the relevant unitary transformations on his mode to attempt to recreate the target state at his location.

The input Fock states are entangled via the beam splitter interaction; described by

$$|\psi\rangle_{AB}=e^{i\beta(a^\dagger b+b^\dagger a)/2}|\mathcal{N}\rangle_A|\mathcal{M}\rangle_B, \tag{1}$$

where a, $a^\dagger$, b, and $b^\dagger$ are the usual boson annihilation and creation operators for modes A and B, respectively. The variable β describes the transmissivity of the beam splitter; $\beta=0$ corresponds to all transmission and no reflection, $\beta=\pi$ corresponds to all reflection and no transmission, and $\beta=\pi/2$ corresponds to a 50:50 beam splitter. When the total photon number is fixed, these states may be written in a pseudo angular momentum algebra, allowing the resource to be expanded in terms of eigenstates of constant number sum. The resource state is

$$\rho_{AB}=\sum_{n,n'=0}^{2N} d_{n-N}d^*_{n'-N}|n\rangle_A\langle n'|\otimes|2N-n\rangle_B\langle 2N-n'| \tag{2}$$

The d_{n-N} are

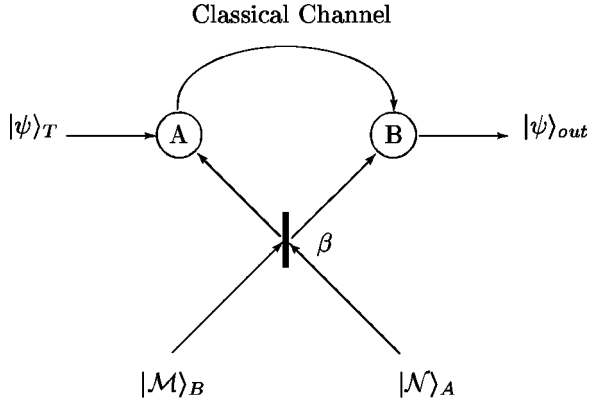

FIG. 1. Schematic experimental setup for teleportation protocol. $|\mathcal{M}\rangle_B$ and $|\mathcal{N}\rangle_A$ are input Fock states to a beam splitter of transmissivity β. The sender of the target state $|\psi\rangle_T$ is at A and the receiver is at B. The state exiting the teleportation process is denoted by $|\psi\rangle_{out}$.

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$$d_{n-N}=e^{-i(\pi/2)(n-N-m)}D^{2N}_{n-N,m}(\beta), \tag{3}$$

where $m=(\mathcal{N}-\mathcal{M})/2$ is the incident photon number difference and the $D^{j}_{m',m}(\beta)$ being the rotation matrix coefficients [23] given by

$$D^{j}_{m',m}(\beta)=[(j+m')!(j-m')!(j+m)!(j-m)!]^{1/2}\sum_{s}\frac{(-1)^{m'-m+s}\left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2s}\left(\sin\frac{\beta}{2}\right)^{m'-m+2s}}{(j+m-s)!s!(m'-m+s)!(j-m'-s)!}. \tag{4}$$

The variable s ranges over all possible values such that the factorials are positive. The resource states are eigenstates of number sum and tend to eigenstates of phase difference in the limit of a large total photon number (for details, see Ref. [13]).

The quality of information transfer is measured by the overlap between the target state and the output state. This is the *fidelity* that we define by

$$F={}_T\langle\psi|\rho_{out,B}|\psi\rangle_T. \tag{5}$$

We now show the mechanics of our teleportation protocol in order to calculate the teleportation fidelity. We teleport an arbitrary state of the form

$$\rho_T=\sum_{m,m'=0}^{\infty}c_m c^{*}_{m'}|m\rangle_T\langle m'|. \tag{6}$$

The subscript T emphasizes that this is the ''target'' state and the c_m are chosen such that the state is normalized. The total state of the system is the tensor product between this and ρ_{AB},

$$\rho_{TAB}=\rho_T\otimes\rho_{AB}. \tag{7}$$

Alice makes joint measurements of number sum (yielding result q) and phase difference (result ϕ_-) on the target and her half of the entangled pair, mode A. The state of the system conditioned on these measurements is

$$\begin{aligned}\rho^{(q,\phi_-)}=&\sum_{w,y,x',z'=0}^{\infty}e^{i(y-w+x'-z')\phi_-}\delta_{w-q+y}\delta_{q-x'-z'}\\&\times|w\rangle_T\langle x'|\otimes|y\rangle_A\langle z'|\\&\otimes\frac{1}{P(q)}\sum_{n,n'=0}^{min(q,2N)}e^{-2i(n-n')\phi_-}\\&\times c_{q-n}c^{*}_{q-n'}d_{n-N}d^{*}_{n'-N}\\&\times|2N-n\rangle_B\langle 2N-n'|,\end{aligned} \tag{8}$$

where

$$P(q)=\sum_{n=0}^{min(q,2N)}|c_{q-n}|^2|d_{n-N}|^2, \tag{9}$$

is the probability of measuring a given number-sum result q. Alice transmits the results of these measurements to Bob via the classical channel. Bob makes the amplification operations

$$\begin{aligned}|2N-n\rangle_B&\rightarrow|q-n\rangle_B\\{}_B\langle 2N-n'|&\rightarrow{}_B\langle q-n'|\end{aligned} \tag{10}$$

and the phase-shift $e^{2i(n-n')\phi_-}$. The unitary amplification operation is described in [24] and in more detail in [25]; other amplification techniques are discussed by Yuen [26] and Björk and Yamamoto [27]. The amplifications and phase transformations complete the protocol. Bob's state is then

$$\begin{aligned}\rho_{out,B}=&\frac{1}{P(q)}\sum_{n,n'=0}^{min(q,2N)}c_{q-n}c^{*}_{q-n'}d_{n-N}d^{*}_{n'-N}\\&\times|q-n\rangle_B{}_B\langle q-n'|,\end{aligned} \tag{11}$$

and the teleportation fidelity is,

$$F(q)=\frac{1}{P(q)}\sum_{n,n'=0}^{min(q,2N)}|c_{q-n}|^2|c_{q-n'}|^2d_{n-N}d^{*}_{n'-N}. \tag{12}$$

Note that the fidelity is dependent upon the number-sum measurement result (q). To obtain an overall figure of merit for the protocol, we remove this dependence by defining the *average fidelity*,

$$\overline{F}=\sum_q P(q)F(q). \tag{13}$$

For our protocol this is

$$\overline{F}=\sum_{q=0}^{\infty}\sum_{n,n'=0}^{min(q,2N)}|c_{q-n}|^2|c_{q-n'}|^2d_{n-N}d^{*}_{n'-N}. \tag{14}$$

If one sets $\mathcal{N}=\mathcal{M}$, this result is identical to that obtained in Ref. [13] without decoherence.

Teleportation fidelity for transmission of an ''even cat'' target state of amplitude $\alpha=3$ is shown in Fig. 2. An even cat state is the even superposition of two coherent states of equal amplitude but opposite phase [28], i.e.,

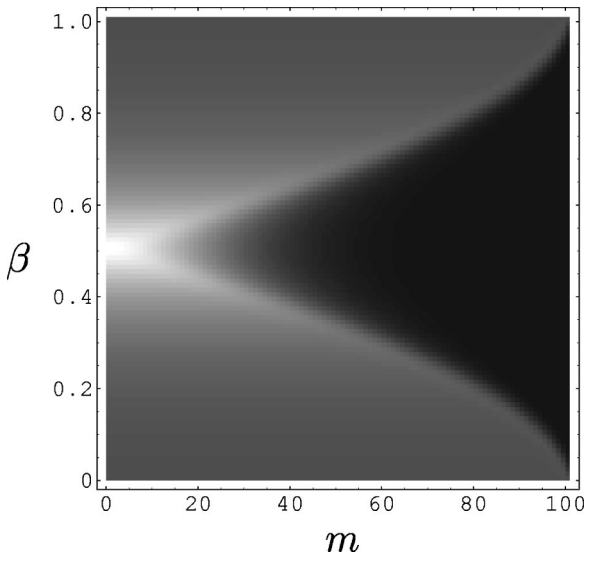

FIG. 2. Density plot of average fidelity as a function of m and β for an ''even cat'' target state of amplitude $\alpha=3$ and total photon number of 100. β is in units of π; $\beta=\pi/2$ corresponds to a 50:50 beam splitter; black corresponds to zero, and white to unity.

$$|\mathrm{cat}\rangle_{even}=\frac{|\alpha\rangle+|-\alpha\rangle}{\sqrt{2+2\exp(-2|\alpha|^2)}}. \tag{15}$$

Many of the trends shown in Fig. 2 can be explained by simple consideration of a beam splitter. As the beam splitter becomes more biased (β tends to either 0 or π), the outgoing photons are partitioned less evenly and the entanglement resource is distorted. This is evident by the average fidelity decreasing to the classical level [34] at $\beta=0$ and $\beta=\pi$. At these extremes, the setup is completely biased with all incident photons being sent in one direction, so there are no phase correlations between the modes above the classical level. Changing the photon number difference also changes the partitioning of outgoing photons, hence, the fidelity decreases with increasing m for the same reasons outlined above. The input photon number difference and beam splitter transmissivity may have opposing photon partitioning effects, thereby keeping the fidelity high. This is evident by the ''ridges'' of the fidelity surface. The ridges decrease in height with increasing m implying that although the two biases are in opposition, the resource is still being distorted.

We may show why the ridges occur in a more quantitative fashion with the aid of the joint phase probability of the resource state. This is

$$P(\phi_1,\phi_2,\beta,m)=|\langle\phi_1|\langle\phi_2|\psi\rangle_{AB}|^2, \tag{16}$$

where the $|\phi_j\rangle$ are the phase states

$$|\phi\rangle_j=\sum_{n=0}^{\infty} e^{-i\phi_j n}|n\rangle. \tag{17}$$

Equation (16) may be written in a more explicit form,

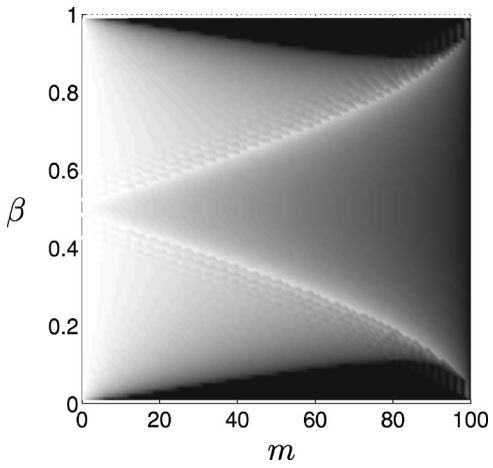

FIG. 3. Density plot of phase difference ϕ_- corresponding to a maximum joint phase probability $P_{max}(\phi_-,\beta,m)$, as a function of beam splitter transmissivity β and input photon number difference m. The ridge structure here helps to explain the ridge structure of the average fidelity $\overline{F}$ as a function of the same variables. Black corresponds to zero, white to $\pi/2$.

$$P(\phi_-,\beta,m)=\left|\sum_{n=0}^{2N} e^{in\phi_-}d_{n-N}(\beta,m)\right|^2, \tag{18}$$

where ϕ_- is the phase difference $\phi_1-\phi_2$. This is a function of three variables: phase difference ϕ_-, beam splitter transmissivity β, and photon number difference m, and is consequently not easy to analyze graphically. However, if one finds the maximum of $P(\phi_-,\beta,m)$ over ϕ_- for given β and m (we call this quantity P_{max}), and the value of ϕ_- that corresponds to this maximum, then we obtain more easily interpretable information. We show in Fig. 3 the value of ϕ_- corresponding to P_{max} as a function of β and m. This function shows the same ridge structure as Fig. 2. When $\beta=\pi/2$ and $m=0$, the average fidelity is a maximum and the joint phase probability density has a maximum at $\phi_-=\pi/2$. For other values of β and m, the ridges in the average fidelity correspond to where the phase distribution has a maximum near $\pi/2$ and where the protocol is therefore better.

Testing our results experimentally will be difficult in the optical regime. However, recent experiments [29–31] showing generation of Fock states, and proposals using alternative technologies [33,32] indicate some future possibility of exploring the ideas presented here.

We have shown how Fock states entangled on a beam splitter may be used as an entanglement resource for teleportation in the case of arbitrary beam splitter properties and arbitrary input Fock states. We have studied how varying the beam splitter transmissivity and input photon number difference influences the average fidelity. The results are consistent with an analysis of how entanglement varies with these parameters.

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